FOURIER TRANSFORMS HAVING ONLY REAL ZEROS

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(Communicated by Joseph A. Ball)

Abstract. Let \( G(z) \) be a real entire function of order less than 2 with only real zeros. Then we classify certain distribution functions \( F \) such that the Fourier transform \( H(z) = \int_{-\infty}^{\infty} G(it)e^{izt}dF(t) \) has only real zeros.

1. Introduction

Pólya [13] suggested that determining the class of functions whose Fourier transforms have only real zeros would be a ‘rather artificial question’ if it were not for the Riemann Hypothesis. For \( \Re(s) > 1 \), the Riemann zeta function is defined by \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \). It has an analytic continuation, and the function

\[
\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)
\]

is entire. The Riemann Hypothesis states that all the zeros of \( \xi(s) \) satisfy \( \Re(s) = 1/2 \). A proof of the Riemann Hypothesis would be a major advance for analytic number theory. Let \( \Xi(z) = \xi\left(\frac{1}{2} + iz\right) \). It is well known (see Titchmarsh [18], chapter 10) that

\[
\Xi(z) = \int_{-\infty}^{\infty} \Phi(x)e^{ixz}dx
\]

where

\[
\Phi(x) = \sum_{n=1}^{\infty} \left(4n^4\pi^2e^{9x/2} - 6n^2\pi e^{5x/2}\right) \exp\left(-n^2\pi e^{2x}\right).
\]

In other words, the Riemann Hypothesis is true if and only if the Fourier transform \( \Xi(z) \) has only real zeros.

Pólya wrote several papers (such as [11, 12, 13, 14], which can all be found in [15]) in which he studied the reality of the zeros of various Fourier transforms. A particularly interesting result is the following:

**Proposition 1** (Pólya [13]). Let \( f \) be an integrable function of a real variable \( t \) such that \( f(t) = f(-t) \) and \( f(t) = O(e^{-bt}) \) for \( t \to \pm\infty \) and \( b > 2 \). Assume that

\[
\int_{-\infty}^{\infty} f(t)e^{izt}dt
\]
has only real zeros. Let \( \phi \) be a real entire function having only real zeros, and assume that \( \phi \) has a Weierstrass product of the form

\[
\phi(z) = cz^m e^{\alpha z - \beta z^2} \prod_k \left(1 - \frac{z}{\alpha_k}\right)^e z^{\alpha_k/n_k}
\]

where \( c, \alpha, \alpha_k \) are real, \( \beta \geq 0 \), and \( m \) is a nonnegative integer. (In other words, \( \phi \) belongs to the Laguerre-Pólya class.) Then

\[
\int_{-\infty}^{\infty} \phi(it)f(t)e^{itz} \, dt
\]

has only real zeros.

In this paper we are concerned with constructing Fourier transforms with only real zeros in which the measure is not assumed to be the ordinary Lebesgue measure \( dt \). The main result is the following theorem:

**Theorem 1.** Suppose \( G \) is an entire function of order \( < 2 \) that is real on the real axis and has only real zeros. Let \( \{a_k\} \) be a nonincreasing sequence of positive real numbers, let \( \{X_k\} \) be the sequence of independent random variables such that \( X_k \) takes values \( \pm 1 \) with equal probability, and let \( F_n \) be the distribution function of the normalized sum \( Y_n = (a_1X_1 + \cdots + a_nX_n)/s_n \) where \( s_n^2 = a_1^2 + \cdots + a_n^2 \). The functions \( F_n \) converge pointwise to a continuous distribution \( F = \lim_{n \to \infty} F_n \). Let \( H \) be the Fourier transform of \( G(it) \) with respect to the measure \( dF \). That is,

\[
H(z) = \int_{-\infty}^{\infty} G(it)e^{itz} \, dF(t).
\]

Then \( H \) is an entire function of order \( \leq 2 \) that is real on the real axis. If \( H \) is not identically zero, then \( H \) has only real zeros.

Theorem 1 includes cases not covered in Proposition 1 because the distribution function \( F(t) \), although continuous, need not be differentiable. If \( F(t) \) is differentiable, we may write

\[
\int_{-\infty}^{\infty} G(it)e^{itz} \, dF(t) = \int_{-\infty}^{\infty} G(it)e^{itz} F'(t) \, dt.
\]

However, since not all functions \( f(t) \) in Proposition 1 are of the form \( f(t) = F'(t) \) for the types of distributions in Theorem 1, Proposition 1 covers cases not included in Theorem 1. So, while there is some overlap between Proposition 1 and Theorem 1, neither implies the other.

The proof of Theorem 1 is given in [2]. Before proceeding with the proof we mention that the proof relies on a result about sums of exponential functions. Let \( h_n(z) \) be the function of a complex variable \( z \) defined by

\[
h_n(z) = \sum G(\pm ia_1 \pm \cdots \pm ia_n) e^{iz(\pm b_1 \pm \cdots \pm b_n)}
\]

where the summation is over all \( 2^n \) possible sign combinations, the same sign combination being used in both the argument of \( G \) and in the exponent. The numbers \( a_1, a_2, a_3, \ldots \) and \( b_1, b_2, b_3, \ldots \) are positive, and \( G \) is as in Theorem 1. The author shows in [3] that all the zeros of the exponential sum \( h_n(z) \) are real. Interestingly, the proof of this fact is similar to the proof of the Lee-Yang Circle Theorem from statistical mechanics (cf. [8] Appendix II or [17] Chapter 5). It should be pointed
out that this result of the author is related to de Branges’ Hilbert spaces of entire functions [5]. Let \( a_k = b_k, s_n^2 = a_1^2 + \cdots + a_n^2 \), and
\[
H_n(z) = 2^{-n} \sum G((\pm ia_1 \pm \cdots \pm ia_n)/s_n) e^{iz(\pm a_1 \pm \cdots \pm a_n)/s_n}.
\]
All of the zeros of \( H_n(z) \) are real. Theorem 1 is established by showing that the limit
\[
H(z) = \lim_{n \to \infty} H_n(z)
\]
is uniform on compact sets.

2. Proof of Theorem 1

The proof of Theorem 1 is presented in this section as a sequence of lemmas. We begin with some notation.

The Laguerre-Pólya class \( \mathcal{LP} \) of functions consists of the entire functions having only real zeros with a Weierstrass factorization of the form
\[
a z^q e^{z - 2b z^2} \prod (1 - z/\alpha_n) e^{z/\alpha_n}
\]
where \( a, \alpha, \beta \) are real, \( \beta \geq 0 \), \( q \) is a nonnegative integer, and the \( \alpha_n \) are nonzero real numbers such that \( \sum \alpha_n^{-2} < \infty \). We shall be most interested in the subset \( \mathcal{LP}^* \) of the Laguerre-Pólya class consisting of all elements of \( \mathcal{LP} \) of order \( < 2 \). Thus, \( \beta \) is necessarily 0 for functions in \( \mathcal{LP}^* \).

The distribution function \( T \) for a random variable \( Y \) is \( T(x) = \Pr(Y \leq x) \). We will consider the following types of random variables and their distribution functions: Let \( \{a_k\} \) be a nonincreasing sequence of positive real numbers. Let \( \{X_k\} \) be a sequence of independent random variables such that \( X_k \) takes values \( \pm 1 \) with equal probability. Let \( Y_n \) be the sum
\[
Y_n = a_1 X_1 + \cdots + a_n X_n
\]
where \( s_n^2 = a_1^2 + \cdots + a_n^2 \). \( F_n \) will denote the distribution function of \( Y_n \), and \( F \) will denote the limit \( F = \lim_{n \to \infty} F_n \). In Theorem 1 the distribution \( F \) has variance 1. However, \( F \) could be rescaled to have any other positive value for its variance. The following lemma describes this \( F \).

**Lemma 1.** The sequence \( F_n \) converges pointwise to a continuous distribution \( F \). If the sequence \( s_n \) is unbounded, \( F \) is the normal distribution. If the sequence \( s_n \) is bounded, \( F \) is not the normal distribution.

**Proof.** This is proved in Lemma 1 of [2].

**Lemma 2** (Pólya [12], Hilfssatz II). Let \( a \) be a positive constant, let \( b \) be real, and let \( G(z) \) be an entire function of genus 0 or 1 that for real \( z \) takes real values, has at least one real zero, and has only real zeros. Then the function
\[
e^{ib}G(z + ia) + e^{-ib}G(z - ia)
\]
has only real zeros.

**Proof.** Pólya’s original statement is Hilfssatz II in [12]. Pólya’s argument is reiterated as Proposition 2 in [3].
We need the following important fact about \( H_n(z) \).

**Lemma 3.** Suppose \( G \in \mathcal{LP}^* \). Let \( a_1, \ldots, a_n, b_1, \ldots, b_n \) be positive real numbers. The exponential sum

\[
     h_n(z) = \sum G(\pm ia_1 \pm \cdots \pm ia_n)e^{iz(\pm b_1 \pm \cdots \pm b_n)}
\]

obtained by summing over all sign combinations, the same combination being used in the argument of \( G \) as in the exponent, is in \( \mathcal{LP}^* \).

**Proof.** It is clear that \( h_n(z) \) is real for real \( z \) and has order 1. The fact that \( h_n(z) \) has real zeros is proved in [3] by a method similar to that of the Lee-Yang Circle Theorem (found in Appendix II in [8]). \( \square \)

If \( s_n^2 = a_1^2 + \cdots + a_n^2 \) and if we use the notation of Riemann-Stieltjes integration, an immediate corollary to Lemma 3 is the following:

**Corollary 4.** The function

\[
     H_n(z) = 2^{-n} \sum G((\pm ia_1 \pm \cdots \pm ia_n)/s_n)e^{iz((\pm a_1 \pm \cdots \pm a_n)/s_n} = \int_{-\infty}^{\infty} G(it)e^{it}dF_n(t)
\]

is in \( \mathcal{LP}^* \).

In Lemmas 3 and 4 we will show that the integrals \( \int_{-\infty}^{\infty} G(it)e^{it}dF_n(t) \) converge uniformly to \( \int_{-\infty}^{\infty} G(it)e^{it}dF_n(t) \) for \( z \) in compact sets. The proof of Lemma 3 will require the following 1994 result of Pinelis, which is an improvement of a conjecture by Eaton [9]:

**Lemma 5** (Pinelis [10], Corollary 2.6). Let \( X_k \) be independent random variables taking values \( \pm 1 \) with equal probability. Let \( s_n^2 = a_1^2 + \cdots + a_n^2 \), and let

\[
     Y_n = \frac{a_1X_1 + \cdots + a_nX_n}{s_n}.
\]

Then

\[
     \Pr(|Y_n| > u) < 2c(1 - \Phi(u))
\]

where \( c = 2e^3/9, \Phi(u) = \int_u^\infty \phi(t)dt \), and \( \phi(t) = (2\pi)^{-1/2}e^{-t^2/2} \).

**Lemma 6.** Let \( \epsilon > 0 \) be given, suppose \( G \in \mathcal{LP}^* \), and let \( K \) be a compact subset of \( \mathbb{C} \). Then there is a positive number \( A \) (depending on \( \epsilon \) and \( K \)) such that

\[
     \int_{|t| > A} |G(it)e^{itz}|dF_n(t) < \epsilon
\]

for all \( n \) and all \( z \in K \).

**Proof.** Let \( \lambda \) denote the order of \( G \). By hypothesis, \( \lambda < 2 \). Choose \( \delta \) with \( \max(1, \lambda) < \delta < 2 \). Then choose \( A > 0 \) large enough so that \( |G(it)e^{itz}| < e^{\delta|t|} \) for all \( z \in K \) and \( |t| > A \). Such an \( A \) exists as follows: Choose \( \delta' \) with \( \lambda < \delta' < \delta \). Then for sufficiently large \( A \), \( |G(it)| < e^{\delta'|t|} \) whenever \( |t| > A \). Since \( K \) is compact, there is an \( R \) so that \( |z| < R \) for all \( z \in K \). For sufficiently large \( A \) and \( |t| > A \),

\[
     |G(it)e^{itz}| \leq |G(it)|e^{\delta'|t|} < e^{\delta'|t| + R|t|} < e^{\delta|t|}.
\]
Thus, $A$ exists as claimed. Then
\[ \int_{A<|t|<B} |G(it)e^{izt}|dF_n(t) < 2 \int_{A} e^{t^2}dF_n(t). \]

After integration by parts the right-hand side becomes
\[ 2e^{B^2}(F_n(B) - 1) - 2e^{A^2}(F_n(A) - 1) - 2 \int_{A} (F_n(t) - 1)d(e^{t^2}) \]
\[ < 2e^{A^2}(1 - F_n(A)) + 2 \int_{A} (1 - F_n(t))d\sqrt{t^2 - 1}e^{t^2}dt. \]

According to Lemma 5 for $t \geq A$ and if $A$ is sufficiently large, then
\[ 1 - F_n(t) \leq \beta \int_{t}^{\infty} e^{-t^2/2}du < \frac{e^{-t^2/2}}{2} \]
where $\beta = 4e^{3/(9\sqrt{2\pi})}$. This gives
\[ \int_{A<|t|<B} |G(it)e^{izt}|dF_n(t) < e^{A^2-A^2/2} + \int_{A} \delta t^{\delta-1}e^{t^2-t^2/2}dt \]
and
\[ \int_{A<|t|} |G(it)e^{izt}|dF_n(t) < e^{A^2-A^2/2} + \int_{A}^{\infty} \delta t^{\delta-1}e^{t^2-t^2/2}dt. \]

For sufficiently large $A$ the last integral is bounded above by $e^{A^2-A^2/2}$. So,
\[ \int_{A<|t|<B} |G(it)e^{izt}|dF_n(t) < 2e^{A^2-A^2/2}. \]

The right-hand side of the last inequality can be made arbitrarily small for sufficiently large $A$. Therefore, we obtain $\int_{|t|>A} |G(it)e^{izt}|dF_n(t) < \epsilon$ as desired. □

**Lemma 7.** Let $K$ be a compact subset of $\mathbb{C}$. Then
\[ \int_{-A}^{A} G(it)e^{izt} dF_n(t) \to \int_{-A}^{A} G(it)e^{izt} dF(t) \]
uniformly as $n \to \infty$ for $z \in K$.

**Proof.** By the Helly-Bray Theorem (see [3] p. 182 or [7] p. 298) it is immediate that convergence occurs pointwise. We must, however, verify uniform convergence for $z \in K$.

Let $\epsilon > 0$ be given, and write $g_z(t) = G(it)e^{izt}$. Choose $\kappa$ such that $\kappa > |g_z(t)|$ and $\kappa > |g'_z(t)|$ for all $z \in K$ and $t \in [-A,A]$. Integration by parts yields
\[ \int_{-A}^{A} G(it)e^{itz} dF(t) = g_z(A)F(A) - g_z(-A)F(-A) - \int_{-A}^{A} F(t)g'_z(t) dt. \]
Then
\[
\left| \int_{-A}^{A} G(it)e^{itz} dF(t) - \int_{-A}^{A} G(it)e^{itz} dF_n(t) \right|
\leq |g_z(A)||F(A) - F_n(A)| + |g_z(-A)||F(-A) - F_n(-A)|
\]
\[+ \int_{-A}^{A} |F(t) - F_n(t)||g_z'(t)| dt\]
\[\leq \kappa|F(A) - F_n(A)| + \kappa|F(-A) - F_n(-A)| + \kappa2A \max_{t \in [-A,A]} |F(t) - F_n(t)|.\]

Since the functions \(F_n\) and \(F\) are distributions functions such that \(F_n\) converges to the continuous distribution \(F\) pointwise on \([-A,A]\), \(F_n\) converges to \(F\) uniformly on \([-A,A]\). Thus for sufficiently large \(n\) and all \(t \in [-A,A]\),
\[|F_n(t) - F(t)| < \min \left( \frac{\epsilon}{6\kappa}, \frac{\epsilon}{3\kappa} \right).\]

Therefore, for all sufficiently large \(n\) and all \(z \in K\),
\[\left| \int_{-A}^{A} G(it)e^{itz} dF(t) - \int_{-A}^{A} G(it)e^{itz} dF_n(t) \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\]

This shows that the convergence \(\int_{-A}^{A} G(it)e^{itz} dF_n(t) \to \int_{-A}^{A} G(it)e^{itz} dF(t)\) is uniform as claimed.

\[\square\]

Lemma 8. Suppose \(G \in \mathcal{LP}^*\). Then \(H(z) = \int_{-\infty}^{\infty} G(it)e^{itz} dF(t)\) is an entire function that is real for real \(z\), and if it does not vanish identically, then it has only real zeros.

Proof. Let \(H_n(z) = \int_{-\infty}^{\infty} G(it)e^{itz} dF_n(t)\). By Lemmas 6 and 7, \(H_n(z)\) converges uniformly to \(H(z)\) on compact sets. Since \(H_n(z)\) is real for real \(z\), its limit \(H(z)\) is real for real \(z\). By Hurwitz’s Theorem (see [11 Thm. 2, p. 178]), if \(H(z)\) is not identically zero, its zeros are limit points of the zeros of the \(H_n(z)\). Since, for each \(n\), \(H_n(z)\) has only real zeros, \(H(z)\) also has only real zeros.

\[\square\]

Lemma 9. Suppose \(G \in \mathcal{LP}^*\). The order of \(H(z) = \int_{-\infty}^{\infty} G(it)e^{itz} dF(t)\) is \(\leq 2\).

Proof. Choose \(\delta\) with \(\lambda < \delta < 2\) where \(\lambda\) is the order of \(G\). Let \(M\) be large enough so that \(|G(z)| < Me^{z|t|}\) for all \(z\). By applying Hölder’s Inequality (see [16 p. 63]) we obtain
\[
\left| \int_{-\infty}^{\infty} G(it)e^{-izt} dF(t) \right| \leq \int_{-\infty}^{\infty} Me^{t|z|}\cdot|t|\cdot dF(t)
\leq M \left( \int_{-\infty}^{\infty} e^{2|z|\cdot|t|} dF(t) \right)^{1/2} \left( \int_{-\infty}^{\infty} e^{2|z|\cdot|t|} dF(t) \right)^{1/2}.
\]

By Lemma 3, both integrals in the product converge. The first integral is independent of \(z\). We will determine a bound for the second integral. Integration by parts
and an application of Lemma 3 give
\[\int_{-\infty}^{\infty} e^{-|zt|} dF(t) = 2 \int_{0}^{\infty} e^{-|zt|} dF(t)\]
\[= 2e^{-|zt|} (F(t) - 1) \bigg|_{0}^{\infty} + 2 \int_{0}^{\infty} 2|z| e^{-|zt|} (1 - F(t)) dt\]
\[= 1 + 4|z| \int_{0}^{\infty} |z| e^{-|zt|} (1 - F(t)) dt\]
\[\leq 1 + K|z| \int_{0}^{\infty} e^{-|zt|} dt \quad \text{where } K = \frac{16|e^{3}|}{9\sqrt{2\pi}}\]

Since
\[\int_{0}^{\infty} e^{-|zt|} dt \leq e^{-|zt|} \int_{0}^{\infty} e^{-(t-2|z|)^2/2} dt = \sqrt{2\pi} e^{2|z|^2},\]
we see that \(\left| \int_{-\infty}^{\infty} G(it)e^{-izt} dF(t) \right|\) is bounded by a constant times \(|z| e^{2|z|^2}\). Thus, the order of \(\int_{-\infty}^{\infty} G(it)e^{-izt} dF(t)\) is \(\leq 2\).

This completes the proof of Theorem 1. \(\square\)

References

10. Iosif Pinelis, Extremal probabilistic problems and Hotelling’s T^2 test under a symmetry condition, Ann. Statist. 22 (1994), no. 1, 357–368. MR1272088 (95m:62116)
11. George Pólya, On the zeros of an integral function represented by Fourier’s integral, Messenger of Math. 52 (1923), 185–188.


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