SAMPLING SETS AND CLOSED RANGE COMPOSITION OPERATORS ON THE BLOCH SPACE

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Abstract. We give a necessary and sufficient condition for a composition operator \( C_\phi \) on the Bloch space to have closed range. We show that when \( \phi \) is univalent, it is sufficient to consider the action of \( C_\phi \) on the set of Möbius transforms. In this case the closed range property is equivalent to a specific sampling set satisfying the reverse Carleson condition.

1. Introduction

An analytic function \( f \) on \( D \) is said to belong to the Bloch space if \( \sup \{|1 - |z|^2|f'(z)|\} \) over \( D \) is finite. Such functions form a complex Banach space \( B \) under the norm \( \| f \|_B = \sup \{|1 - |z|^2|f'(z)|, z \in D\} + |f(0)| \). Functions belonging to the little Bloch space \( B_0 \) (consisting of the closure of polynomials in \( B \)) are characterized by the property: \( \lim_{|z| \to 1} (1 - |z|^2)f'(z) = 0 \).

Observe that \( \sup \{|1 - |z|^2|f'(z)|, z \in D\} \) is a pseudonorm, which coincides with the Bloch-norm on the closed subspace of functions that vanish at the origin. In general it coincides with the quotient norm on \( B/C \) where \( C \) denotes the closed subspace of constant functions.

The following concept is what all our criteria are based on.

We say that a subset \( H \) of \( D \) is called a sampling set for the Bloch space \( B \) if \( \exists k > 0 \) such that \( \sup \{|1 - |z|^2|f'(z)|, z \in D\} \leq k \sup \{|1 - |z|^2|f'(z)|, z \in H\} \) holds \( \forall f \in B \).

This is equivalent to \( H \) being a sampling set for the \( L^\infty \) version of the weighted Bergman space, denoted by \( A^{-1} \) [5, p. 22]. There are other definitions of sampling set for the Bloch space, but this one suits our purpose the best.

For each \( z \) belonging to the unit disk \( D \), let \( \phi_z \) denote the Möbius transformation of \( D \), given by

\[
\phi_z(w) = \frac{z - w}{1 - \overline{z}w},
\]

for \( w \in D \). The pseudohyperbolic distance (between \( z \) and \( w \)) on \( D \) is defined by

\[
\rho(z, w) = |\phi_z(w)|.
\]

\( D(a, s) \) stands for the set \( \{z \in D, \rho(z, a) < s\} \).

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Theorem 0. If \( \phi(0) = 0 \), the composition operator \( C_\phi \) is bounded below on \( B \) (equivalently, has closed range on \( B \)) if and only if it is bounded below on the subspace of functions that vanish at the origin. This is equivalent to the condition that \( \| f \circ \phi \|_{B/C} \geq k \| f \|_{B/C} \).

Remark 1. If \( \phi(0) = a \) and \( \psi = \phi \circ \phi \), then \( C_\phi \) is bounded below on \( B \) if and only if \( C_\psi \) is bounded below on \( B \). Moreover, \( \tau_\psi = \tau_\phi \).

So we assume from now on that \( \phi(0) = 0 \) and that \( C_\phi \) is acting on the subspace of functions that vanish at the origin. It is natural that the sets \( \Omega_\varepsilon = \{ z ; |\tau_\phi(z)| \geq \varepsilon \} \) and \( G_\varepsilon = \phi(\Omega_\varepsilon) \) play a pivotal role in our investigation.

Theorem 1. The composition operator \( C_\phi \) is bounded below on \( B \) if and only if \( \exists \varepsilon > 0 \) such that if \( \Omega_\varepsilon = \{ z \in D ; |\tau_\phi(z)| \geq \varepsilon \} \), then \( G_\varepsilon \) is a sampling set for \( B \).

Proof. Assume that \( G_\varepsilon \) is a sampling set for \( B \), for some \( \varepsilon > 0 \). Then \( \forall f \in B \) with \( f(0) = 0 \),

\[
\| f \|_B \leq k \sup \{ (1 - |\phi(z)|^2)|f'(\phi(z))|, z \in \Omega_\varepsilon \} \\
\leq k \sup \{ |\tau_\phi(z)|^{-1}(1 - |z|^2)|f'(\phi(z))||\phi'(z)|, z \in \Omega_\varepsilon \} \\
\leq k \varepsilon^{-1} \sup \{ (1 - |z|^2)|(f \circ \phi)'(z)|, z \in D \} \\
\leq k \varepsilon^{-1} \| f \circ \phi \|_B.
\]

Conversely suppose that \( C_\phi \) is bounded below on \( B \). Then \( \exists k > 0 \) such that whenever \( f(0) = 0 \) and \( \| f \|_B = 1 \), \( \sup \{ (1 - |z|^2)|(f \circ \phi)'(z)| \} \geq k \).

Suppose \( \| f \|_B = 1 \), \( f(0) = 0 \) and choose \( z_f \) such that \( (1 - |z_f|^2)|(f \circ \phi)'(z_f)| \geq k/2 \), i.e. \( |\tau_\phi(z_f)||1 - |\phi(z_f)|^2||f'(\phi(z_f))| \geq k/2 \). But each of the two factors is no larger than 1. Hence each is at least as large as \( k/2 \). Thus if \( \varepsilon = k/2 \), then \( G_\varepsilon \) is a sampling set for \( B \). \( \square \)

As a subset \( H \) of \( D \) is said to satisfy the reverse Carleson condition if \( \exists s > 0 \) and \( c > 0 \) such that \( |D(a,s) \cap H| \geq c|D(a,s)| \) for all \( a \in D \), or equivalently if \( \int_H |f(z)|^2dA(z) \geq c \int_D |f(z)|^2dA(z) \forall f \), which are analytic and square integrable on \( D \).

This definition and the techniques we use are found in \( [5] \).
We show that if $G_z$ satisfies the reverse Carleson condition, then $G_z$ is a sampling set for the Bloch space. In order to do that we need to note an equivalent form for the Bloch-norm. We include a short proof for completeness.

Observation 1. For an analytic function $f$ on $D$ with $f(0) = 0$, 
\[ \|f\|_B^2 \approx \sup \{ \int_D |f'(z)|^2(1 - |\phi_a(z)|^2)^2dA(z), a \in D \}. \]

Proof. If $a \in D$, then,
\[ \int_D |f'(z)|^2(1 - |\phi_a(z)|^2)^2dA(z) = \int_D |f'(z)|^2(1 - |z|^2)^2|\phi'_a(z)|^2dA(z) \leq (\|f\|_B^2) \int_D |\phi'_a(z)|^2dA(z) \leq \|f\|_B^2. \]

In order to prove the reverse inequality, first choose $s > 0$ such that $\rho(z, w) < s$ implies $|(1 - |z|^2)f'(z) - (1 - |w|^2)f'(w)| \leq \frac{1}{4}\forall f \in B$ with $\|f\|_B \leq 1$. See [2] Proposition 2. Now given $f \in B$ with $\|f\|_B = 1$ and $f(0) = 0$, we have $\sup \{(1 - |z|^2)|f'(z)|, z \in D \} \geq 1$. Choose $z_f$ such that $\forall z \in D(z_f, s), (1 - |z|^2)|f'(z)| \geq \frac{1}{4}$. Hence
\[ \int_{D(z_f, s)} |f'(z)|^2(1 - |\phi_{z_f}(z)|^2)^2dA(z) \geq \int_{D(z_f, s)} |f'(z)|^2(1 - |\phi_{z_f}(z)|^2)^2dA(z) \geq \frac{1}{16} \int_{D(z_f, s)} \left(\frac{1 - |\phi_{z_f}(z)|^2}{1 - |z|^2}\right)^2 dA(z). \]

Note that $z \in D(z_f, s) \Rightarrow (1 - |z|^2) \approx c_s(1 - |z_f|^2)$ and $|\phi_{z_f}(z)| = |\rho(z_f, z)| < s$. Thus $\int_{D(z_f, s)} |f'(z)|^2(1 - |\phi_{z_f}(z)|^2)^2dA(z) \geq c_s$. \(\square\)

**Proposition 1.** If $H \subseteq D$ and $H$ satisfies the reverse Carleson condition, then $H$ is a sampling set for the Bloch space.

Proof. Suppose $\|f_n\|_B \leq 1$, $f_n(0) = 0$ and $\delta_n = \sup \{(1 - |z|^2)|f'_n(z), z \in \mathbb{H}\} \to 0$ as $n \to \infty$.

If $a \in D$, then
\[ \int_D |f'_n(z)|^2(1 - |\phi_a(z)|^2)^2dA(z) = \int_D (1 - |z|^2)^2|f'_n(z)|^2|\phi'_a(z)|^2dA(z) \leq c \int_H (1 - |z|^2)^2|f'_n(z)|^2|\phi'_a(z)|^2dA(z) \quad [6] \text{ p. 10} \]
\[ \leq c\delta_n \int_H |\phi'_a(z)|^2dA(z) \leq c\delta_n \int_D |\phi'_a(z)|^2dA(z) \leq c\delta_n. \]

Hence, $\|f_n\|_B^2 \leq c\delta_n$ (by the observation above), and $\|f_n\|_B \to 0$. \(\square\)

Our next proposition is an expanded form of the main result in [2]. We include a short proof of a part of it for completeness.

**Proposition 2.** Suppose $\phi$ is an analytic self-map of $D$ and assume that $\|\phi_w \circ \phi\|_{B/C} \geq k \forall w \in D$. Then the following conditions hold.
Let $\forall\, \varepsilon < k, \rho(z, G_\varepsilon) \leq \sqrt{1 - \varepsilon} = r \, \forall z \in D$.

Moreover, $\exists$ constants $s$ and $r', 0 < s < 1$ and $r' \in [r, 1)$ such that given $w \in D \exists z_w \in D$ such that $\phi(D(z_w, s)) \subseteq D(w, r') \cap G_\varepsilon$.

**Proof.** (1) Suppose that $\varepsilon < k$ and $w \in D$. Choose $z_w \in D$ such that

$$\left| 1 - |z_w|^2 \right| |\phi'(z_w)||\phi'(z_w)| \geq \varepsilon.$$

But $\left(1 - |z_w|^2\right)|\phi'(z_w)||\phi'(z_w)| = |\tau_\phi(z_w)|(1 - \rho^2(w, \phi(z_w)))$. Each factor on the right-hand side is no larger than 1; hence each is at least $\varepsilon$. Thus $z_w \in \Omega_\varepsilon$ and $\rho(w, \phi(z_w)) \leq r < 1$ where $r = \sqrt{1 - \varepsilon}$.

(2) In [3] Theorem 6 it is shown that $\phi$ is Lipschitz with respect to the pseudo-hyperbolic metric on the domain and the Euclidean one on the range. We denote the Lipschitz constant by $\alpha$. Fix $\varepsilon < \varepsilon'$, choose $s < \frac{\varepsilon}{\varepsilon'}$ and let $s < \varepsilon/(2\alpha)$. If $\lambda \in D(z_w, s)$, then $|\tau_\phi(\lambda)| \geq \varepsilon'$ and (by the Schwarz-Pick lemma) $\rho(\phi(z_w), \phi(\lambda)) \leq \rho(z_w, \lambda) < s$. Thus for $\lambda \in D(z_w, s)$ we have

$$\rho(w, \phi(\lambda)) \leq \frac{\rho(\phi(z_w), w) + \rho(\phi(z_w), \phi(\lambda))}{1 + \rho(w, \phi(z_w))}\rho(\phi(z_w), \phi(\lambda)) < r + s \frac{1 + rs}{1 + rs}$$

since $\frac{1 + rs}{1 + rs}$ is an increasing function of $s$ and $r$ if they both lie in $(0, 1)$. Let $r' = \frac{1 + rs}{1 + rs}$.

We have shown that $\phi(D(z_w, s)) \subseteq D(w, r')$. By the choice of $s, |\tau_\phi(\lambda)| \geq \varepsilon' \forall \lambda \in D(z_w, s)$, i.e. $D(z_w, s) \subseteq \Omega_{\varepsilon'}$ and hence $\phi(D(z_w, s)) \subseteq \Omega_{\varepsilon'}$.

We conclude that $\phi(D(z_w, s)) \subseteq \Omega_{\varepsilon'} \cap D(w, r')$.

In [2] it is shown that condition (1) of the previous proposition implies that $C_\phi$ is bounded below on the set of Möbius transforms, and that in case $C_\phi$ is close to being an isometry on the set of Möbius transforms ($k > \frac{15}{16}$), then $C_\phi$ is bounded below on the Bloch space.

We now show that in case $\phi$ is univalent, then no lower bound on $k$ (except $k > 0$) is necessary.

### 3. Univalence

**Corollary 1.** If $\phi$ is a univalent self-map of $D$ and $\|\phi_w \circ \phi\|_{B/C} \geq k \forall w \in D$, then $\forall \varepsilon < k, G_\varepsilon$ satisfies the reverse Carleson condition.

**Proof.** Let $\varepsilon < k$. Pick $\varepsilon' \in (\varepsilon, k)$ and apply the conclusion of Proposition 2 to $\varepsilon'$. $\exists s, r' \in (0, 1)$ such that given $w \in D \exists z_w$ such that $\phi(D(z_w, s)) \subseteq D(w, r') \cap G_\varepsilon$.

We use the fact that $\forall \lambda \in D(z_w, s), |\tau_\phi(\lambda)| \geq \varepsilon$ and the univalence of $\phi$ to conclude that $|\phi(D(z_w, s))| = \int_{D(z_w, s)} |\phi'(\lambda)|^2 A(\lambda) \geq \varepsilon^2 \int_{D(z_w, s)} \left(\frac{1 - |\phi'(\lambda)|^2}{1 - |\lambda|^2}\right)^2 dA(\lambda)$.

But $1 - |\lambda|^2 \approx c_s(1 - |z_w|^2)$ and $1 - |\phi'(\lambda)|^2 \approx c_s(1 - |\phi(z_w)|^2) \approx c_s c_t(1 - |w|^2)$.

Since $|D(a, r)| \approx c_s(1 - |a|^2)$, we have $|D(w, r') \cap G_\varepsilon| \geq \varepsilon |D(z_w, s)| \geq c |D(w, r')|$

where $c$ is independent of $w$.

We summarize the main result for the univalent case in the following proposition.

**Theorem 2.** Suppose $\phi$ is a univalent self-map of $D$. Then the following are equivalent.

1. $C_\phi$ is bounded below on $B$.
2. $\|\phi_w \circ \phi\|_{B/C} \geq k \forall w \in D$.
3. $\forall \varepsilon < k, \rho(G_\varepsilon, z) \leq r < 1 \forall z \in D, r$ depending only on $\varepsilon$.
4. $\forall \varepsilon < k, G_\varepsilon$ satisfies the reverse Carleson condition.
Corollary 2. (1) If \( \phi \) is univalent and \( C_\phi \) is bounded below on BMOA, then it is bounded below on the Bloch space.

(2) If \( \phi \) is univalent and \( C_\phi \) is bounded below on the Bloch space, then it is bounded below on the Dirichlet space.

Proof. (1) If \( \phi \) is univalent, then \( \forall \, w \in D, \phi_w \circ \phi \) is univalent and in this case
\[
\| \phi_w \circ \phi \|_{BMOA} \approx \| \phi_w \circ \phi \|_B
\]
\[\tag{10}\] Hence if \( C_\phi \) is bounded below on BMOA, then \( \| \phi_w \circ \phi \|_B \geq k \forall w \in D \). Hence by Proposition 3, \( C_\phi \) is bounded below on the Bloch space.

(2) By Theorem 2, if \( C_\phi \) is bounded below on the Bloch space, then \( \exists \varepsilon > 0 \) such that \( G_\varepsilon \) satisfies the reverse Carleson condition; hence so does \( G = \phi(D) \). By \[4\] \( C_\phi \) is bounded below on the Dirichlet space. \( \square \)

Remark. W. Smith has given an example of a univalent map \( \phi \) with \( \overline{\phi(D)} = D \) such that \( \tau_\phi(z) \to 0 \) as \( |z| \to 1 \) \[9, 6.5\]. Hence \( C_\phi \) is compact on \( B_0 \) \[6\]. But \( C_\phi \) has closed range on the Dirichlet space \[3\].

We now show that in case \( \phi \) is univalent and \( C_\phi \) is bounded below, then \( G \) has no generalized cusps \[7\] p. 256] and in fact a somewhat stronger condition holds. We also give an example to show that it is not sufficient.

Observation 2. It is a simple consequence of Koebe’s one-quarter theorem \[7\] p. 9] that if \( \phi \) is univalent, then \( \tau_\phi \approx \frac{\text{dist}(\phi(z), \partial G)}{\text{dist}(\phi(z), \partial D)} \).

Proposition 3. Suppose \( \phi \) is univalent and there exists \( \varepsilon > 0 \) such that \( G_\varepsilon \) satisfies the reverse Carleson condition. Then there exists \( \delta > 0 \) such that \( \forall \omega \in \partial D, \)
\[
\lim_{\phi(z) \to \omega} \frac{\text{dist}(\phi(z), \partial G)}{|\phi(z) - \omega|} \geq \delta.
\]

Proof. If \( \eta < 1 \) let \( \Delta(a, \eta) = \{ z \in D, |z - a| \leq \eta(1 - |a|) \} \) for \( a \in D \). By \[6\] p. 4], \( \exists \eta < 1 \) such that the following holds:
\[
G_\varepsilon \cap \Delta(a, \eta) \neq \emptyset \forall a \in D.
\]
Suppose \( \omega \in \partial D \) and choose \( \{a_n\} \) along the radius ending in \( \omega \) such that \( a_n \to \omega \).

Choose \( z_n \in \Omega_\varepsilon \) such that \( \phi(z_n) \in \Delta(a_n, \eta) \). Then \( |\phi(z_n)| \leq |a_n| + \eta(1 - |a_n|) \leq \eta + |a_n|(1 - \eta) \) and hence \( 1 - |\phi(z_n)| \geq (1 - \eta)(1 - |a_n|) \). Note that \( |\omega - a_n| = 1 - |a_n| \).

On the other hand, \( |\phi(z_n) - \omega| \leq |\phi(z_n) - a_n| + |a_n - \omega| \leq \eta(1 - |a_n|) + 1 - |a_n| = (1 + \eta)(1 - |a_n|) \).

Now by observation 2,
\[
\frac{\text{dist}(\phi(z_n), \partial G)}{|\phi(z_n) - \omega|} \geq \frac{1}{4} \frac{\tau_\phi(z_n)(1 - |\phi(z_n)|)}{|\phi(z_n) - \omega|} \geq \frac{\varepsilon}{4} \frac{1 - \eta}{1 + \eta} \square.
\]


Next we give an example to show that the above condition is not sufficient for $C_\phi$ to be bounded below.

**Example 1.** Let

$$G = D \setminus \left( \bigcup_i D_i \cup l_i \right)$$

where $D_i = D(a_i, r_i)$ is the pseudohyperbolic disk with $|a_i|$ close enough to 1, $r_i \to 1$ and $l_i$ is a line segment connecting $D_i$ to $\partial D$.

Let $\phi$ be the Riemann map onto $G$. If $\phi(z)$ approaches a point $\omega$ on $\partial D$ that is not an endpoint of the line segment $l_i$ or if $\omega \neq 1$, then $\tau_\phi(z) \approx 1$ by observation 2. It is clear that the conclusion of the previous lemma holds for all $\omega \in \partial D$ that are not endpoints of line segments $l_i$ or 1. For $\omega = 1$, choose $\phi(z_n) \in G_\epsilon$, approaching 1 non-tangentially, from below the $x$-axis.

For $\omega_i = \text{endpoint of the line segment } l_i$, $G_\epsilon$ has an extra non-tangential region taken away from $G$ but with the same angle opening for every $i$. Thus we may pick $\phi(z_n)$ approaching $\omega_i$ through an angle with a slightly larger opening. So the conclusion of the previous proposition is satisfied, but no pseudohyperbolic neighbourhood of $G_\epsilon$ covers $D$. Hence $C_\phi$ is not bounded below on $B$.

The next example deals with a non-automorphic univalent $\phi$ that induces a closed-range composition operator on the Bloch space.

**Example 2.** Let $G = D \setminus [0, 1)$, and let $\phi$ be the Riemann map onto $G$. Since

$$\tau_\phi(z) \approx \frac{\text{dist}(\phi(z), \partial G)}{1 - |\phi(z)|}$$

and the ratio on the right approaches 1 as $\phi(z)$ approaches any point $\omega \neq 1$ on the unit circle, $G_\epsilon$ includes all of $D$, except a pseudohyperbolic neighbourhood of $[0, 1)$. Hence with a suitable value of $r$, every point of $D$ is within pseudohyperbolic distance $r$ of $G_\epsilon$ and hence $C_\phi$ is bounded below.

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