MEASURES OF CONCORDANCE
DETERMINED BY $D_4$-INVARIANT MEASURES ON $(0, 1)^2$

H. H. EDWARDS, P. MIKUSIŃSKI, AND M. D. TAYLOR

Abstract. A measure, $\mu$, on $(0, 1)^2$ is said to be $D_4$-invariant if its value for any Borel set is invariant with respect to the symmetries of the unit square. A function, $\kappa$, generated in a certain way by a measure, $\mu$, on $(0, 1)^2$ is shown to be a measure of concordance if and only if the generating measure is positive, regular, $D_4$-invariant, and satisfies certain inequalities. The construction examined here includes Blomqvist’s beta as a special case.

Let $I = [0, 1]$ and $I^2 = [0, 1] \times [0, 1]$. $\eta$ is a doubly stochastic measure on $I^2$ if it is a probability measure on the Borel sets of $I^2$ such that $\eta(A \times I) = \eta(I \times A) = \lambda(A)$ where $A$ is a Borel set of $I$ and $\lambda$ is the one-dimensional Lebesgue measure. A copula (more precisely a 2-copula) is a function, $C : I^2 \rightarrow I$, that is related to some doubly stochastic measure, $\eta$, by $C(x, y) = \eta([0, x] \times [0, y])$. (See [3].) There is a one-to-one correspondence between Cop(2), the set of copulas, and the set of doubly stochastic measures. The doubly stochastic measure corresponding to the copula, $C$, will be denoted $\eta_C$.

Besides being associated with a doubly stochastic measure, a copula can be uniquely determined by a pair of continuous random variables. By Sklar’s theorem, for any continuous random vector, $(X, Y)$, with marginals, $F_X$ and $F_Y$ respectively, and joint distribution function, $F_{X,Y}$, there exists a unique copula, $C$, such that $F_{X,Y}(x, y) = C(F_X(x), F_Y(y))$. (See [1] or [6].)

The simplest examples are as follows. If $Y$ is an increasing function of $X$, then the associated copula is $M(x, y) = \min(x, y)$. If $Y$ is a decreasing function of $X$, then the associated copula is $W(x, y) = \max(x + y - 1, 0)$. Finally, if $X$ and $Y$ are independent, then the associated copula is $\Pi(x, y) = xy$. (See again [6].)

When considering two random variables it can be useful to know how much large values of one random variable correspond with large values of the other. Such a property can be gauged by a measure of concordance, a concept developed by Scarsini [7] and presented in Nelsen’s book [6], An Introduction to Copulas.

A measure of concordance associated with a continuous random vector $(X, Y)$ is a real number $\kappa_{X,Y}$ satisfying certain axioms, which we shall exhibit below. It can be shown that this value depends only on the copula, $C$, associated with $(X, Y)$. Because of this, $\kappa_C$ is often used instead of $\kappa_{X,Y}$. In fact, since $\kappa$ is a functional on Cop(2), we will write $\kappa(C)$.

Received by the editors August 1, 2003 and, in revised form, November 11, 2003 and January 13, 2004.

2000 Mathematics Subject Classification. Primary 62H05, 62H20.
The star product, developed by Darsow, Nguyen, and Olsen (see [2]) of two given copulas results in a copula. Given \( A, B \in \text{Cop}(2) \), the star product is defined as

\[
(A * B)(x, y) = \int_0^1 D_2 A(x, t) D_1 B(t, y) \, dt
\]

where \( D_i \) is the differential operator on the \( i \)th variable for \( i = 1, 2 \). Straightforward calculations yield \((M * C)(x, y) = (C * M)(x, y) = C(x, y), (\Pi * C)(x, y) = (C * \Pi)(x, y) = \Pi(x, y), (W * C)(x, y) = y - C(1 - x, y), \) and \((C * W)(x, y) = x - C(x, 1 - y)\). Note that if \( C \) is the copula associated with the random vector, \((X, Y)\), then \( y - C(1 - x, y) \) is associated with \((-X, Y)\). Similarly, \( x - C(x, 1 - y) \) is associated with \((X, -Y)\). Keeping the previous observations in mind and letting \( C^T(x, y) = C(y, x) \) so that \( C^T \) is associated with \((Y, X)\), we can write the definition for a measure of concordance as follows.

**Definition 0.1.** A functional \( \kappa : \text{Cop}(2) \to [-1, 1] \) is called a measure of concordance if:

1. \( \kappa(M) = 1 \) and \( \kappa(\Pi) = 0 \),
2. \( \kappa(W * C) = \kappa(C * W) = -\kappa(C) \),
3. \( \kappa(C) = \kappa(C^T) \),
4. \( C_1 \leq C_2 \) pointwise implies \( \kappa(C_1) \leq \kappa(C_2) \), and
5. \( C_n \to C \) pointwise implies \( \kappa(C_n) \to \kappa(C) \).

In our previous paper [3] we discussed measures of concordance determined by \( D_4 \)-invariant copulas. This type of measure of concordance is expressed as

\[
\kappa(C) = \frac{\int_{I^2} (C - \Pi) \, dA}{\int_{I^2} (M - \Pi) \, dA} \quad \text{or} \quad \kappa(C) = \frac{\int_{I^2} (C - \Pi) \, d\eta_A}{\int_{I^2} (M - \Pi) \, d\eta_A}
\]

where \( A \) is any fixed \( D_4 \)-invariant copula. For example, if \( A = \Pi \), then Spearman’s rho is obtained. However, there are other commonly known measures of concordance that are not of this type, but are similar in form with the exception of \( \eta \) being a doubly stochastic measure. One such example is Blomqvist’s beta.

Blomqvist’s beta, \( \beta \), can be expressed as

\[
\beta(C) = 4C \left( \frac{1}{2}, \frac{1}{2} \right) - 1
\]

or can alternatively be written in the form

\[
\beta(C) = \frac{\int_{(0,1)^2} (C - \Pi) \, d\nu}{\int_{(0,1)^2} (M - \Pi) \, d\nu}
\]

where \( \nu \) is a measure such that a mass of 4 is concentrated only at \((\frac{1}{2}, \frac{1}{2})\). Notice that \( \nu \) certainly cannot be generated by a copula since it is not a doubly stochastic measure.

Given a measure, \( \mu \), on the Borel sets of \((0,1)^2\), we consider in this paper the conditions on \( \mu \) that allow the mapping

\[
(0.1) \quad C \mapsto \frac{\int_{(0,1)^2} (C - \Pi) \, d\mu}{\int_{(0,1)^2} (M - \Pi) \, d\mu}
\]

to be a measure of concordance. Scarsini implicitly anticipated measures of concordance of this form by giving what amounted to special cases of (0.1) in [7].
Definition 0.2. A measure, $\mu$, is $D_4$-invariant if $\mu$ is invariant under the symmetries of $I^2$. That is, if $S$ is a Borel set of $I^2$ and $g : I^2 \rightarrow I^2$ is a symmetry of $I^2$, then $\mu(g(S)) = \mu(S)$.

It is sufficient when applying this definition to consider the symmetries
\[
\tau(x, y) = (y, x) \quad \text{and} \quad \sigma(x, y) = (1 - x, y).
\]
Note that if $\eta_C$ is a doubly stochastic measure corresponding to a copula, $C$, then $\eta_C(\sigma(S)) = \eta_{W\ast C}(S)$ and $\eta_C(\tau(S)) = \eta_{C\tau}(S)$ for any Borel set $S$ in $[0, 1]^2$. Thus, if $\eta_C = \eta_{W\ast C} = \eta_{C\tau}$, then $C$ is a $D_4$-invariant copula. (See [3].)

Theorem 0.3. Let $\kappa$ be a measure of concordance. There exists a regular Borel measure, $\mu$, on $I^2$ such that
\[
\kappa(C) = \int_{I^2} (C - \Pi) \, d\mu
\]
if and only if
\begin{align}
(0.2) \quad & \kappa(tA + (1-t)B) = t\kappa(A) + (1-t)\kappa(B) \quad \text{for} \quad t \in (0, 1) \quad \text{and} \quad A, B \in \text{Cop}(2), \\
(0.3) \quad & \text{there exists} \quad \alpha > 0 \quad \text{such that} \quad |\kappa(C)| \leq \alpha ||C - \Pi||_\infty \quad \text{for} \quad C \in \text{Cop}(2).
\end{align}

Proof. Let $\kappa$ be a measure of concordance such that $\kappa(C) = \int_{I^2} (C - \Pi) \, d\mu$ for some regular Borel measure on $I^2$. Clearly, (0.2) is satisfied. Since $\mu(I^2) < \infty$, we have $|\kappa(C)| = |\int_{I^2} (C - \Pi) \, d\mu| \leq \mu(I^2)||C - \Pi||_\infty$ for any copula $C$.

Now assume that $\kappa$ is a measure of concordance such that (0.2) and (0.3) hold. Letting $E = \{C - \Pi : C \text{ is a copula}\}$, we have a compact, convex subset of $C(I^2)$, the space of continuous functions on $I^2$, with respect to $||\cdot||_\infty$. Define a functional $\Lambda : E \rightarrow \mathbb{R}$ via
\[
\Lambda(C - \Pi) = \kappa(C).
\]
Since $\Lambda$ is uniformly bounded on $E$ and linear, it can be extended to a bounded linear functional on $C(I^2)$. By the Riesz Representation Theorem, there exists a regular Borel measure, $\mu$, on $I^2$ such that
\[
\Lambda(f) = \int_{I^2} f \, d\mu
\]
for all $f \in C(I^2)$. Consequently, $\kappa(C) = \int_{I^2} (C - \Pi) \, d\mu$ for any copula, $C$. \hfill \Box

We say that a measure of concordance defined by $\kappa(C) = \int_{I^2} (C - \Pi) \, d\mu$ is copular if and only if $\mu$ is a scalar multiple of a doubly stochastic measure. (See [3].)

If we continue to consider measures of concordance of the form
\[
\kappa(C) = \int_{I^2} (C - \Pi) d\mu
\]
with $\mu$ being a regular measure, then we are restricting ourselves to only consider such $\mu$ where $\mu(I^2) < \infty$ since $I^2$ is compact. However, if we adjust the form of our measure of concordance to
\[
\kappa(C) = \gamma \int_{(0,1)^2} (C - \Pi) d\mu,
\]
then we can also use $\mu$ where $\mu(I^2) = \infty$ since $C - \Pi = 0$ on $\partial(I^2)$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
We first describe a construction that will be useful. Choose a rectangle, \( R = [x_1, x_2] \times [y_1, y_2] \) in \((0, 1)^2\) and \( \delta > 0 \) such that the expanded rectangle, \([x_1 - \delta, x_2 + \delta] \times [y_1 - \delta, y_2 + \delta]\), is also contained in \((0, 1)^2\). The boundaries of \( I^2 \) and the lines \( x = x_1 - \delta, x = x_1, x = x_2, x = x_2 + \delta, y = y_1 - \delta, y = y_1, y = y_2, y = y_2 + \delta \) partition \( I^2 \) into 25 rectangular cells. We define a doubly stochastic measure on \( I^2 \) by distributing a unit mass with density 2 in the cells, \([x_1 - \delta, x_1] \times [y_1 - \delta, y_1]\) and \([x_2, x_2 + \delta] \times [y_2, y_2 + \delta]\), and density 0 in the cells, \([x_1 - \delta, x_1] \times [y_2, y_2 + \delta]\) and \([x_2, x_2 + \delta] \times [y_1 - \delta, y_1]\). Otherwise, the density of any cell is 1. See Figure 1. Let \( C_{R,\delta} \) be the copula associated with this doubly stochastic measure. We make use of \( C_{R,\delta} \) in some of the following proofs.

![Figure 1](image)

**Lemma 0.4.** If \( \mu \) and \( \nu \) are positive, regular Borel measures such that
\[
\int_{(0,1)^2} (C - \Pi) \, d\mu = \int_{(0,1)^2} (C - \Pi) \, d\nu
\]
for all \( C \), then \( \mu = \nu \).

**Proof.** If for any rectangle \( R = [x_1, x_2] \times [y_1, y_2] \) in \((0, 1)^2\) we have \( \mu(R) = \nu(R) \), then we are done.

Consider \( Q_{R,\delta}(x, y) \) where \( Q_{R,\delta}(x, y) = (C_{R,\delta}(x, y) - \Pi(x, y))/\delta^2 \). For any \((x, y) \notin R\), there exists a sufficiently small \( \delta_0 > 0 \) such that \( Q_{R,\delta_0}(x, y) = 0 \).

For any \((x, y) \in R\), \( Q_{R,\delta}(x, y) = 1 \). So, by letting \( \delta \to 0 \) we see that \( Q_{R,\delta} \to \chi(R) \).

Since \(|Q_{R,\delta}| \leq 1\) for any \( R \) and for any \( \delta > 0 \), then by the dominated convergence theorem,
\[
\mu(R) = \int_{(0,1)^2} \chi(R) \, d\mu = \lim_{\delta \to 0} \int_{(0,1)^2} Q_{R,\delta} \, d\mu
= \lim_{\delta \to 0} \int_{(0,1)^2} Q_{R,\delta} \, d\nu = \int_{(0,1)^2} \chi(R) \, d\nu
= \nu(R).
\]

\[ \square \]
Lemma 0.5. If \( \mu \) is a Borel measure on \((0,1)^2\), then \( \mu \) is \( D_4 \)-invariant if and only if

\[
\int_{(0,1)^2} (C^T - \Pi) \, d\mu = \int_{(0,1)^2} (C - \Pi) \, d\mu
\]

and

\[
\int_{(0,1)^2} ((W \ast C) - \Pi) \, d\mu = -\int_{(0,1)^2} (C - \Pi) \, d\mu
\]

for any copula, \( C \).

Proof. Recall that \( \tau(x, y) = (y, x) \) and \( \sigma(x, y) = (1 - x, y) \).

For any copula, \( C \), we have

\[
\int_{(0,1)^2} (C(x, y) - xy) \, d(\mu \circ \tau) = \int_{(0,1)^2} (C(y, x) -yx) \, d\mu = \int_{(0,1)^2} (C^T(x, y) -xy) \, d\mu
\]

and

\[
\int_{(0,1)^2} (C(x, y) - xy) \, d(\mu \circ \sigma) = \int_{(0,1)^2} (C(1-x, y) - (1-x)y) \, d\mu
\]

\[
= -\int_{(0,1)^2} (y - C(1-x, y) -xy) \, d\mu
\]

\[
= -\int_{(0,1)^2} ((W \ast C)(x, y) - xy) \, d\mu.
\]

In other words,

\[
\int_{(0,1)^2} (C - \Pi) \, d(\mu \circ \tau) = \int_{(0,1)^2} (C^T - \Pi) \, d\mu
\]

and

\[
\int_{(0,1)^2} (C - \Pi) \, d(\mu \circ \sigma) = -\int_{(0,1)^2} ((W \ast C) - \Pi) \, d\mu
\]

for any copula, \( C \). If \( \mu \) is \( D_4 \)-invariant, the above reduces to (0.4) and (0.5). On the other hand, if (0.4) and (0.5) hold, then \( \mu \) is \( D_4 \)-invariant by Lemma 0.4. \( \square \)

Theorem 0.6. Let \( \mu \) be a Borel measure on \((0,1)^2\). Then

\[
\kappa(C) = \gamma \int_{(0,1)^2} (C - \Pi) \, d\mu
\]

is a measure of concordance for some \( \gamma > 0 \) if and only if \( \mu \) is positive, regular, and \( D_4 \)-invariant, \( 0 < \int_{(0,1)^2} (M - \Pi) \, d\mu < \infty \), and \( \gamma = (\int_{(0,1)^2} (M - \Pi) \, d\mu)^{-1} \).

Proof. Note that in the proof of this theorem, \( Q_{R,\delta} \) is as defined in Lemma 0.4 where \( R \) is a rectangle in \((0,1)^2\).

Suppose that \( \kappa(C) \) is a measure of concordance.

Clearly \( 0 < \int_{(0,1)^2} (M - \Pi) \, d\mu < \infty \) and \( \gamma = (\int_{(0,1)^2} (M - \Pi) \, d\mu)^{-1} \) since \( \kappa(M) = 1 \).

\( \mu(R) \geq 0 \) for any \( R \) in \((0,1)^2\) since for any \( \delta > 0 \),

\[
\int_{(0,1)^2} Q_{R,\delta} \, d\mu \geq \frac{1}{\delta^2} \int_{(0,1)^2} (\Pi - \Pi) \, d\mu = 0,
\]

and by letting \( \delta \to 0 \) we have \( \mu(R) \geq 0 \).
If we suppose \( \mu \) is not regular, then there exists a compact set \( K \) in \((0,1)^2\) such that \( \mu(K) = \infty \). However, setting \( m = \min_{(x,y) \in K} (M - \Pi)(x,y) > 0 \), we have
\[
\infty > \int_{(0,1)^2} (M - \Pi) \, d\mu \geq \int_K (M - \Pi) \, d\mu \geq m \int_K d\mu = m\mu(K) = \infty.
\]
So \( \mu \) must be regular.

Finally, since \( \kappa(C) \) is a measure of concordance we have
\[
\int_{(0,1)^2} (C^T - \Pi) \, d\mu = \int_{(0,1)^2} (C - \Pi) \, d\mu
\]
and
\[
\int_{(0,1)^2} ((W \ast C) - \Pi) \, d\mu = -\int_{(0,1)^2} (C - \Pi) \, d\mu.
\]
So \( \mu \) is \( D_4 \)-invariant by Lemma \[\text{[1.5]}\]

Now suppose \( \mu \) is positive, regular, and \( D_4 \)-invariant, \( 0 < \int_{(0,1)^2} (M - \Pi) \, d\mu < \infty \), and \( \gamma = (\int_{(0,1)^2} (M - \Pi) \, d\mu)^{-1} \).

Since by Lemma \[\text{[0.5]}\] we have
\[
\int_{(0,1)^2} (W - \Pi) \, d\mu = \int_{(0,1)^2} ((W \ast M) - \Pi) \, d\mu = -\int_{(0,1)^2} (M - \Pi) \, d\mu
\]
and for any copula, \( C \), we have \( W \leq C \leq M \) pointwise by \[\text{[5]}\], we obtain \(-1 \leq \kappa(C) \leq 1 \).

Clearly \( \kappa(M) = 1 \) and \( \kappa(\Pi) = 0 \).

Since \( \mu \) is \( D_4 \)-invariant, by Lemma \[\text{[0.5]}\] we have \( \kappa(W \ast C) = \kappa(C \ast W) = -\kappa(C) \) and \( \kappa(C^T) = \kappa(C) \).

Since \( \mu \) is a positive measure, \( C_1 \leq C_2 \) pointwise implies \( \kappa(C_1) \leq \kappa(C_2) \).

Finally, if \( C_n \to C \) pointwise, the dominated convergence theorem yields \( \kappa(C_n) \to \kappa(C) \). \( \square \)

Note that Theorem \[\text{[0.3]}\] does not characterize measures of concordance that preserve convex sums of copulas. While we have given a form for a measure of concordance preserving convex sums when \[\text{[0.3]}\] holds, we have not yet examined what happens when the condition \[\text{[0.3]}\] is relaxed.

**Proposition 0.7.** Let \( \kappa(C) = \int_{(0,1)^2} (C - \Pi) \, d\mu \) be a measure of concordance. If \( \mu((0,1)^2) = \infty \), then there exists no \( \alpha > 0 \) such that \( |\kappa(C)| \leq \alpha ||C - \Pi||_\infty \) for all \( C \in \text{Cop}(2) \).

This can be easily shown as follows. Let \( R_n \) be a sequence of closed rectangles in \((0,1)^2\) such that \( R_n \to (0,1)^2 \). Note for the sequence \( C_{R_n, \delta_n} \) that it is necessary for \( \delta_n \to 0 \) as \( n \to \infty \). Then suppose there exists an \( \alpha > 0 \) such that \( |\kappa(C)| \leq \alpha ||C - \Pi||_\infty \) for all \( C \in \text{Cop}(2) \). Then \( |\kappa(C_{R_n, \delta_n})| \leq \alpha \delta_n^2 \). Dividing by \( \delta_n^2 \) and letting \( n \to \infty \) gives \( \mu((0,1)^2) \leq \alpha \), which is impossible since \( \mu((0,1)^2) = \infty \).

We want to look at examples where condition \[\text{[0.3]}\] fails, yet \( \kappa \) is a measure of concordance which preserves convex sums of copulas. To construct examples, we will take the region in \((0,1)^2\) where \( 0 < x < \frac{1}{2} \) and \( x \leq y \leq \frac{1}{2} \). Let us call this region \( \Delta \). Consider any positive, regular Borel measure on \( \Delta \), call it \( \mu_\Delta \), such that
\[
0 < \int_\Delta (M - \Pi) \, d\mu_\Delta < \infty.
\]
To construct a positive, regular, $D_4$-invariant Borel measure on $(0,1)^2$ we repeatedly reflect the distributed mass on $\Delta$ determined by $\mu_\Delta$ about the lines $y = x$ and $x = \frac{1}{2}$. By considering each of the eight regions of $(0,1)^2$ separated by $x = \frac{1}{2}$, $y = \frac{1}{2}$, $y = x$, and $y = 1 - x$ and then using the transformations $(x,y) \mapsto (y,x)$, $(x,y) \mapsto (1-x,y)$, and $(x,y) \mapsto (x,1-y)$ it can be seen that

$$0 < \int_{(0,1)^2} (M - \Pi) \, d\mu < \infty.$$  

Therefore, by Theorem 0.6, $\mu$ can be used to define a measure of concordance.

Observe in the following examples that $\mu((0,1)^2)$ is not necessarily finite. Thus by Proposition 0.7 condition (0.3) is not always satisfied, yet measures of concordance are determined by $\mu$ that preserve convex sums of copulas.

**Example 0.8.** Let $\{x_n\}$ be a decreasing sequence in $(0,\frac{1}{2})$ where $x_n \to 0$, and let $\alpha_n > 0$ be such that $\sum_{n=1}^\infty \alpha_n x_n < \infty$. Also, define $\mu_\Delta = \sum_{n=1}^\infty \alpha_n \delta(x_n,\frac{1}{2})$ where $\delta(x,y)$ indicates the unit point measure at $(x,y)$, thereby insuring that condition (0.6) is satisfied since

$$0 < \int_{\Delta} (M - \Pi) \, d\mu_\Delta = \frac{1}{2} \sum_{n=1}^\infty \alpha_n x_n < \infty$$

while the measure, $\mu$, generated by $\mu_\Delta$ does not necessarily define a measure of concordance satisfying the condition (0.3).

Indeed, this is the case if $x_n = \frac{1}{n}$ and $\alpha_n = n$ since $\mu((0,1)^2) = 4 \sum_{i=1}^\infty n = \infty$, which implies that condition (0.3) is not satisfied by Proposition 0.7.

**Example 0.9.** This example examines a situation where the mass is distributed in a continuous rather than discrete fashion as in the previous example. Consider a continuous function, $f^* : (0,\frac{1}{2}] \to (0,\infty)$ where $\int_{0}^{\frac{1}{2}} tf^*(t) \, dt < \infty$. Let $f^*$ define a density distribution on the line segment $x = \frac{1}{2}$, $0 < y \leq \frac{1}{2}$. If we consider an extension, $f$, of $f^*$ where

$$f(t) = \begin{cases} f^*(t), & 0 < t \leq \frac{1}{2}, \\ f^*(1-t), & \frac{1}{2} < t < 1, \end{cases}$$

then $f$ can be used to define a $D_4$-invariant density distribution on the line segments $x = \frac{1}{2}$ and $y = \frac{1}{2}$ in $(0,1)^2$. Since $f$ is continuous and maps to $(0,\infty)$, the density distribution defined by $f$ induces a measure, say $\mu$, that is both regular and positive. Furthermore, since

$$0 < \int_{(0,1)^2} (M - \Pi) d\mu$$

$$= \frac{1}{2} \left[ \int_{0}^{\frac{1}{2}} M \left( t, \frac{1}{2} \right) + M \left( 1-t, \frac{1}{2} \right) + M \left( \frac{1}{2}, t \right) + M \left( \frac{1}{2}, 1-t \right) - 1 \right] f(t) dt$$

$$= 2 \int_{0}^{\frac{1}{2}} tf(t) dt = 2 \int_{0}^{\frac{1}{2}} tf^*(t) dt < \infty,$$

we have by Theorem 0.6 that

$$\kappa(C) = \gamma \int_{0}^{\frac{1}{2}} \left[ C \left( t, \frac{1}{2} \right) + C \left( 1-t, \frac{1}{2} \right) + C \left( \frac{1}{2}, t \right) + C \left( \frac{1}{2}, 1-t \right) - 1 \right] f(t) dt$$

is a measure of concordance where $\gamma = (\int_{(0,1)^2} (M - \Pi) d\mu)^{-1} = (2 \int_{0}^{\frac{1}{2}} tf(t) dt)^{-1}$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Indeed, this is the case if $f^*(t) = \frac{1}{t}$, yet
\[ \mu((0,1)^2) = 4 \int_0^{\frac{1}{2}} \frac{dt}{t} = \infty, \]
which implies that condition (0.3) is not satisfied by Proposition 0.7.

**Example 0.10.** We give a probabilistic interpretation of our measure of concordance in the case when the mass of $\mu$ is concentrated at a finite number of points. (This is a variation of the generalization of Blomqvist’s beta to a multivariate measure of concordance given by H. Joe in [4].) We begin with a particularly simple such $\mu$.

Let $0 < y_0 < x_0 < 1 - x_0 < 1 - y_0 < 1$. We construct a measure, $\mu$, on the Borel sets of $I^2$ by placing a mass at $(x_0, y_0)$ and the other seven points in $I^2$ generated by the symmetries of the unit square. It is necessary for the mass placed at each of these points to be $\frac{1}{4y_0}$ in order to ensure that $C \mapsto \int_{I^2} (C - \Pi) \, d\mu$ is a measure of concordance.

Allowing the eight massed points to set up a partition of $I^2$ into rectangular cells, we let $f$ be the function that assigns to each of these cells the value indicated in Figure 2. By direct calculation it can be seen that
\[ (0.7) \quad \int_{(0,1)^2} (C - \Pi) \, d\mu = \int_{(0,1)^2} \left( f - \frac{1}{2y_0} \right) \, dC. \]

![Figure 2.](image_url)

Letting $X_1$ and $X_2$ be random variables uniformly distributed over $I$ such that the random vector $(X_1, X_2)$ is associated with the copula, $C$, by (0.7) we have the probabilistic interpretation
\[
\int_{(0,1)^2} (C - \Pi) \, d\mu = \frac{6}{4y_0} P((X_1, X_2) \in [0, y_0] \times [0, y_0]) \\
+ \frac{4}{4y_0} \{ P((X_1, X_2) \in [0, y_0] \times [y_0, x_0]) + P((X_1, X_2) \in [y_0, x_0] \times [0, y_0]) \} + \ldots \\
- \frac{2}{4y_0} \{ P((X_1, X_2) \in [0, 1] \times [1 - y_0, 1]) + P((X_1, X_2) \in [1 - y_0, 1] \times [0, 1 - y_0]) \\
+ P((X_1, X_2) \in [1 - x_0, 1 - y_0] \times [1 - x_0, 1 - y_0]) \}.
\]
The general case of $\mu$ with mass at a finite number of points can be realized as weighted sums or degenerate cases of examples where the mass is distributed over eight points.

ACKNOWLEDGEMENT

The authors wish to convey their thanks for the referee's helpful suggestions.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, P.O. BOX 161364, ORLANDO, FLORIDA 32816-1364
E-mail address: newcopulae@yahoo.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, P.O. BOX 161364, ORLANDO, FLORIDA 32816-1364
E-mail address: piotrm@mail.ucf.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, P.O. BOX 161364, ORLANDO, FLORIDA 32816-1364
E-mail address: mtaylor@pegasus.cc.ucf.edu