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MEASURES OF CONCORDANCE DETERMINED BY D_4-INVARIANT MEASURES ON $(0,1)^2$

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Abstract. A measure, μ, on $(0,1)^2$ is said to be D_4-invariant if its value for any Borel set is invariant with respect to the symmetries of the unit square. A function, κ, generated in a certain way by a measure, μ, on $(0,1)^2$ is shown to be a measure of concordance if and only if the generating measure is positive, regular, D_4-invariant, and satisfies certain inequalities. The construction examined here includes Blomqvist's beta as a special case.

Let $I = [0,1]$ and $I^2 = [0,1] \times [0,1]$. η is a doubly stochastic measure on I^2 if it is a probability measure on the Borel sets of I^2 such that $\eta(A \times I) = \eta(I \times A) = \lambda(A)$ where A is a Borel set of I and λ is the one-dimensional Lebesgue measure. A copula (more precisely a 2-copula) is a function, $C : I^2 \longrightarrow I$, that is related to some doubly stochastic measure, η, by $C(x,y) = \eta([0,x] \times [0,y])$. (See [6].) There is a one-to-one correspondence between Cop(2), the set of copulas, and the set of doubly stochastic measures. The doubly stochastic measure corresponding to the copula, C, will be denoted η_C.

Besides being associated with a doubly stochastic measure, a copula can be uniquely determined by a pair of continuous random variables. By Sklar's theorem, for any continuous random vector, (X,Y), with marginals, F_X and F_Y respectively, and joint distribution function, $F_{X,Y}$, there exists a unique copula, C, such that $F_{X,Y}(x,y) = C(F_X(x), F_Y(y))$. (See [1] or [6].)

The simplest examples are as follows. If Y is an increasing function of X, then the associated copula is $M(x,y) = \min(x,y)$. If Y is a decreasing function of X, then the associated copula is $W(x,y) = \max(x+y-1, 0)$. Finally, if X and Y are independent, then the associated copula is $\Pi(x,y) = xy$. (See again [6].)

When considering two random variables it can be useful to know how much large values of one random variable correspond with large values of the other. Such a property can be gauged by a measure of concordance, a concept developed by Scarsini [7] and presented in Nelsen's book [6], *An Introduction to Copulas*.

A measure of concordance associated with a continuous random vector (X,Y) is a real number $\kappa_{X,Y}$ satisfying certain axioms, which we shall exhibit below. It can be shown that this value depends only on the copula, C, associated with (X,Y). Because of this, κ_C is often used instead of $\kappa_{X,Y}$. In fact, since κ is a functional on Cop(2), we will write $\kappa(C)$.

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The star product, developed by Darsow, Nguyen, and Olsen (see [2]) of two given copulas results in a copula. Given $A, B \in \mathrm{Cop}(2)$, the star product is defined as

$$(A * B)(x, y) = \int_0^1 D_2 A(x, t) D_1 B(t, y)\, dt$$

where D_i is the differential operator on the ith variable for $i = 1, 2$. Straightforward calculations yield $(M * C)(x, y) = (C * M)(x, y) = C(x, y)$, $(\Pi * C)(x, y) = (C * \Pi)(x, y) = \Pi(x, y)$, $(W * C)(x, y) = y - C(1 - x, y)$, and $(C * W)(x, y) = x - C(x, 1 - y)$.

Note that if C is the copula associated with the random vector, (X, Y), then $y - C(1-x, y)$ is associated with $(-X, Y)$. Similarly, $x - C(x, 1-y)$ is associated with $(X, -Y)$. Keeping the previous observations in mind and letting $C^T(x, y) = C(y, x)$ so that C^T is associated with (Y, X), we can write the definition for a measure of concordance as follows.

Definition 0.1. A functional $\kappa : \mathrm{Cop}(2) \to [-1, 1]$ is called a measure of concordance if:

1. $\kappa(M) = 1$ and $\kappa(\Pi) = 0$,
2. $\kappa(W * C) = \kappa(C * W) = -\kappa(C)$,
3. $\kappa(C) = \kappa(C^T)$,
4. $C_1 \leq C_2$ pointwise implies $\kappa(C_1) \leq \kappa(C_2)$, and
5. $C_n \to C$ pointwise implies $\kappa(C_n) \to \kappa(C)$.

In our previous paper [3] we discussed measures of concordance determined by D_4-invariant copulas. This type of measure of concordance is expressed as

$$\kappa(C) = \frac{\int_{I^2} (C - \Pi) dA}{\int_{I^2} (M - \Pi) dA} \quad \text{or} \quad \kappa(C) = \frac{\int_{I^2} (C - \Pi) d\eta_A}{\int_{I^2} (M - \Pi) d\eta_A}$$

where A is any fixed D_4-invariant copula. For example, if $A = \Pi$, then Spearman's rho is obtained. However, there are other commonly known measures of concordance that are not of this type, but are similar in form with the exception of η being a doubly stochastic measure. One such example is Blomqvist's beta.

Blomqvist's beta, β, can be expressed as

$$\beta(C) = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1$$

or can alternatively be written in the form

$$\beta(C) = \frac{\int_{(0,1)^2} (C - \Pi)\, d\nu}{\int_{(0,1)^2} (M - \Pi)\, d\nu}$$

where ν is a measure such that a mass of 4 is concentrated only at $(\frac{1}{2}, \frac{1}{2})$. Notice that ν certainly cannot be generated by a copula since it is not a doubly stochastic measure.

Given a measure, μ, on the Borel sets of $(0, 1)^2$, we consider in this paper the conditions on μ that allow the mapping

$$C \mapsto \frac{\int_{(0,1)^2} (C - \Pi)\, d\mu}{\int_{(0,1)^2} (M - \Pi)\, d\mu} \tag{0.1}$$

to be a measure of concordance. Scarsini implicitly anticipated measures of concordance of this form by giving what amounted to special cases of (0.1) in [7].

Definition 0.2. A measure, μ, is D_4-*invariant* if μ is invariant under the symmetries of I^2. That is, if S is a Borel set of I^2 and $g : I^2 \longrightarrow I^2$ is a symmetry of I^2, then $\mu(g(S)) = \mu(S)$.

It is sufficient when applying this definition to consider the symmetries

$$\tau(x, y) = (y, x) \quad \text{and} \quad \sigma(x, y) = (1 - x, y).$$

Note that if η_C is a doubly stochastic measure corresponding to a copula, C, then $\eta_C(\sigma(S)) = \eta_{W*C}(S)$ and $\eta_C(\tau(S)) = \eta_{C^T}(S)$ for any Borel set S in $[0, 1]^2$. Thus, if $\eta_C = \eta_{W*C} = \eta_{C^T}$, then C is a D_4-invariant copula. (See [3].)

Theorem 0.3. *Let κ be a measure of concordance. There exists a regular Borel measure, μ, on I^2 such that*

$$\kappa(C) = \int_{I^2} (C - \Pi)\ d\mu$$

if and only if

(0.2) $\kappa(tA+(1-t)B) = t\kappa(A)+(1-t)\kappa(B)$ *for* $t \in (0, 1)$ *and* $A, B \in \mathrm{Cop}(2)$,

and

(0.3) *there exists* $\alpha > 0$ *such that* $|\kappa(C)| \leq \alpha||C - \Pi||_\infty$ *for* $C \in \mathrm{Cop}(2)$.

Proof. Let κ be a measure of concordance such that $\kappa(C) = \int_{I^2}(C-\Pi)\ d\mu$ for some regular Borel measure on I^2. Clearly, (0.2) is satisfied. Since $\mu(I^2) < \infty$, we have $|\kappa(C)| = |\int_{I^2}(C - \Pi)\ d\mu| \leq \mu(I^2)||C - \Pi||_\infty$ for any copula C.

Now assume that κ is a measure of concordance such that (0.2) and (0.3) hold. Letting $E = \{C - \Pi : C \text{ is a copula}\}$, we have a compact, convex subset of $\mathcal{C}(I^2)$, the space of continuous functions on I^2, with respect to $||\cdot||_\infty$. Define a functional $\Lambda : E \to \mathbb{R}$ via

$$\Lambda(C - \Pi) = \kappa(C).$$

Since Λ is uniformly bounded on E and linear, it can be extended to a bounded linear functional on $\mathcal{C}(I^2)$. By the Riesz Representation Theorem, there exists a regular Borel measure, μ, on I^2 such that

$$\Lambda(f) = \int_{I^2} f\ d\mu$$

for all $f \in \mathcal{C}(I^2)$. Consequently, $\kappa(C) = \int_{I^2}(C - \Pi)\ d\mu$ for any copula, C. □

We say that a measure of concordance defined by $\kappa(C) = \int_{I^2}(C - \Pi)\ d\mu$ is *copular* if and only if μ is a scalar multiple of a doubly stochastic measure. (See [3].)

If we continue to consider measures of concordance of the form

$$\kappa(C) = \int_{I^2} (C - \Pi) d\mu$$

with μ being a regular measure, then we are restricting ourselves to only consider such μ where $\mu(I^2) < \infty$ since I^2 is compact. However, if we adjust the form of our measure of concordance to

$$\kappa(C) = \gamma \int_{(0,1)^2} (C - \Pi) d\mu,$$

then we can also use μ where $\mu(I^2) = \infty$ since $C - \Pi = 0$ on $\partial(I^2)$.

We first describe a construction that will be useful. Choose a rectangle, $R = [x_1, x_2] \times [y_1, y_2]$ in $(0,1)^2$ and $\delta > 0$ such that the expanded rectangle, $[x_1 - \delta, x_2 + \delta] \times [y_1 - \delta, y_2 + \delta]$, is also contained in $(0,1)^2$. The boundaries of I^2 and the lines $x = x_1 - \delta$, $x = x_1$, $x = x_2$, $x = x_2 + \delta$, $y = y_1 - \delta$, $y = y_1$, $y = y_2$, and $y = y_2 + \delta$ partition I^2 into 25 rectangular cells. We define a doubly stochastic measure on I^2 by distributing a unit mass with density 2 in the cells, $[x_1 - \delta, x_1] \times [y_1 - \delta, y_1]$ and $[x_2, x_2 + \delta] \times [y_2, y_2 + \delta]$, and density 0 in the cells, $[x_1 - \delta, x_1] \times [y_2, y_2 + \delta]$ and $[x_2, x_2 + \delta] \times [y_1 - \delta, y_1]$. Otherwise, the density of any cell is 1. See Figure 1. Let $C_{R,\delta}$ be the copula associated with this doubly stochastic measure. We make use of $C_{R,\delta}$ in some of the following proofs.

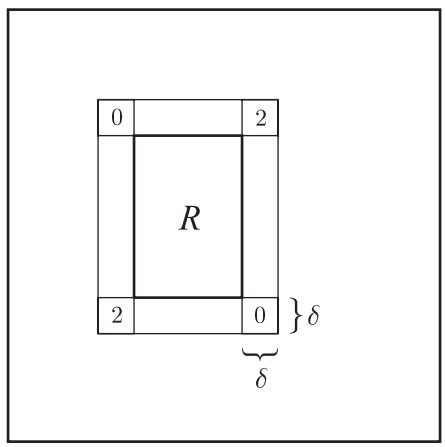

FIGURE 1.

Lemma 0.4. *If μ and ν are positive, regular Borel measures such that*

$$\int_{(0,1)^2} (C - \Pi)\, d\mu = \int_{(0,1)^2} (C - \Pi)\, d\nu$$

for all C, then $\mu = \nu$.

Proof. If for any rectangle $R = [x_1, x_2] \times [y_1, y_2]$ in $(0,1)^2$ we have $\mu(R) = \nu(R)$, then we are done.

Consider $Q_{R,\delta}(x,y)$ where $Q_{R,\delta}(x,y) = (C_{R,\delta}(x,y) - \Pi(x,y))/\delta^2$. For any $(x,y) \notin R$, there exists a sufficiently small $\delta_0 > 0$ such that $Q_{R,\delta_0}(x,y) = 0$.

For any $(x,y) \in R$, $Q_{R,\delta}(x,y) = 1$. So, by letting $\delta \to 0$ we see that $Q_{R,\delta} \to \chi(R)$.

Since $|Q_{R,\delta}| \leq 1$ for any R and for any $\delta > 0$, then by the dominated convergence theorem,

$$\begin{aligned}
\mu(R) &= \int_{(0,1)^2} \chi(R)\, d\mu = \lim_{\delta \to 0} \int_{(0,1)^2} Q_{R,\delta}\, d\mu \\
&= \lim_{\delta \to 0} \int_{(0,1)^2} Q_{R,\delta}\, d\nu = \int_{(0,1)^2} \chi(R)\, d\nu \\
&= \nu(R).
\end{aligned}$$

□

Lemma 0.5. *If μ is a Borel measure on $(0,1)^2$, then μ is D_4-invariant if and only if*

$$\int_{(0,1)^2} (C^T - \Pi)\ d\mu = \int_{(0,1)^2} (C - \Pi)\ d\mu \tag{0.4}$$

and

$$\int_{(0,1)^2} ((W * C) - \Pi)\ d\mu = -\int_{(0,1)^2} (C - \Pi)\ d\mu \tag{0.5}$$

for any copula, C.

Proof. Recall that $\tau(x,y) = (y,x)$ and $\sigma(x,y) = (1-x,y)$.

For any copula, C, we have

$$\int_{(0,1)^2} (C(x,y) - xy)\ d(\mu \circ \tau) = \int_{(0,1)^2} (C(y,x) - yx)\ d\mu = \int_{(0,1)^2} (C^T(x,y) - xy)\ d\mu$$

and

$$\begin{aligned}\int_{(0,1)^2} (C(x,y) - xy)\ d(\mu \circ \sigma) &= \int_{(0,1)^2} (C(1-x,y) - (1-x)y)\ d\mu \\ &= -\int_{(0,1)^2} (y - C(1-x,y) - xy)\ d\mu \\ &= -\int_{(0,1)^2} ((W * C)(x,y) - xy)\ d\mu.\end{aligned}$$

In other words,

$$\int_{(0,1)^2} (C - \Pi)\ d(\mu \circ \tau) = \int_{(0,1)^2} (C^T - \Pi)\ d\mu$$

and

$$\int_{(0,1)^2} (C - \Pi)\ d(\mu \circ \sigma) = -\int_{(0,1)^2} ((W * C) - \Pi)\ d\mu$$

for any copula, C. If μ is D_4-invariant, the above reduces to (0.4) and (0.5). On the other hand, if (0.4) and (0.5) hold, then μ is D_4-invariant by Lemma 0.4. □

Theorem 0.6. *Let μ be a Borel measure on $(0,1)^2$. Then*

$$\kappa(C) = \gamma \int_{(0,1)^2} (C - \Pi)\ d\mu$$

is a measure of concordance for some $\gamma > 0$ if and only if μ is positive, regular, and D_4-invariant, $0 < \int_{(0,1)^2} (M - \Pi)\ d\mu < \infty$, and $\gamma = (\int_{(0,1)^2} (M - \Pi)\ d\mu)^{-1}$.

Proof. Note that in the proof of this theorem, $Q_{R,\delta}$ is as defined in Lemma 0.4 where R is a rectangle in $(0,1)^2$.

Suppose that $\kappa(C)$ is a measure of concordance.

Clearly $0 < \int_{(0,1)^2} (M - \Pi)\ d\mu < \infty$ and $\gamma = (\int_{(0,1)^2} (M - \Pi)\ d\mu)^{-1}$ since $\kappa(M) = 1$.

$\mu(R) \geq 0$ for any R in $(0,1)^2$ since for any $\delta > 0$,

$$\int_{(0,1)^2} Q_{R,\delta}\ d\mu \geq \frac{1}{\delta^2} \int_{(0,1)^2} (\Pi - \Pi)\ d\mu = 0,$$

and by letting $\delta \to 0$ we have $\mu(R) \geq 0$.

If we suppose μ is not regular, then there exists a compact set K in $(0,1)^2$ such that $\mu(K) = \infty$. However, setting $m = \min_{(x,y)\in K}(M - \Pi)(x,y) > 0$, we have

$$\infty > \int_{(0,1)^2} (M - \Pi)\ d\mu \geq \int_K (M - \Pi)\ d\mu \geq m \int_K d\mu = m\mu(K) = \infty.$$

So μ must be regular.

Finally, since $\kappa(C)$ is a measure of concordance we have

$$\int_{(0,1)^2} (C^T - \Pi)\ d\mu = \int_{(0,1)^2} (C - \Pi)\ d\mu$$

and

$$\int_{(0,1)^2} ((W * C) - \Pi)\ d\mu = -\int_{(0,1)^2} (C - \Pi)\ d\mu.$$

So μ is D_4-invariant by Lemma 0.5.

Now suppose μ is positive, regular, and D_4-invariant, $0 < \int_{(0,1)^2}(M - \Pi)\ d\mu < \infty$, and $\gamma = (\int_{(0,1)^2}(M - \Pi)\ d\mu)^{-1}$.

Since by Lemma 0.5 we have

$$\int_{(0,1)^2} (W - \Pi)\ d\mu = \int_{(0,1)^2} ((W * M) - \Pi)\ d\mu = -\int_{(0,1)^2} (M - \Pi)\ d\mu$$

and for any copula, C, we have $W \leq C \leq M$ pointwise by [5], we obtain $-1 \leq \kappa(C) \leq 1$.

Clearly $\kappa(M) = 1$ and $\kappa(\Pi) = 0$.

Since μ is D_4-invariant, by Lemma 0.5 we have $\kappa(W * C) = \kappa(C * W) = -\kappa(C)$ and $\kappa(C^\tau) = \kappa(C)$.

Since μ is a positive measure, $C_1 \leq C_2$ pointwise implies $\kappa(C_1) \leq \kappa(C_2)$.

Finally, if $C_n \to C$ pointwise, the dominated convergence theorem yields $\kappa(C_n) \to \kappa(C)$. □

Note that Theorem 0.3 does not characterize measures of concordance that preserve convex sums of copulas. While we have given a form for a measure of concordance preserving convex sums when (0.3) holds, we have not yet examined what happens when the condition (0.3) is relaxed.

Proposition 0.7. *Let $\kappa(C) = \int_{(0,1)^2}(C - \Pi)\ d\mu$ be a measure of concordance. If $\mu((0,1)^2) = \infty$, then there exists no $\alpha > 0$ such that $|\kappa(C)| \leq \alpha||C - \Pi||_\infty$ for all $C \in \mathrm{Cop}(2)$.*

This can be easily shown as follows. Let R_n be a sequence of closed rectangles in $(0,1)^2$ such that $R_n \to (0,1)^2$. Note for the sequence C_{R_n,δ_n} that it is necessary for $\delta_n \to 0$ as $n \to \infty$. Then suppose there exists an $\alpha > 0$ such that $|\kappa(C)| \leq \alpha||C - \Pi||_\infty$ for all $C \in \mathrm{Cop}(2)$. Then $|\kappa(C_{R_n,\delta_n})| \leq \alpha{\delta_n}^2$. Dividing by ${\delta_n}^2$ and letting $n \to \infty$ gives $\mu((0,1)^2) \leq \alpha$, which is impossible since $\mu((0,1)^2) = \infty$.

We want to look at examples where condition (0.3) fails, yet κ is a measure of concordance which preserves convex sums of copulas. To construct examples, we will take the region in $(0,1)^2$ where $0 < x < \frac{1}{2}$ and $x \leq y \leq \frac{1}{2}$. Let us call this region Δ. Consider any positive, regular Borel measure on Δ, call it μ_Δ, such that

$$0 < \int_\Delta (M - \Pi)\ d\mu_\Delta < \infty.$$

To construct a positive, regular, D_4-invariant Borel measure on $(0,1)^2$ we repeatedly reflect the distributed mass on Δ determined by μ_Δ about the lines $y = x$ and $x = \frac{1}{2}$. By considering each of the eight regions of $(0,1)^2$ separated by $x = \frac{1}{2}$, $y = \frac{1}{2}$, $y = x$, and $y = 1 - x$ and then using the transformations $(x,y) \mapsto (y,x)$, $(x,y) \mapsto (1-x,y)$, and $(x,y) \mapsto (x,1-y)$ it can be seen that

$$0 < \int_{(0,1)^2} (M - \Pi)\, d\mu < \infty. \tag{0.6}$$

Therefore, by Theorem 0.6, μ can be used to define a measure of concordance.

Observe in the following examples that $\mu((0,1)^2)$ is not necessarily finite. Thus by Proposition 0.7 condition (0.3) is not always satisfied, yet measures of concordance are determined by μ that preserve convex sums of copulas.

Example 0.8. Let $\{x_n\}$ be a decreasing sequence in $(0, \frac{1}{2})$ where $x_n \to 0$, and let $\alpha_n > 0$ be such that $\sum_{n=1}^{\infty} \alpha_n x_n < \infty$. Also, define $\mu_\Delta = \sum_{n=1}^{\infty} \alpha_n \delta(x_n, \frac{1}{2})$ where $\delta(x,y)$ indicates the unit point measure at (x,y), thereby insuring that condition (0.6) is satisfied since

$$0 < \int_\Delta (M - \Pi)\, d\mu_\Delta = \frac{1}{2} \sum_{n=1}^{\infty} \alpha_n x_n < \infty$$

while the measure, μ, generated by μ_Δ does not necessarily define a measure of concordance satisfying the condition (0.3).

Indeed, this is the case if $x_n = \frac{1}{2^{n+1}}$ and $\alpha_n = n$ since $\mu((0,1)^2) = 4\sum_{i=1}^{\infty} n = \infty$, which implies that condition (0.3) is not satisfied by Proposition 0.7.

Example 0.9. This example examines a situation where the mass is distributed in a continuous rather than discrete fashion as in the previous example. Consider a continuous function, $f^* : (0, \frac{1}{2}] \to (0, \infty)$ where $\int_0^{\frac{1}{2}} t f^*(t)\, dt < \infty$. Let f^* define a density distribution on the line segment $x = \frac{1}{2}$, $0 < y \le \frac{1}{2}$. If we consider an extension, f, of f^* where

$$f(t) = \begin{cases} f^*(t), & 0 < t \le \frac{1}{2}, \\ f^*(1-t), & \frac{1}{2} < t < 1, \end{cases}$$

then f can be used to define a D_4-invariant density distribution on the line segments $x = \frac{1}{2}$ and $y = \frac{1}{2}$ in $(0,1)^2$. Since f is continuous and maps to $(0,\infty)$, the density distribution defined by f induces a measure, say μ, that is both regular and positive. Furthermore, since

$$\begin{aligned} 0 &< \int_{(0,1)^2} (M - \Pi) d\mu \\ &= \int_0^{\frac{1}{2}} \left[M\left(t, \frac{1}{2}\right) + M\left(1-t, \frac{1}{2}\right) + M\left(\frac{1}{2}, t\right) + M\left(\frac{1}{2}, 1-t\right) - 1 \right] f(t) dt \\ &= 2\int_0^{\frac{1}{2}} t f(t) dt = 2\int_0^{\frac{1}{2}} t f^*(t) dt < \infty, \end{aligned}$$

we have by Theorem 0.6 that

$$\kappa(C) = \gamma \int_0^{\frac{1}{2}} \left[C\left(t, \frac{1}{2}\right) + C\left(1-t, \frac{1}{2}\right) + C\left(\frac{1}{2}, t\right) + C\left(\frac{1}{2}, 1-t\right) - 1 \right] f(t) dt$$

is a measure of concordance where $\gamma = (\int_{(0,1)^2} (M - \Pi) d\mu)^{-1} = (2\int_0^{\frac{1}{2}} t f(t) dt)^{-1}$.

Indeed, this is the case if $f^*(t) = \frac{1}{t}$, yet

$$\mu((0,1)^2) = 4\int_0^{\frac{1}{2}} \frac{dt}{t} = \infty,$$

which implies that condition (0.3) is not satisfied by Proposition 0.7.

Example 0.10. We give a probabilistic interpretation of our measure of concordance in the case when the mass of μ is concentrated at a finite number of points. (This is a variation of the generalization of Blomqvist's beta to a multivariate measure of concordance given by H. Joe in [4].) We begin with a particularly simple such μ.

Let $0 < y_0 < x_0 < 1 - x_0 < 1 - y_0 < 1$. We construct a measure, μ, on the Borel sets of I^2 by placing a mass at (x_0, y_0) and the other seven points in I^2 generated by the symmetries of the unit square. It is necessary for the mass placed at each of these points to be $\frac{1}{4y_0}$ in order to insure that $C \mapsto \int_{I^2}(C - \Pi)\, d\mu$ is a measure of concordance.

Allowing the eight massed points to set up a partition of I^2 into rectangular cells, we let f be the function that assigns to each of these cells the value indicated in Figure 2. By direct calculation it can be seen that

$$\int_{(0,1)^2} (C - \Pi)\, d\mu = \int_{(0,1)^2} \left(f - \frac{1}{2y_0}\right) dC. \tag{0.7}$$

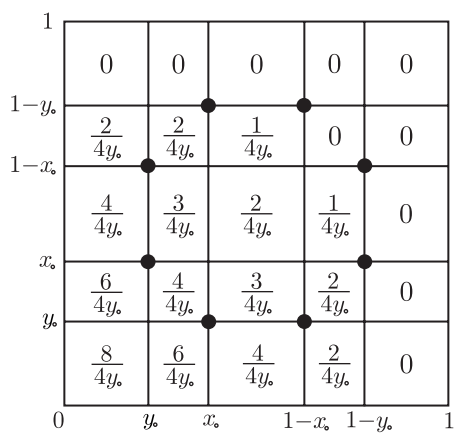

FIGURE 2.

Letting X_1 and X_2 be random variables uniformly distributed over I such that the random vector (X_1, X_2) is associated with the copula, C, by (0.7) we have the probabilistic interpretation

$$\begin{aligned}
\int_{(0,1)^2} (C - \Pi)\, d\mu &= \frac{6}{4y_0} P((X_1, X_2) \in [0, y_0] \times [0, y_0]) \\
&+ \frac{4}{4y_0}\{P((X_1, X_2) \in [0, y_0] \times [y_0, x_0]) + P((X_1, X_2) \in [y_0, x_0] \times [0, y_0])\} + \ldots \\
&- \frac{2}{4y_0}\{P((X_1, X_2) \in [0, 1] \times [1 - y_0, 1]) + P((X_1, X_2) \in [1 - y_0, 1] \times [0, 1 - y_0]) \\
&\qquad + P((X_1, X_2) \in [1 - x_0, 1 - y_0] \times [1 - x_0, 1 - y_0])\}.
\end{aligned}$$

The general case of μ with mass at a finite number of points can be realized as weighted sums or degenerate cases of examples where the mass is distributed over eight points.

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