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SUBELLIPTIC CORDES ESTIMATES

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ABSTRACT. We prove Cordes type estimates for subelliptic linear partial differential operators in non-divergence form with measurable coefficients in the Heisenberg group. As an application we establish interior horizontal $W^{2,2}$-regularity for p-harmonic functions in the Heisenberg group $\mathbb{H}^1$ for the range $\frac{\sqrt{17}-1}{2} \le p < \frac{5+\sqrt{5}}{2}$.

1. Introduction

The main goal of this paper is to prove some estimates of Cordes type for subelliptic partial differential operators in non-divergence form with measurable coefficients in the Heisenberg group, including the linearized p-Laplacian. To show the applicability of our methods let us state the following theorem that constitutes a special case of our results.

Theorem 1.1. *Let $\frac{\sqrt{17}-1}{2} \le p < \frac{5+\sqrt{5}}{2}$. Then any p-harmonic function in the Heisenberg group $\mathbb{H}^1$ initially in $HW^{1,p}_{\text{loc}}$ is in $HW^{2,2}_{\text{loc}}$.*

We build on previous regularity results obtained by Marchi [7, 8] and extended by the first author [3], which give non-uniform bounds of the $HW^{2,2}$ (or $HW^{2,p}$) norm of the approximate p-harmonic functions. Using the Cordes condition [2, 11] and Strichartz's spectral analysis [10] we establish $HW^{2,2}$ estimates for linear subelliptic partial differential operators with measurable coefficients. As an application we obtain uniform $HW^{2,2}$ bounds for the approximate p-harmonic functions for p in a range that depends on the dimension of the Heisenberg group $\mathbb{H}^n$.

Consider the Heisenberg group $\mathbb{H}^n$, that is, $\mathbb{R}^{2n+1}$ with the group multiplication

$$(x_1, ..., x_{2n}, t)\cdot(y_1, ..., y_{2n}, u) = (x_1+y_1, ..., x_{2n}+y_{2n}, t+u-\frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i)) .$$

For $i \in \{1, ..., n\}$ consider the vector fields

$$X_i = \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial t} , \quad X_{n+i} = \frac{\partial}{\partial x_{n+i}} + \frac{x_i}{2}\frac{\partial}{\partial t} , \quad T = \frac{\partial}{\partial t} .$$

The nontrivial commutators are $[X_i, X_{n+i}] = T$; otherwise $[X_i, X_j] = 0$.

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Let $\Omega \subset \mathbb{H}^n$ be a domain. Consider the following Sobolev space with respect to the horizontal vector fields X_i as

$$HW^{2,2}(\Omega) = \{u \in L^2(\Omega) \ : \ X_iX_ju \in L^2(\Omega)\,, \text{ for all } i,j \in \{1,...,2n\}\}$$

endowed with the inner product

$$(u,v)_{HW^{2,2}(\Omega)} = \int_\Omega \Big(u(x)v(x) + \sum_{i,j=1}^{2n} X_iX_ju(x) \cdot X_iX_jv(x) \Big) dx\,.$$

$HW^{2,2}(\Omega)$ is a Hilbert space and let $HW_0^{2,2}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in this Hilbert space.

We denote by X^2u the matrix of second-order horizontal derivatives whose entries are $(X^2u)_{ij} = X_j(X_iu)$, and by $\Delta_H u = \sum_{i=1}^{2n} X_iX_iu$ the subelliptic Laplacian associated to the horizontal vector fields X_i.

Lemma 1.1. *For all $u \in HW_0^{2,2}(\Omega)$ we have*

$$||X^2u||_{L^2(\Omega)} \leq c_n ||\Delta_H u||_{L^2(\Omega)}\,,$$

where

$$c_n = \sqrt{1 + \frac{2}{n}}\,.$$

The constant c_n is sharp when $\Omega = \mathbb{H}^n$.

Proof. We follow the spectral analysis of Δ_H developed by Strichartz [10]. Let us recall the fact that $-\Delta_H$ and iT commute and share the same system of eigenvectors

$$\begin{aligned}\Phi_{\lambda,k,l}(z,t) \quad &= \quad \frac{\lambda^n}{(2\pi)^{n+1}(n+2k)^{n+1}} \cdot \exp\left(-\frac{il\lambda t}{n+2k}\right) \\ &\quad\cdot \exp\left(-\frac{\lambda|z|^2}{4(n+2k)}\right) \cdot L_k^{n-1}\left(\frac{\lambda|z|^2}{2(n+2k)}\right)\,,\end{aligned}$$

where $l = \pm 1$, $k \in \{0,1,2,...\}$ and L_k^{n-1} is the Laguerre polynomial

$$L_k^{n-1}(t) = \frac{e^t}{t^{n-1}} \cdot \frac{1}{k!} \cdot \frac{d^k}{dt^k}\left(e^{-t}t^{k+n-1}\right)\,.$$

For the eigenvalues, we have the following relations:

$$iTu * \Phi_{\lambda,k,l} \quad = \quad \frac{l\lambda}{n+2k} u * \Phi_{\lambda,k,l}, \tag{1.1}$$

$$-\Delta_H u * \Phi_{\lambda,k,l} \quad = \quad \lambda u * \Phi_{\lambda,k,l}\,, \tag{1.2}$$

where $*$ denotes the group convolution. Therefore, the spectral decomposition of $\Delta_H u$ for $u \in C_0^\infty(\Omega)$, the Plancherel formula, and relations (1.1)-(1.2) give

$$\begin{aligned}||\Delta_H u||^2_{L^2(\Omega)} \quad &= \quad 2\pi \sum_{k=0}^\infty \sum_{l=\pm 1} (n+2k) \int_0^\infty \int_{\mathbb{C}^n} |\Delta_H u * \Phi_{\lambda,k,l}(z,0)|^2 \, dz d\lambda \\ &= \quad 2\pi \sum_{k=0}^\infty \sum_{l=\pm 1} (n+2k) \int_0^\infty \int_{\mathbb{C}^n} \left| \frac{n+2k}{l} iTu * \Phi_{\lambda,k,l}(z,0) \right|^2 dz d\lambda \\ &\geq \quad n^2 ||Tu||^2_{L^2}(\Omega).\end{aligned}$$

Therefore, for all $u \in C_0^\infty(\Omega)$ we have

$$||Tu||_{L^2(\Omega)} \leq \frac{1}{n} ||\Delta_H u||_{L^2(\Omega)}\,. \tag{1.3}$$

In the following we will use the fact that the formal adjoint of X_k is $-X_k$. Let $u \in C_0^\infty(\Omega)$. For $k \in \{1, ..., n\}$ and $j \neq k+n$, X_k and X_j commute; therefore,

$$\int_\Omega (X_k X_j u(x))^2 \, dx = \int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) dx \,.$$

For $j = k+n$ we have

$$\begin{aligned}
& \int_\Omega (X_k X_j u(x))^2 \; dx = \int_\Omega X_k X_j u(x) \cdot (X_j X_k u(x) + T u(x)) \; dx \\
= & \int_\Omega X_k X_j u(x) \cdot X_j X_k u(x) \, dx + \int_\Omega X_k X_j u(x) \cdot T u(x) \, dx \\
= & -\int_\Omega X_j u(x) \cdot X_k X_j X_k u(x) \, dx + \int_\Omega X_k X_j u(x) \cdot T u(x) \, dx \\
= & -\int_\Omega X_j u(x) \cdot (X_j X_k + T) \, X_k u(x) \, dx + \int_\Omega X_k X_j u(x) \cdot T u(x) \, dx \\
= & -\int_\Omega X_j u(x) \cdot X_j X_k X_k u(x) \, dx + 2 \int_\Omega X_k X_j u(x) \cdot T u(x) \, dx \\
= & \int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) \, dx + 2 \int_\Omega X_k X_j u(x) \cdot T u(x) \, dx \,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_\Omega (X_j X_k u(x))^2 \; dx \\
= & \int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) \, dx - 2 \int_\Omega X_j X_k u(x) \cdot T u(x) \, dx \,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& ||X^2 u||^2_{L^2(\Omega)} = \sum_{k,j=1}^{2n} ||X_k X_j u||^2_{L^2(\Omega)} \\
= & \sum_{k,j=1}^{2n} \int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) \, dx + 2 \sum_{k=1}^{n} \int_\Omega [X_k, X_{k+n}] u(x) \cdot T u(x) \, dx \\
= & \int_\Omega \left(\sum_{k=1}^{2n} X_k X_k u(x) \right)^2 \, dx + 2n \int_\Omega (T u(x))^2 \; dx \\
\leq & \left(1 + 2n \frac{1}{n^2} \right) ||\Delta_H u||^2_{L^2(\Omega)} = \left(1 + \frac{2}{n} \right) ||\Delta_H u||^2_{L^2(\Omega)} \,.
\end{aligned}$$

The constant $\sqrt{1+\frac{2}{n}}$ is sharp when $\Omega = \mathbb{H}^n$, because for $v = \Phi_{\lambda,0,1}$ we have $Tv = \frac{i}{n} \Delta_H v$. □

2. Cordes conditions for second-order subelliptic PDE operators in non-divergence forms with measurable coefficients

Let us consider now

$$\mathcal{A} u = \sum_{i,j=1}^{2n} a_{ij}(x) X_i X_j u$$

where the functions $a_{ij} \in L^\infty(\Omega)$. Let us denote by $A = (a_{ij})$ the $2n \times 2n$ matrix of coefficients.

Definition 2.1 ([2, 11])**.** We say that A satisfies the Cordes condition $K_{\varepsilon,\sigma}$ if there exist $\varepsilon \in (0,1]$ and $\sigma > 0$ such that

$$0 < \frac{1}{\sigma} \le \sum_{i,j=1}^{2n} a_{ij}^2(x) \le \frac{1}{2n-1+\varepsilon} \left(\sum_{i=1}^{2n} a_{ii}(x) \right)^2 , \text{ a.e. } x \in \Omega . \tag{2.1}$$

Theorem 2.1. *Let $0 < \varepsilon \le 1$, $\sigma > 0$ such that $\gamma = \sqrt{1-\varepsilon}\, c_n < 1$ and A satisfies the Cordes condition $K_{\varepsilon,\sigma}$. Then for all $u \in HW_0^{2,2}(\Omega)$ we have*

$$||X^2 u||_{L^2} \le \sqrt{1 + \frac{2}{n}}\, \frac{1}{1-\gamma}\, ||\alpha||_{L^\infty} ||\mathcal{A} u||_{L^2} , \tag{2.2}$$

where

$$\alpha(x) = \frac{\langle A(x), I \rangle}{||A(x)||^2} .$$

Proof. We denote by I the identity $2n \times 2n$ matrix, by $\langle A, B \rangle = \sum_{i,j=1}^{2n} a_{ij} b_{ij}$ the inner product and by $||A|| = \sqrt{\sum_{i,j=1}^{2n} a_{ij}^2}$ the Euclidean norm in $\mathbb{R}^{2n \times 2n}$ for matrices A and B. The Cordes condition $K_{\varepsilon,\sigma}$ implies that

$$\frac{\langle A(x), I \rangle^2}{||A(x)||^2} \ge 2n - (1-\varepsilon) \tag{2.3}$$

for all $x \in \Omega' \subset \Omega$, where the Lebesgue measure of $\Omega \setminus \Omega'$ is 0.

Now let $x \in \Omega'$ be arbitrary, but fixed. Consider the quadratic polynomial

$$P(\alpha) = ||A(x)||^2 \alpha^2 - 2\langle A(x), I \rangle \alpha + 2n - (1-\varepsilon) .$$

Inequality (2.3) shows that

$$\min_{\alpha \in \mathbb{R}} P(\alpha) = P\left(\frac{\langle A(x), I \rangle}{||A(x)||^2} \right) \le 0 . \tag{2.4}$$

Therefore there exists

$$\alpha(x) = \frac{\langle A(x), I \rangle}{||A(x)||^2} \tag{2.5}$$

such that $P(\alpha(x)) \le 0$. Observing that

$$||I - \alpha(x) A(x)||^2 = ||A(x)||^2 \alpha^2(x) - 2\langle A(x), I \rangle \alpha(x) + 2n$$

we get that (2.4) implies that

$$||I - \alpha(x) A(x)||^2 \le 1 - \varepsilon ,$$

which is equivalent to

$$|\langle I - \alpha(x) A(x), M \rangle| \le \sqrt{1-\varepsilon} ||M|| , \text{ for all } M \in \mathcal{M}_{2n}(\mathbb{R}) . \tag{2.6}$$

Condition (2.6) can also be written as

$$\left| \sum_{i=1}^{n} m_{ii} \, - \, \alpha(x) \sum_{i,j=1}^{n} a_{ij}(x) m_{ij} \right| \le \sqrt{1-\varepsilon} \left(\sum_{i,j=1}^{n} m_{ij}^2 \right)^{1/2} \tag{2.7}$$

for all $M \in \mathcal{M}_{2n}(\mathbb{R})$.

Formula (2.7) and Lemma 1.1 imply that for all $u \in HW_0^{2,2}(\Omega)$ we have

$$\int_\Omega |\Delta_H u(x) - \alpha(x)\mathcal{A}u(x)|^2 \, dx \le (1-\varepsilon) \int_\Omega \sum_{i,j=1}^{2n} (X_i X_j u(x))^2 \, dx$$
$$\le (1-\varepsilon) c_n^2 \int_\Omega |\Delta_H u(x)|^2 \, dx .$$

Therefore, for $\gamma = \sqrt{1-\varepsilon}\, c_n < 1$ we get

$$||\Delta_H u - \alpha \mathcal{A} u||_{L^2(\Omega)} \le \gamma ||\Delta_H u||_{L^2(\Omega)},$$

which shows that

$$||X^2 u||_{L^2(\Omega)} \le c_n ||\Delta_H u||_{L^2(\Omega)}$$
$$\le \frac{c_n}{1-\gamma} ||\alpha \mathcal{A} u||_{L^2(\Omega)} \le \frac{c_n}{1-\gamma} ||\alpha||_{L^\infty(\Omega)} ||\mathcal{A} u||_{L^2(\Omega)} .$$

□

3. $HW^{2,2}$-interior regularity for p-harmonic functions in $\mathbb{H}^n$

Let $\Omega \in \mathbb{H}^n$ be a domain, $h \in HW^{1,p}(\Omega)$ and $p > 1$. Consider the problem of minimizing the functional

$$\Phi(u) = \int_\Omega |Xu(x)|^p \, dx$$

over all $u \in HW^{1,p}(\Omega)$ such that $u - h \in HW_0^{1,p}(\Omega)$. The Euler equation for this problem is the p-Laplace equation

$$\sum_{i=1}^{2n} X_i \left(|Xu|^{p-2} X_i u \right) = 0 \, , \text{ in } \Omega . \tag{3.1}$$

A function $u \in HW^{1,p}(\Omega)$ is called a weak solution for (3.1) if

$$\sum_{i=1}^{2n} \int_\Omega |Xu(x)|^{p-2} X_i u(x) \cdot X_i \varphi(x) dx = 0 \, , \ \forall \, \varphi \in HW_0^{1,p}(\Omega) . \tag{3.2}$$

Φ is a convex functional on $HW^{1,p}$; therefore weak solutions are minimizers for Φ and vice-versa.

For $m \in \mathbb{N}$ let us now define the approximating problems of minimizing the functionals

$$\Phi_m(u) = \int_\Omega \left(\frac{1}{m} + |Xu(x)|^2 \right)^{\frac{p}{2}}$$

and the corresponding Euler equations

$$\sum_{i=1}^{2n} X_i \left(\left(\frac{1}{m} + |Xu|^2 \right)^{\frac{p-2}{2}} X_i u \right) = 0 \, , \text{ in } \Omega . \tag{3.3}$$

The weak form of this equation is

(3.4)
$$\sum_{i=1}^{2n} \int_\Omega \left(\frac{1}{m} + |Xu(x)|^2 \right)^{\frac{p-2}{2}} X_i u(x) \cdot X_i \varphi(x) dx = 0 \, , \ \text{ for all } \ \varphi \in HW_0^{1,p}(\Omega) .$$

The differentiated version of equation (3.3) has the form

$$\sum_{i,j=1}^{2n} a_{ij}^m \, X_i X_j u = 0 \,, \text{ in } \Omega, \tag{3.5}$$

where

$$a_{ij}^m(x) = \delta_{ij} + (p-2)\frac{X_i u(x)\, X_j u(x)}{\frac{1}{m} + |Xu(x)|^2} \,.$$

Let us consider a weak solution $u_m \in HW^{1,p}(\Omega)$ of equation (3.3). Then $a_{ij}^m \in L^\infty(\Omega)$. Define the mapping $L_m : W_0^{2,2}(\Omega) \to L^2(\Omega)$ by

$$L_m(v)(x) \;=\; \sum_{i,j=1}^{2n} a_{ij}^m(x) X_i X_j v(x) \,. \tag{3.6}$$

We will check the validity of Theorem 2.1 for L_m. We have

$$\sum_{i=1}^{2n} a_{ii}^m(x) = 2n + (p-2)\frac{|Xu_m|^2}{\frac{1}{m} + |Xu_m|^2}$$

and

$$\sum_{i,j=1}^{2n} \left(a_{ij}^m(x)\right)^2 = 2n + 2(p-2)\frac{|Xu_m|^2}{\frac{1}{m} + |Xu_m|^2} + (p-2)^2 \frac{|Xu_m|^4}{\left(\frac{1}{m} + |Xu_m|^2\right)^2} \,.$$

Denote

$$(p-2)\frac{|Xu_m|^2}{\frac{1}{m} + |Xu_m|^2} = \Lambda \,.$$

Therefore, for an $\varepsilon \in (1 - \frac{1}{c_n^2}, 1)$ we need

$$2n + 2\Lambda + \Lambda^2 \le \frac{1}{2n - 1 + \varepsilon}\left(2n + \Lambda\right)^2 \,.$$

This leads to

$$\begin{aligned} (2n-1)\Lambda^2 &\le (1-\varepsilon)\left(2n + 2\Lambda + \Lambda^2\right) \\ &< \frac{1}{c_n^2}\left(2n + 2\Lambda + \Lambda^2\right) \,. \end{aligned}$$

Hence,

$$\left((2n-1)c_n^2 - 1\right)\Lambda^2 - 2\Lambda - 2n < 0 \,.$$

Solving this inequality we get

$$\Lambda \in \left(\frac{1 - \sqrt{2n\left((2n-1)\,c_n^2 - 1\right) + 1}}{(2n-1)c_n^2 - 1} \,,\; \frac{1 + \sqrt{2n\left((2n-1)\,c_n^2 - 1\right) + 1}}{(2n-1)c_n^2 - 1} \right) \,. \tag{3.7}$$

Using $c_n^2 = \frac{n+2}{n}$ and the fact that $\frac{|Xu_m|^2}{\frac{1}{m} + |Xu_m|^2} < 1$ for all $m \in \mathbb{N}$ we have that

$$p - 2 \in \left(\frac{n - n\sqrt{4n^2 + 4n - 3}}{2n^2 + 2n - 2} \,,\; \frac{n + n\sqrt{4n^2 + 4n - 3}}{2n^2 + 2n - 2} \right) \tag{3.8}$$

and that the operators L_m satisfy the assumptions of Theorem 2.1 uniformly in m.

Let us remark that in the case $n = 1$ we have

$$p - 2 \in \left(\frac{1 - \sqrt{5}}{2} \,,\; \frac{1 + \sqrt{5}}{2} \right) \,.$$

Theorem 3.1. *Let*

$$2 \le p < 2 + \frac{n + n\sqrt{4n^2+4n-3}}{2n^2+2n-2}\,.$$

If $u \in HW^{1,p}(\Omega)$ is a minimizer for the functional Φ, then $u \in HW^{2,2}_{\mathrm{loc}}(\Omega)$.

Proof. The case $p = 2$ is well known, so let us suppose $p \ne 2$. Let $u \in HW^{1,p}(\Omega)$ be a minimizer for Φ. Consider $x_0 \in \Omega$ and $r > 0$ such that $B_{4r} = B(x_0, 4r) \subset\subset \Omega$. We need a cut-off function $\eta \in C_0^\infty(B_{2r})$ such that $\eta = 1$ on B_r. Also consider minimizers u_m for Φ_m on $HW^{1,p}(B_{2r})$ subject to $u_m - u \in HW_0^{1,p}(B_{2r})$. Then $u_m \to u$ in $HW^{1,p}(B_{2r})$ as $m \to \infty$.

By [3, 7] we get that for $2 \le p < 4$ we have $u_m \in HW^{2,2}_{\mathrm{loc}}(\Omega)$, but with bounds depending on m, and also that u_m satisfies the equation $L_m(u_m) = 0$ a.e. in B_{2r}. So, in B_{2r} we have a.e.

$$X_iX_j(\eta^2 u_m) = X_iX_j(\eta^2)u_m + X_j(\eta^2)X_iu_m + X_i(\eta^2)X_ju_m + \eta^2 X_iX_ju_m$$

and hence

$$L_m(\eta^2 u_m) = u_m\, L_{m,u_m}(\eta^2) + \sum_{i,j=1}^{2n} a_{ij}^m(x)\Big(X_j(\eta^2)X_iu_m + X_i(\eta^2)X_ju_m\Big)\,.$$

By Theorem 2.1 it follows that

$$\begin{aligned} ||X^2u_m||_{L^2(B_r)} &\le ||X^2(\eta^2u_m)||_{L^2(B_{2r})} \le c||L_m(\eta^2u_m)||_{L^2(B_{2r})} \\ &\le c||u_m||_{HW^{1,p}(B_{2r})} \le c||u||_{HW^{1,p}(B_{2r})} \end{aligned}$$

where c is independent of m. Therefore, $u \in HW^{2,2}(B_r)$. □

Remark 3.1. Observe that the range for p given by Theorem 3.1 is shrinking from $[2, \frac{5+\sqrt{5}}{2})$ to $[2, 3]$ as n increases from 1 to ∞.

For the case $p < 2$ we need the following lemmas. The first lemma is an interpolation result and its proof is based on integration by parts.

Lemma 3.1. *For all $u \in C_0^\infty(\Omega)$ and for all $\delta > 0$ there exists $c(\delta) > 0$ such that*

$$||Xu||^2_{L^2(\Omega)} \le \delta||X^2u||^2_{L^2(\Omega)} + c(\delta)||u||^2_{L^2(\Omega)}\,.$$

Proof.

$$\begin{aligned} ||Xu||^2_{L^2(\Omega)} &= \sum_{i=1}^{2n}\int_\Omega X_iu(x)\ X_iu(x)\,dx = -\sum_{i=1}^{2n}\int_\Omega u(x)\ X_iX_iu(x)dx \\ &= -\int_\Omega u(x)\ \Delta_H u(x)\,dx \le \frac{\delta}{2n}\int_\Omega |\Delta_H u(x)|^2\,dx + c(\delta)\int_\Omega u^2(x)\,dx \\ &\le \delta\int_\Omega |X^2u(x)|^2\,dx + c(\delta)\int_\Omega u^2(x)\,dx. \end{aligned}$$

□

From Lemma 3.1 and the higher-order extension results available for the Sobolev spaces on the Heisenberg group [6, 9] we get the following result.

Lemma 3.2. *For all $u \in HW^{2,2}(B_r)$ and all $\delta > 0$ there exists $c(\delta) > 0$ such that*

$$||Xu||^2_{L^2(B_r)} \le \delta||X^2u||^2_{L^2(B_r)} + c(\delta)||u||^2_{L^2(B_r)}\,.$$

By Lemmas 3.1 and 3.2 we can use a method similar to the proof of Theorem 9.11 [5] to get the following result.

Lemma 3.3. *Let us suppose that the operator $\mathcal{A}$ satisfies the assumptions of Theorem 4.1 and that $B_{3r} \subset \Omega$. Then*

$$||X^2u||_{L^2(B_r)} \le c\Big(||\mathcal{A}u||_{L^2(B_{2r})} + ||u||_{L^2(B_{2r})}\Big)\,,$$

for all $u \in HW^{2,2}_{\mathrm{loc}}(B_{3r})$.

Proof. Let $\eta \in C_0^\infty(B_{2r})$, $0 < \sigma < 1$ and $\sigma' = \frac{1+\sigma}{2}$ such that η is a cut-off function between $B_{\sigma 2r}$ and $B_{\sigma' 2r}$ satisfying

$$|X\eta| \le \frac{4}{(1-\sigma)r} \;\text{ and }\; |X^2\eta| \le \frac{16}{(1-\sigma)^2r^2}\,.$$

Then we can use Theorem 2.1 for ηu to get

$$\begin{aligned}
&||X^2u||_{L^2(B_{\sigma 2r})} \le ||X^2(\eta u)||_{L^2(B_{2r})} \le c||\mathcal{A}(\eta u)||_{L^2(B_{2r})} \\
&\le c\left\|\eta\mathcal{A}u + u\,\mathcal{A}(\eta) + \sum_{i,j=1}^{2n} a_{ij}\Big(X_j(\eta)X_iu + X_i(\eta)X_ju\Big)\right\|_{L^2(B_{2r})} \\
&\le c\Big(||\mathcal{A}u||_{L^2(B_{2r})} + \frac{1}{(1-\sigma)r}||Xu||_{L^2(B_{\sigma' 2r})} + \frac{1}{(1-\sigma)^2r^2}||u||_{L^2(B_{\sigma' 2r})}\Big).
\end{aligned}$$

For $k \in \{0,1,2\}$ let us use the seminorms

$$|||u|||_k = \sup_{0<\sigma<1}(1-\sigma)^k r^k ||X^ku||_{L^2(B_{\sigma 2r})}\,.$$

Then

$$|||u|||_2 \le c\Big(r^2||\mathcal{A}u||_{L^2(B_{2r})} + |||u|||_1 + |||u|||_0\Big)\,.$$

Lemma 3.2 implies that for $\delta > 0$ small we have

$$|||u|||_1 \le \delta|||u|||_2 + c(\delta)|||u|||_0\,.$$

Therefore,

$$|||u|||_2 \le c\Big(r^2||\mathcal{A}u||_{L^2(B_{2r})} + |||u|||_0\Big)$$

and hence

$$||X^2u||_{L^2(B_{\sigma 2r})} \le \frac{c}{(1-\sigma)^2r^2}\Big(r^2||\mathcal{A}u||_{L^2(B_{2r})} + ||u||_{L^2(B_{2r})}\Big)\,.$$

For $\sigma = \frac{1}{2}$ we get the desired inequality. □

Theorem 3.2. *Let us consider the Heisenberg group $\mathbb{H}^1$ and*

$$\frac{\sqrt{17}-1}{2} \le p \le 2\,.$$

If $u \in HW^{1,p}(\Omega)$ is a minimizer for the functional Φ, then $u \in HW^{2,2}_{\mathrm{loc}}(\Omega)$.

Proof. We start the proof in the same way as we did in the proof of Theorem 3.1. Let $u \in HW^{1,p}(\Omega)$ be a minimizer for Φ. Consider $x_0 \in \Omega$ and $r > 0$ such that $B_{4r} = B(x_0, 4r) \subset\subset \Omega$. We need a test function $\eta \in C_0^\infty(B_{3r})$. Also consider minimizers u_m for Φ_m on $HW^{1,p}(B_{3r})$ subject to $u_m - u \in HW_0^{1,p}(B_{3r})$. Then $u_m \to u$ in $HW^{1,p}(B_{3r})$ as $m \to \infty$. We use the facts that

$$\frac{4}{3} < \frac{5-\sqrt{5}}{2} < \frac{\sqrt{17}-1}{2} < 2,$$

the homogeneous dimension of $\mathbb{H}^1$ is $Q = 4$, and

$$2 \leq \frac{4p}{4-p} \quad \text{for all } \frac{4}{3} \leq p < 2\,.$$

The Sobolev embeddings result in the subelliptic setting [1] says that

$$HW_0^{1,p}\big(B_{3r}\big) \hookrightarrow L^q\big(B_{3r}\big)\,, \ \text{ for } \ 1 \leq q \leq \frac{4p}{4-p}\,.$$

Therefore, $u_m \to u$ in $L^2\big(B_{3r}\big)$. Also, using that (see [3]) for $\frac{\sqrt{17}-1}{2} \leq p \leq 2$ we have $u_m \in HW^{2,p}_{\text{loc}}\big(B_{3r}\big)$ we get that $Xu_m \in L^2_{\text{loc}}\big(B_{3r}\big)$. Let us remark that these bounds of $X^2 u_m$ in L^p may depend on m and that $L_m(u_m) = 0$ a.e. in B_{3r}. Moreover,

$$\begin{aligned} &||L_m\big(\eta^2 u_m\big)||_{L^2(B_{3r})} \\ &= c\left\| u_m\, L_m(\eta^2) + \sum_{i,j=1}^{2n} a_{ij}^{m,u}(x)\Big(X_j(\eta^2)X_i u_m + X_i(\eta^2)X_j u_m\Big)\right\|_{L^2(B_{3r})} \\ &\leq c\Big(||u_m||_{L^2(\text{supp}\eta)} + ||Xu_m||_{L^2(\text{supp}\eta)}\Big) < +\infty, \end{aligned}$$

and hence $u_m \in HW^{2,2}_{\text{loc}}\big(B_{3r}\big)$. By Lemma 3.3 for all m sufficiently large we have

$$||X^2\big(u_m\big)||_{L^2(B_r)} \leq c||u_m||_{L^2(B_{2r})} \leq 2c||u||_{L^2(B_{2r})},$$

which shows that $X^2 u_m$ is uniformly bounded in $HW^{2,2}(B_r)$; hence $u \in HW^{2,2}(B_r)$. □

In the forthcoming article [4] we establish the $C^{1,\alpha}$ regularity for p-harmonic functions in $\mathbb{H}^n$ when p is in a neighborhood of 2.

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