

## PURELY PERIODIC $\beta$ -EXPANSIONS WITH PISOT UNIT BASE

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ABSTRACT. Let  $\beta > 1$  be a Pisot unit. A family of sets  $\{X_i\}_{1 \leq i \leq q}$  defined by a  $\beta$ -numeration system has been extensively studied as an atomic surface or Rauzy fractal. For the purpose of constructing a Markov partition, a domain  $\hat{X} = \bigcup_{i=1}^q \hat{X}_i$  constructed by an atomic surface has appeared in several papers. In this paper we show that the domain  $\hat{X}$  completely characterizes the set of purely periodic  $\beta$ -expansions.

### 1. INTRODUCTION

**1.1. Purely periodic  $\beta$ -expansion.** Let  $\beta > 1$  be a real number. The  $\beta$ -expansion of a real number  $x \in [0, 1]$  is defined as the sequence  $(x_i)_{i \geq 1}$  with values in  $\{0, 1, \dots, [\beta]\}$  produced by the  $\beta$ -transformation  $T_\beta : x \mapsto \beta x \pmod{1}$  as follows:

$$\forall i \geq 1, x_i = \lfloor \beta T_\beta^{(i-1)} x \rfloor, \text{ and thus } x = \sum_{i=1}^{\infty} x_i \beta^{-i} = 0.x_1 \dots x_n \dots$$

An expansion is *finite* if  $(x_i)_{i \geq 1}$  is eventually 0. A  $\beta$ -expansion is *periodic* if there exists  $p \geq 1$  and  $M \geq 1$  such that  $x_k = x_{k+p}$  holds for all  $k \geq M$ ; if  $x_k = x_{k+p}$  holds for all  $k \geq 1$ , then it is *purely periodic*. We denote by  $\text{Per}(\beta)$  the numbers in  $[0, 1)$  with periodic  $\beta$ -expansions and by  $\text{Pur}(\beta)$  the numbers in  $[0, 1)$  with purely periodic  $\beta$ -expansions.

Let  $\mathbb{Q}(\beta)$  be the smallest fields containing  $\mathbb{Q}$  and  $\beta$ . An easy argument shows that  $\text{Per}(\beta) \subseteq \mathbb{Q}(\beta) \cap [0, 1)$  for every real number  $\beta > 1$ . K. Schmidt [17] showed that if  $\beta$  is a *Pisot number* (an algebraic integer whose conjugates have modulus  $< 1$ ), then  $\text{Per}(\beta) = \mathbb{Q}(\beta) \cap [0, 1)$ . The purely periodic  $\beta$ -expansions are also discussed in [17].

**Property 1** (Schmidt [17]). *Suppose that  $\beta$  satisfies  $\beta^2 = n\beta + 1$  for some integer  $n \geq 1$ . Then every  $x \in \mathbb{Q} \cap [0, 1)$  has purely periodic  $\beta$ -expansion.*

A number is called a *Pisot unit* if it is a Pisot number as well as an algebraic unit. The purpose of this paper is to characterize the set  $\text{Pur}(\beta)$  when  $\beta$  is a Pisot unit. To state our result, we need some preparations.

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**1.2. Automaton of the  $\beta$ -numeration system.** Let  $d_\beta(1) = (b_i)_{i \geq 1}$  denote the  $\beta$ -expansion of 1. Let  $d_\beta^*(1) = d_\beta(1)$  if  $d_\beta(1)$  is infinite, and let  $d_\beta^*(1) = (t_1 \dots t_{m-1}(t_m - 1))^\infty$  if  $d_\beta(1) = t_1 \dots t_{m-1}t_m$  is finite (with  $t_m \neq 0$ ). A sequence is *admissible* if and only if starting from any place in the sequence, the right side truncation is lexicographically strictly less than  $d_\beta^*(1)$ . If the right side truncations are less than or equal to  $d_\beta^*(1)$ , then the sequence is called *weakly admissible*.

When  $d_\beta^*(1)$  is eventually periodic (this holds for any Pisot number), the set of weakly admissible  $\beta$ -representations can be recognized by the following automaton  $M_\beta$ . Assume that

$$d_\beta^*(1) = (b_i)_{i \geq 1} = b_1 \dots b_{q-p} \overline{b_{q-p+1} \dots b_q},$$

so  $b_{k+p} = b_k$  for all  $k > q - p$ . Let  $r_k = b_k b_{k+1} \dots$ ,  $k \geq 1$ . (We denote by  $\bar{r}_k = 0.b_k b_{k+1} \dots$  the corresponding real number.) Then  $\{r_k\}_{k \geq 1}$  is a finite set and actually it is  $\{r_1, \dots, r_q\}$ . Let the states of  $M_\beta$  be the integers  $1, \dots, q$  together with a fail state  $F$ . The initial state is 1. From a state  $j < q$ ,  $b_j$  leads to state  $j + 1$ , while the arrows with labels less than  $b_j$  lead to state 1, and all arrows whose labels are greater than  $b_j$  fail. From state  $q$ ,  $b_q$  leads to state  $q - p + 1$ , while all arrows with lower labels lead back to 1 and all arrows with higher labels fail (cf. for example [13, 14, 20]).

**1.3. Canonical embedding.** We denote by  $\mathbb{Z}(\beta)$  the smallest ring containing  $\mathbb{Z}$  and  $\beta$ . From now on we assume that  $\beta > 1$  is a Pisot unit with minimal polynomial  $x^d = a_{d-1}x^{d-1} + \dots + a_1x + a_0$ , where  $a_0 = \pm 1$ . Let

$$A_\beta = \begin{pmatrix} 0 & \cdots & 0 & a_0 \\ 1 & & 0 & a_1 \\ & \ddots & & \vdots \\ 0 & & 1 & a_{d-1} \end{pmatrix}$$

be the *companion matrix* of  $\beta$ . Since  $\{1, \beta, \dots, \beta^{d-1}\}$  is a basis of the field  $\mathbb{Q}(\beta)$ , we define a mapping  $\phi : \mathbb{Q}(\beta) \mapsto \mathbb{Q}^d$  by

$$\phi(z_1 + z_2\beta + \dots + z_d\beta^{d-1}) = (z_1, z_2, \dots, z_d)^T,$$

where  $v^T$  is the transpose of a vector  $v$ . Clearly for  $x \in \mathbb{Q}(\beta)$ , it follows that

$$(1.1) \quad \phi(\beta x) = A_\beta \phi(x).$$

We consider  $A_\beta$  as a linear transformation on  $\mathbb{R}^d$ . Each real root of the minimal polynomial of  $\beta$  is an eigenvalue for this transformation, with a 1-dimensional eigenspace, and for each pair of complex conjugate roots there is a 2-dimensional invariant subspace. Denote by  $V$  the eigenspace of  $\beta$ . Then  $V$  is the expanding eigenspace and the other eigenspaces are contractive. Let  $P$  be the direct sum of all these contractive eigenspaces. Then  $\mathbb{R}^d = V \oplus P$ ,  $\dim P = d - 1$  and  $\dim V = 1$ . According to this direct sum we define two natural projections  $\pi : \mathbb{R}^d \mapsto P$  and  $\pi' : \mathbb{R}^d \mapsto V$ . It is easy to see that

$$(1.2) \quad A_\beta \circ \pi = \pi \circ A_\beta, \quad A_\beta \circ \pi' = \pi' \circ A_\beta.$$

**1.4. Construction of Markov partition.** Let  $w_n \dots w_1 w_0$  be an admissible sequence. We say  $w_n \dots w_1 w_0$  stops at state  $i$  if there is a path with label  $w_n \dots w_1 w_0$  which starts from state 1 and ends at state  $i$ . Let

$$Y = \left\{ \sum_{i=0}^n w_i \beta^i : w_n \dots w_1 w_0 \text{ is admissible, } n \geq 0 \right\},$$

$$Y_i = \left\{ \sum_{i=0}^n w_i \beta^i : w_n \dots w_1 w_0 \text{ is admissible and stops at state } i, n \geq 0 \right\}.$$

Clearly  $\{Y_1, \dots, Y_q\}$  is a partition of  $Y$ . Define

$$X = \overline{\pi \circ \phi(Y)} \text{ and } X_i = \overline{\pi \circ \phi(Y_i)}, \quad 1 \leq i \leq q.$$

Then  $X = \bigcup_{i=1}^q X_i$  is a subset of the contractive subspace  $P$  and is called the *atomic surface* by [11, 8]. (It is called a *Rauzy fractal* by [6, 7, 19] and is called a central tile by [20, 3, 4].) Denote by  $e_1, \dots, e_d$  the canonical basis of  $\mathbb{R}^d$ , and let

$$I_i = \{ \theta \pi'(e_1) : 0 \leq \theta \leq r_i \}, \quad 1 \leq i \leq q,$$

be subsets of  $V$ . Then

$$\hat{X}_i = -X_i + I_i := \{ -x + y : x \in X_i, y \in I_i \}, \quad 1 \leq i \leq q,$$

are  $d$ -dimensional tubes. Taking their union, we get

$$\hat{X} := \bigcup_{i=1}^q \hat{X}_i.$$

(See Figure 1.1.) This construction has appeared in several papers (cf. [15, 18, 11, 8]), where its purpose is to construct Markov partitions for the group automorphism  $A_\beta$ . However, in this paper, we will show that  $\hat{X}$  completely characterizes the set of purely periodic  $\beta$ -expansions.

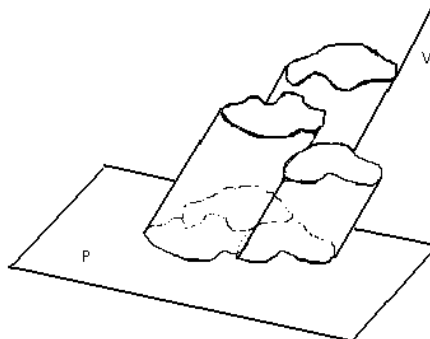


FIGURE 1.1.

**1.5. Main result.** Our purpose is to study  $\text{Pur}(\beta)$ , the set of purely periodic  $\beta$ -representations which are admissible. However, if we consider  $\text{Pur}'(\beta)$ , the set of purely periodic  $\beta$ -representations which are *weakly* admissible, it is slightly different from  $\text{Pur}(\beta)$ . A purely periodic sequence is weakly admissible but not admissible if and only if it is a shift of the sequence  $d_\beta^*(1)$  and  $d_\beta^*(1)$  is purely periodic. Notice that  $d_\beta^*(1)$  is purely periodic if and only if  $d_\beta(1)$  is finite. So

$$\text{Pur}'(\beta) = \begin{cases} \text{Pur}(\beta), & \text{when } d_\beta(1) \text{ is not finite,} \\ \text{Pur}(\beta) \cup \{r_1, \dots, r_q\}, & \text{when } d_\beta(1) \text{ is finite.} \end{cases}$$

Since  $\text{Pur}(\beta)$  and  $\text{Pur}'(\beta)$  have very small differences, so in the following we will study  $\text{Pur}'(\beta)$  instead of  $\text{Pur}(\beta)$ .

**Main Theorem.** *Let  $\beta > 1$  be a Pisot unit. Then  $x \in \text{Pur}'(\beta)$  if and only if  $x \in \mathbb{Q}(\beta) \cap [0, 1)$  and  $\phi(x) \in \hat{X}$ .*

This theorem has been proved when  $\beta$  is of degree 2 by Hara and Ito [10], and proved by Sano and Ito [12] when  $\beta$  is a Pisot unit with the minimal polynomial  $x^d = a_{d-1}x^{d-1} + \dots + a_1x + a_0$  where  $a_{d-1} \geq a_{d-2} \geq \dots \geq a_1 \geq a_0 = 1$ . In the present paper, we use a simpler argument and obtain a complete result.

## 2. ALGEBRAIC VERSION OF THE MAIN THEOREM

A two-sided sequence  $\{a_i\}_{i \in \mathbb{Z}}$  is (weakly) admissible provided each left truncation  $\{a_i\}_{i \geq M}$  is (weakly) admissible. Let  $(\Omega, \sigma)$  denote the two-sided symbolic dynamical system associated with  $\beta$ , where  $\Omega$  is the set of admissible two-sided sequences and  $\sigma$  the shift operator. We write a two-sided sequence in the form  $(w, u)$ , where  $w = \dots w_2w_1w_0$  is a backward sequence and  $u = u_1u_2\dots$  is a forward sequence.

**2.1. First realization of  $(\Omega, \sigma)$ .** Let  $\beta$  be a Pisot unit of degree  $d$ . Denote by

$$(2.1) \quad \beta = \beta^{(1)}, \beta^{(2)}, \dots, \beta^{(d)}$$

the algebraic conjugates of  $\beta$ . We arrange the sequence  $\beta^{(1)}, \beta^{(2)}, \dots, \beta^{(d)}$  in such an order: the real roots are ahead of the complex roots, and for a complex root  $\beta^{(j)}$  we put its complex conjugate  $\overline{\beta^{(j)}}$  next to it to make a pair. Suppose  $\beta^{(1)}, \dots, \beta^{(r)}$  are real roots and the others are complex roots. For a backward sequence  $w$ , we define

$$\rho(w) := \left( - \sum_{i=0}^{\infty} w_i(\beta^{(2)})^i, \dots, - \sum_{i=0}^{\infty} w_i(\beta^{(d)})^i \right).$$

Abusing the notation a little, we define a projection  $\rho : \Omega \mapsto \mathbb{R}^r \times \mathbb{C}^{d-r}$  as

$$(2.2) \quad \rho(w, u) := \left( \sum_{i=1}^{\infty} u_i \beta^{-i}, \rho(w) \right).$$

Let  $K := \{\rho(w, u) : (w, u) \in \Omega\}$  be the projection of  $\Omega$ . Clearly  $K$  is a bounded subset of  $[0, 1) \times (\mathbb{R}^{r-1} \times \mathbb{C}^{d-r})$ . We write a point of  $K$  as  $(x, y)$ , where  $x \in [0, 1)$  and  $y \in \mathbb{R}^{r-1} \times \mathbb{C}^{d-r}$ . The last ingredient we need is a mapping  $S : K \mapsto K$  defined by

$$(2.3) \quad S \begin{pmatrix} x \\ y \end{pmatrix} = I_\beta \begin{pmatrix} x \\ y \end{pmatrix} - \lfloor \beta x \rfloor \cdot \mathbf{1},$$

where  $\mathbf{1}$  denotes the vector  $(1, 1, \dots, 1)^T$  and  $I_\beta$  is the diagonal matrix

$$I_\beta = \begin{pmatrix} \beta^{(1)} & & 0 \\ & \ddots & \\ 0 & & \beta^{(d)} \end{pmatrix}.$$

**Proposition 2.1.**  $(K, S)$  is a realization of  $(\Omega, \sigma)$ . Precisely, (i)  $S \circ \rho = \rho \circ \sigma$ , (ii)  $S(K) = K$ , that is,

$$\begin{array}{ccc} & \sigma & \\ \Omega & \longrightarrow & \Omega \\ \rho \downarrow & & \downarrow \rho \\ K & \longrightarrow & K \\ & S & \end{array}$$

*Proof.* (i) By direct calculation.

(ii)  $S(K) = S \circ \rho(\Omega) = \rho \circ \sigma(\Omega) = \rho(\Omega) = K$ . □

**2.2. Characterization of  $\text{Pur}(\beta)$ .** For  $x = c_0 + c_1\beta + \dots + c_{d-1}\beta^{d-1}$  with  $c_j \in \mathbb{Q}$ , the  $j$ -th conjugate of  $x$  in the field  $\mathbb{Q}(\beta)$  is defined by

$$x^{(j)} = c_0 + c_1\beta^{(j)} + \dots + c_{d-1}(\beta^{(j)})^{d-1}.$$

We define  $x'$ , the (total) conjugate of  $x$ , to be  $x' = (x^{(2)}, \dots, x^{(d)})$ . Particularly,  $\beta' = (\beta^{(2)}, \dots, \beta^{(d)})$ . Clearly

$$\begin{pmatrix} \beta x \\ (\beta x)' \end{pmatrix} = I_\beta \begin{pmatrix} x \\ x' \end{pmatrix}.$$

Proposition 2.2 illustrates the relation between  $S$  and the  $\beta$ -transformation  $T_\beta$ .

**Proposition 2.2.** For any  $x \in \mathbb{Q}(\beta) \cap [0, 1)$ ,  $S(x, x') = (T_\beta(x), (T_\beta(x))')$ .

*Proof.*  $S \begin{pmatrix} x \\ x' \end{pmatrix} = I_\beta \begin{pmatrix} x \\ x' \end{pmatrix} - \lfloor \beta x \rfloor \cdot \mathbf{1} = \begin{pmatrix} \beta^{(1)}x^{(1)} - \lfloor \beta x \rfloor \\ \vdots \\ \beta^{(d)}x^{(d)} - \lfloor \beta x \rfloor \end{pmatrix} = \begin{pmatrix} T_\beta(x) \\ (T_\beta(x))' \end{pmatrix}$ . □

**Theorem 2.1.** Let  $\beta > 1$  be a Pisot unit. Then  $x \in \text{Pur}(\beta)$  if and only if  $x \in \mathbb{Q}(\beta) \cap [0, 1)$  and  $(x, x') \in K$ .

*Proof.* (i) Suppose  $x = 0.\overline{a_1 \dots a_L} \in \text{Pur}(\beta)$ . Then  $x \in \mathbb{Q}(\beta) \cap [0, 1)$ . Let  $w = \dots \overline{a_1 \dots a_L}$  and  $u = \overline{a_1 \dots a_L} \dots$ . Clearly  $(w, u) \in \Omega$ . The first coordinate of  $\rho(w, u)$  is

$$x = 0.\overline{a_1 \dots a_L} = \frac{a_1\beta^{L-1} + \dots + a_{L-1}\beta + a_L}{\beta^L - 1}.$$

The  $j$ -th coordinate of  $\rho(w, u)$  is ( $j \geq 2$ )

$$\begin{aligned} & -(a_L + a_{L-1}\beta^{(j)} + \dots + a_1(\beta^{(j)})^{L-1})(1 + (\beta^{(j)})^L + (\beta^{(j)})^{2L} + \dots) \\ &= \frac{a_1(\beta^{(j)})^{L-1} + \dots + a_{L-1}(\beta^{(j)}) + a_L}{(\beta^{(j)})^L - 1} = x^{(j)}. \end{aligned}$$

Thus  $(x, x') = \rho(w, u) \in K$ .

(ii) Now we prove the other direction. Suppose  $x \in \mathbb{Q}(\beta) \cap [0, 1)$  and  $(x, x') \in K$ . Let  $b$  be the smallest integer such that  $b x \in \mathbb{Z}(\beta)$  and set

$$\mathcal{R}_b := \{(x, x') : x \in b^{-1}\mathbb{Z}(\beta)\} \cap K.$$

Then  $\mathcal{R}_b$  is a finite set because  $K$  is bounded.

First we assert that  $S(\mathcal{R}_b) \subseteq \mathcal{R}_b$ . For  $(x, x') \in \mathcal{R}_b$ ,  $S(x, x') = (T_\beta x, (T_\beta x)')$  by Proposition 2.2. Since  $T_\beta x = \beta x - \lfloor \beta x \rfloor \in b^{-1}\mathbb{Z}(\beta)$  and  $S(K) = K$ , we have that  $S(\mathcal{R}_b) \subseteq \mathcal{R}_b$ .

Secondly we claim that  $S$  is surjective on  $\mathcal{R}_b$ . For  $(x, x') \in \mathcal{R}_b$ , there exists at least one sequence  $(w, u) \in \Omega$  such that  $\rho(w, u) = (x, x')$ . Let  $w_0$  be the first symbol of  $w$  and let  $y = \beta^{-1}(x + w_0)$ . We argue that  $\rho \circ \sigma^{-1}(w, u) = (y, y')$ . Write

$$\rho \circ \sigma^{-1}(w, u) = \rho(\dots w_2 w_1 . w_0 u_1 u_2 \dots) = (t_1, t_2, \dots, t_d).$$

Then

$$t_1 = \frac{w_0}{\beta} + \frac{u_1}{\beta^2} + \frac{u_2}{\beta^3} + \dots = \frac{x + w_0}{\beta} = y,$$

and for  $j \geq 2$ ,

$$\begin{aligned} t_j &= -w_1 - \beta^{(j)} w_2 - (\beta^{(j)})^2 w_3 - \dots \\ &= (\beta^{(j)})^{-1} ((-w_0 - \beta^{(j)} w_1 - (\beta^{(j)})^2 w_2 - \dots) + w_0) \\ &= (\beta^{(j)})^{-1} (x^{(j)} + w_0) = y^{(j)}. \end{aligned}$$

So  $\rho \circ \sigma^{-1}(w, u) = (y, y')$ . This implies  $(y, y') \in K$  and hence  $S(y, y') = (x, x')$ . On the other hand,  $y \in b^{-1}\mathbb{Z}(\beta)$  because  $\beta$  is an algebraic unit. So  $(y, y')$  belongs to  $\mathcal{R}_b$  and it is a preimage of  $(x, x')$ . This proves that  $S$  is surjective on  $\mathcal{R}_b$ .

Hence  $S|_{\mathcal{R}_b}$  is a one-to-one mapping, and thus there exists an integer  $n$  such that

$$(x, x') = S^n(x, x') = (T_\beta^n x, (T_\beta^n x)').$$

Therefore  $x = T_\beta^n x$ ; namely, the  $\beta$ -expansion of  $x$  is purely periodic. □

**2.3. Example 2.1.** We apply Theorem 2.1 to Pisot units of degree two (cf. [10]). the domain  $K$  is illustrated by Figure 2.1. We leave the easy calculations to the readers.

**Case  $\beta^2 = n\beta + 1, n \geq 1$ .** In this case  $d_\beta^*(1) = \overline{n0}$ , and  $M_\beta$  has two states  $\bar{r}_1 = 1$  and  $\bar{r}_2 = 0.\overline{0n} = \frac{1}{\beta}$ . The domain  $K$  is illustrated by Figure 2.1.a. Since the line segment from  $o$  to  $(1, 1)$  is contained in  $K$ ,  $(x, x') = (x, x)$  belongs to  $K$  for any rational  $0 \leq x < 1$ . Hence  $\mathbb{Q} \cap [0, 1) \subseteq \text{Pur}(\beta)$ , which is the result of K. Schmidt (cf. [17]).

**Case  $\beta^2 = n\beta - 1, n \geq 3$ .** In this case  $d_\beta^* = (n-1)\overline{(n-2)}$ ,  $M_\beta$  has two states  $\bar{r}_1 = 1$  and  $\bar{r}_2 = 0.\overline{(n-1)} = \frac{\beta-1}{\beta}$ . The domain  $K$  is illustrated by Figure 2.1.b. Now for any rational number  $x \in [0, 1)$ ,  $(x, x) \notin K$  and the  $\beta$ -expansion of  $x$  is not purely periodic.

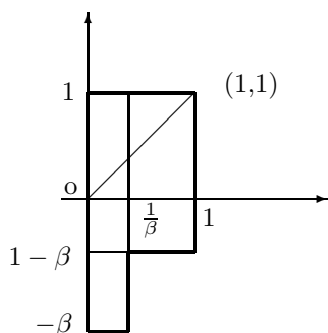


FIGURE 2.1.a

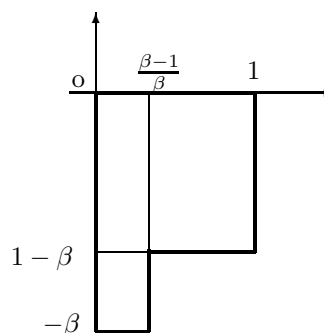


FIGURE 2.1.b

**2.4. Admissible and weakly admissible.** Let  $\bar{\Omega}$  be the closure of  $\Omega$  in the symbolic metric. Then  $\bar{\Omega}$  is the set of all weakly admissible sequences. The set  $\bar{\Omega}$  is compact and  $\bar{K} = \rho(\bar{\Omega})$ . Let  $\lfloor \beta x \rfloor$  denote the largest integer which is *strictly* less than  $\beta x$ . Replacing  $\lfloor \beta x \rfloor$  in (2.3) by  $\lfloor \beta x \rfloor$ , we get a new operator  $\bar{S}$ . Similar to the above discussion, we have

**Theorem 2.1'.** *Let  $\beta > 1$  be a Pisot unit. Then  $x \in \text{Pur}'(\beta)$  if and only if  $x \in \mathbb{Q}(\beta) \cap [0, 1]$  and  $(x, x') \in \bar{K}$ .*

### 3. PROOF OF THE MAIN THEOREM

**3.1. Second realization of  $(\Omega, \sigma)$ .** Recall that a sequence  $w_n \dots w_1 w_0$  stops at state  $i$  if there is a path from state 1 to state  $i$  with label  $w_n \dots w_1 w_0$ . It is equivalent to  $w_n \dots w_1 w_0 \cdot r_i$  being weakly admissible, but for any  $r' \succ r$  (in lexicographical order),  $w_n \dots w_1 w_0 \cdot r'$  is not weakly admissible. We generalize this definition to an infinite backward sequence. We say  $w = (\dots w_n \dots w_1 w_0)$  stops at state  $i$  if  $w \cdot r_i$  is weakly admissible but  $w \cdot r'$  is not weakly admissible for any  $r' \succ r$ . We claim that

$$(3.1) \quad X_i = \left\{ \sum_{k=0}^{\infty} w_k A_{\beta}^k \pi(e_1) : (\dots w_n \dots w_1 w_0) \text{ stops at state } i \right\}.$$

By the definition of  $X_i$ , we have

$$X_i = \overline{\pi \circ \phi \left\{ \sum_{k=0}^n w_k \beta^k : w_n \dots w_1 w_0 \text{ stops at state } i \right\}}.$$

Since  $\pi$  and  $\phi$  are linear, by (1.1) and (1.2) we see that

$$\begin{aligned} \pi \circ \phi \left( \sum_{k=0}^n w_k \beta^k \right) &= \pi \left( \sum_{k=0}^n w_k A_{\beta}^k \phi(1) \right) \\ &= \sum_{k=0}^n w_k A_{\beta}^k \pi(e_1). \end{aligned}$$

Hence the right side of (3.1) is a subset of  $X_i$ . On the other hand, for any  $x \in X_i$ , there is a sequence of finite words  $\{w^{[k]}\}_{k \geq 1}$  such that  $w^{[k]}$  stops at state  $i$  and the  $\pi \circ \phi(w^{[k]})$  converge to  $x$ . Since  $\bar{\Omega}$  is compact, there is at least one limit point  $w$ .

One can show that  $w$  stops at state  $i$  and  $\sum_{k=0}^\infty w_k A_\beta^k \pi(e_1) = x$ . This proves the other direction of (3.1).

From

$$[0, \bar{r}_i] = \left\{ \sum_{k=1}^\infty u_k \beta^k : u = (u_1 u_2 \dots) \text{ weakly admissible and } u \preceq r_i \right\}$$

we get

$$(3.2) \quad I_i = \left\{ \sum_{k=1}^\infty u_k A_\beta^{-k} \pi'(e_1) : u = (u_1 u_2 \dots) \text{ weakly admissible and } u \preceq r_i \right\}.$$

Motivated by (3.1) and (3.2), we define  $\psi : \bar{\Omega} \mapsto \hat{X}$  as

$$\psi(w, u) := - \sum_{k=0}^\infty w_k A_\beta^k \pi(e_1) + \sum_{k=1}^\infty u_k A_\beta^{-k} \pi'(e_1).$$

**Proposition 3.1.**  $\psi(\bar{\Omega}) = \hat{X}$ .

*Proof.* Pick any  $(w, u) \in \bar{\Omega}$ . There is a state  $i$  such that  $w$  stops at state  $i$  and  $u \preceq r_i$ . Therefore  $\sum_{k=0}^\infty w_k A_\beta^k \pi(e_1) \in X_i$  and  $\sum_{k=1}^\infty u_k A_\beta^{-k} \pi'(e_1) \in I_i$ . So  $\psi(w, u) \in X_i + I_i = \hat{X}_i \subset \hat{X}$ . The other direction can be proved similarly.  $\square$

We will show in §3.4 that  $(\hat{X}, \hat{T}_\beta)$  is another realization of  $(\Omega, \sigma)$ .

**3.2. Proof of the Main Theorem.** Let  $v = v_1, v_2, \dots, v_d$  be eigenvectors of  $A_\beta$  corresponding to eigenvalues  $\beta, \beta^{(2)}, \dots, \beta^{(d)}$  respectively. The order is arranged by (2.1). Every vector of  $\mathbb{R}^d$  can be written in a unique way as a linear combination of  $v_1, v_2, \dots, v_d$ . If we assume that

$$(3.3) \quad e_1 = v_1 + v_2 + \dots + v_d,$$

then  $v_1, v_2, \dots, v_d$  are uniquely determined. Let  $U = (v_1, v_2, \dots, v_d)$  be a  $d \times d$  matrix with entries in  $\mathbb{C}$ . Regarding  $U$  as a linear transformation, we have

**Proposition 3.2.**  $U(\bar{K}) = \hat{X}$ .

*Proof.* Equation (3.3) implies that  $\pi'(e_1) = v_1$  and  $\pi(e_1) = v_2 + \dots + v_d$ . Hence

$$A_\beta^k \pi(e_1) = (\beta^{(2)})^k v_2 + \dots + (\beta^{(d)})^k v_d.$$

Therefore,

$$\begin{aligned} U(\bar{K}) &= U \circ \rho(\bar{\Omega}) \\ &= \left\{ U \left( \sum_{k=1}^\infty u_k \beta^{-k}, - \sum_{k=0}^\infty w_k (\beta^{(2)})^k, \dots, - \sum_{k=0}^\infty w_k (\beta^{(d)})^k \right) : (w, u) \in \bar{\Omega} \right\} \\ &= \left\{ \sum_{k=1}^\infty u_k \beta^{-k} v_1 - \sum_{k=0}^\infty w_k ((\beta^{(2)})^k v_2 + \dots + (\beta^{(d)})^k v_d) : (w, u) \in \bar{\Omega} \right\} \\ &= \left\{ \sum_{k=1}^\infty u_k A_\beta^{-k} \pi'(e_1) - \sum_{k=0}^\infty w_k A_\beta^k \pi(e_1) : (w, u) \in \bar{\Omega} \right\} \\ &= \psi(\bar{\Omega}) = \hat{X}. \end{aligned}$$

$\square$

**Lemma 3.3.** For  $x \in \mathbb{Q}(\beta)$ ,

$$(3.4) \quad U(x, x') = \phi(x).$$

*Proof.* We claim that if (3.4) is true for  $x$ , then it is also true for  $\beta x$ . For

$$\begin{aligned} U \begin{pmatrix} \beta x \\ (\beta x)' \end{pmatrix} &= UI_\beta \begin{pmatrix} x \\ x' \end{pmatrix} = (\beta^{(1)}v_1, \dots, \beta^{(d)}v_d) \begin{pmatrix} x \\ x' \end{pmatrix} \\ &= A_\beta(v_1, \dots, v_d) \begin{pmatrix} x \\ x' \end{pmatrix} = A_\beta U \begin{pmatrix} x \\ x' \end{pmatrix} = A_\beta \phi(x) = \phi(\beta x). \end{aligned}$$

By (3.3), we see that (3.4) is true for  $x = 1$ . Hence (3.4) holds for  $x = \beta^k$ ,  $0 \leq k \leq d - 1$ . Notice that both  $\phi$  and  $x \mapsto x'$  are linear mappings; therefore, (3.4) hold for all  $x \in \mathbb{Q}(\beta)$ .  $\square$

*Proof of the Main Theorem.* Notice that the matrix  $U$  is non-singular; hence,

$$(x, x') \in \bar{K} \Leftrightarrow U(x, x') \in U\bar{K} \Leftrightarrow \phi(x) \in \hat{X}.$$

Now our Main Theorem follows from Theorem 2.1'.  $\square$

**3.3. Example 3.1.** Again we consider a Pisot unit of degree two.

**Case  $\beta^2 = n\beta + 1$ .** The domain  $\hat{X}$  is illustrated by Figure 3.1.a. Since the line segment from  $o$  to  $e_1$  is contained in  $\hat{X}$ , so  $\phi(x) \in \hat{X}$  holds for any rational  $x$  in  $[0, 1)$ . Hence  $\mathbb{Q} \cap [0, 1) \subseteq \text{Pur}'(\beta)$ .

**Case  $\beta^2 = n\beta - 1$ .** The domain  $\hat{X}$  is illustrated by Figure 3.1.b.

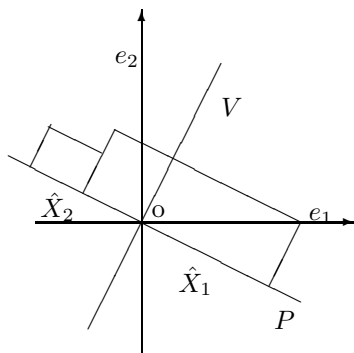


FIGURE 3.1.a

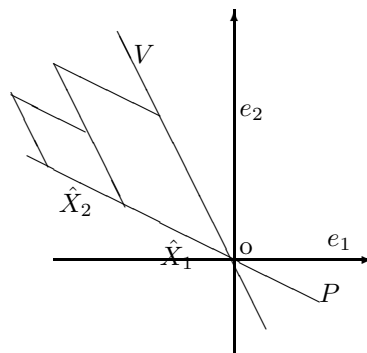


FIGURE 3.1.b

**3.4. Transformation  $\hat{T}_\beta$ .** Thurston [20] described the following dynamics: Let  $t \in \mathbb{R}^d$  be a point in the tube between  $P$  and the hyperplane  $P + e_1$ . Multiply  $t$  by  $A_\beta$  and subtract the largest multiple of  $e_1$  which keeps  $t$  on the same side of  $P$ . (Remember that  $e_1 = \phi(1)$ .) Let us denote this transformation by  $\hat{T}_\beta$ . If  $t = \phi(x)$  with  $x \in \mathbb{Q}(\beta)$  (we say  $t$  is algebraic), then it follows that

$$\hat{T}_\beta(t) = \phi(T_\beta x).$$

Observe that  $\hat{T}_\beta^k(t)$  always remains in a bounded region near the origin as  $\hat{T}_\beta$  is iterated. It can never escape very far from  $P$  since we always guide it back, and it can never escape very far from  $V$  since  $A_\beta$  squeezes every point toward  $V$ .

If  $t$  starts out as an algebraic integer in  $\mathbb{Q}(\beta)$ , then it always remains an algebraic integer. The set of all algebraic integers forms a lattice in  $\mathbb{R}^d$  by mapping  $\phi$ , so  $\hat{T}_\beta^k(t)$  can only take a finite number of values. Therefore, its orbit eventually arrives back at a previous point, and from then on it repeats. If  $t$  is in  $\mathbb{Q}(\beta)$ , then there is some integer  $b$  such that  $bt$  is an algebraic integer. Then we apply the above argument. This proves that every number in  $\mathbb{Q}(\beta)$  has eventually a periodic  $\beta$ -expansion.

From the discussion of Section 2 and Section 3, we can easily confirm the following commutative diagram, and thus  $(\hat{X}, \hat{T}_\beta)$  is a realization of  $(\bar{\Omega}, \sigma)$ .

$$(3.5) \quad \begin{array}{ccccc} \hat{X} & \xleftarrow{\psi} & \bar{\Omega} & \xrightarrow{\rho} & \bar{K} \\ \hat{T}_\beta \downarrow & & \sigma \downarrow & & \downarrow \bar{s} \\ \hat{X} & \xleftarrow{\psi} & \bar{\Omega} & \xrightarrow{\rho} & \bar{K} \end{array}$$

We obtain a more exact picture about the behavior of  $\hat{T}_\beta$ : starting from an algebraic point  $t$ , sooner or later it will fall into  $\hat{X}$ . Then it moves inside  $\hat{X}$ , and comes back to the entering point after a finite number of steps, and from then on it repeats.

#### 4. SOME REMARKS

**4.1. (F)-property.** We denote by  $\text{Fin}(\beta)$  the set of all  $\alpha \in \mathbb{Z}(\beta)_{\geq 0}$  that have finite  $\beta$ -expansion. We say  $\beta$  has the *(F)-property* (finite expansion property) provided  $\mathbb{Z}(\beta)_{\geq 0} \subseteq \text{Fin}(\beta)$ . The (F)-property of algebraic numbers is introduced by C. Frougny and B. Solomyak [9]. Akiyama found the connection between the (F)-property and tiling.

**Lemma 4.1** (Akiyama [3]). *If  $\beta > 1$  is a Pisot unit with the (F)-property, then  $0 \in P$  is an interior point of the atomic surface  $X = \bigcup_{i=1}^q X_i$ .*

The following result of Akiyama can be derived from the Main Theorem. (Akiyama’s original proof is an algebraic argument.)

**Theorem A** (Akiyama [2]). *If  $\beta > 1$  is a Pisot unit with the (F)-property, then there exists a constant  $c = c(\beta) > 0$  such that  $\mathbb{Q} \cap [0, c) \subseteq \text{Pur}(\beta)$ .*

*Proof.* Let  $\Delta$  be the line passing through 0 and  $e_1$ . Since  $0 \in P$  is an interior point of  $X$ ,  $\hat{X}$  contains a small half-ball with center 0 and sitting on the positive side of  $P$  (the side containing  $e_1$ ). Hence a small part of the line  $\Delta$  near 0 is contained in  $\hat{X}$ . We start from the origin and walk along the line  $\Delta$ . Let  $ce_1$  be the first point we meet on the boundary of  $\hat{X} \cap \Delta$ . For a rational number  $x \in [0, c)$ ,  $\phi(x) = x\phi(1) = xe_1$  belongs to the line segment from 0 to  $c\phi(1)$  and thus is contained in  $\hat{X}$ . So by the Main Theorem,  $x \in \text{Pur}'(\beta)$ . Since  $x$  is not an algebraic integer, it cannot be one of  $r_1, \dots, r_q$ . Therefore  $x \in \text{Pur}(\beta)$ . Moreover, this number  $c$  is the best bound.  $\square$

Let  $\beta > 1$  satisfy  $\beta^2 = n\beta + 1$ . Then  $\beta$  has the (F)-property, and 0 is an interior point of  $X$  as we see in Figure 3.1.a. In this case  $c = 1$ . If  $\beta > 1$  satisfies  $\beta^2 = n\beta - 1$ , then  $\beta$  does not have the (F)-property. From Figure 3.1.b we see that 0 sits on the boundary of  $X$ , and Theorem A is false. When  $d \geq 3$ , it is not so easy to get the best bound  $c$  in Theorem A because of the fractal nature of atomic surfaces.

**4.2. Atomic surfaces and Markov partitions.** The family  $\{X_i\}_{1 \leq i \leq q}$  has been studied extensively in many papers. It is proved that  $\{X_i\}_{1 \leq i \leq q}$  are the invariant sets of a graph iterated function system (graph-IFS). The graph-IFS satisfies the open set condition, and the  $X_i$  have non-empty interiors. So  $\{X_i\}_{1 \leq i \leq q}$  forms a self-similar tiling system (cf. [16, 6, 2, 3, 4, 7, 19, 11, 8]). For detailed discussions of the tiling and dynamical properties of  $\{X_i\}_{1 \leq i \leq q}$ , we refer to [11, 8].

Akiyama [4] introduced the *weakly finite property*:  $\beta$  is said to satisfy the weakly finite property if for any  $x \in \mathbb{Z}(\beta)_{\geq 0}$  and any  $\epsilon > 0$ , there exist  $y, z \in \text{Fin}(\beta)$  such that  $x = y - z$  and  $z < \epsilon$ . Notice that the (F)-property implies the weakly finite property since we can always choose  $z = 0$ . It is proved that many tiling and dynamical properties are rooted in the weakly finite property of  $\beta$  (cf. [4, 11, 8]).

Notice that the domains in Figure 3.1 are essentially the Markov partitions of  $A_\beta$  constructed by Adler and Weiss [1]. It is easy to check that in Example 3.1,  $\hat{X} + \mathbb{Z}^2$  is a lattice tiling of  $\mathbb{R}^2$  and thus  $\hat{X}$  is a two-dimensional torus. Moreover, if we regard  $A_\beta$  as a group automorphism acting on  $\hat{X}$ , then it coincides with  $\hat{T}_\beta$ , and  $\{\hat{X}_1, \hat{X}_2\}$  is a Markov partition for  $A_\beta$ . In general, we have the following result.

**Theorem B** (Ei, Ito and Rao [8]). *Let  $\beta$  be a Pisot unit. Then the following statements are equivalent:*

- (1)  $\beta$  satisfies the weakly finite property;
- (2)  $\hat{X} + \mathbb{Z}^d$  is a tiling of  $\mathbb{R}^d$ , and consequently  $\{\hat{X}_1, \dots, \hat{X}_d\}$  is a Markov partition of the group automorphism  $A_\beta$ .

Akiyama [4] conjectured that any Pisot unit satisfies the weakly finite property. Some partial results are obtained in [5]. If the conjecture is true, then  $\{\hat{X}_1, \dots, \hat{X}_q\}$  always gives a Markov partition of  $A_\beta$  and it is a generalization of the construction of Adler and Weiss [1]. It is interesting that the Markov partition of  $A_\beta$  characterizes the set of purely periodic  $\beta$ -expansions.

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