Semi-continuity of metric projections in $\ell_\infty$-direct sums

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Abstract. Let $Y$ be a proximinal subspace of finite codimension of $c_0$. We show that $Y$ is proximinal in $\ell_\infty$ and the metric projection from $\ell_\infty$ onto $Y$ is Hausdorff metric continuous. In particular, this implies that the metric projection from $\ell_\infty$ onto $Y$ is both lower Hausdorff semi-continuous and upper Hausdorff semi-continuous.

1. Preliminaries

Let $X$ be a real Banach space. For $x$ in $X$ and $r > 0$, we denote by $B_X(x, r)$ ($B_X[x, r]$), the open (closed) ball in $X$, with $x$ as center and $r$ as radius. The closed unit ball of $X$ will be denoted by $B_X$ and the unit sphere of $X$ by $S_X$. Also, $X^*$ denotes the dual of $X$. The collection of norm attaining functionals in $X^*$ would be denoted by $\text{NA}(X)$. That is, a functional $f$ in $X^*$ is in $\text{NA}(X)$ if and only if there exists $x$ in $S_X$ such that $f(x) = \|f\|$.

For a subspace $Y$ of $X$, let

$$Y^\perp = \{ f \in X^* : f(x) = 0 \quad \forall \ x \in Y \}.$$ 

If $A$ is a closed subset of $X$ and $x$ is in $X$, $d(x, A) = \inf\{\|x - y\| : y \in A\}$. If $\mathbb{C}(Y)$ denotes the class of non-empty, bounded and closed subsets of $Y$, then the Hausdorff metric on $\mathbb{C}(Y)$ is given by

$$h(A, B) = \max_{x \in A} \{ \sup_{y \in B} d(x, B), \sup_{y \in B} d(y, A) \}$$

for $A$ and $B$ in $\mathbb{C}(Y)$.

Let $D \subseteq X$ and $F$ be a map from $D$ into a collection of non-empty subsets of $X$. If $x$ is in $D$, the set-valued map $F$ is lower semi-continuous at $x$ if given $\epsilon > 0$ and $z$ in $F(x)$, there exists $\delta > 0$ such that for all $y$ in $D \cap B(x, \delta)$, there exists $w$ in $F(y) \cap B(z, \epsilon)$. If the choice of $\delta$ is independent of the choice of $z \in F(x)$, or equivalently

$$F(y) \cap B(z, \epsilon) \neq \emptyset, \quad \forall \ z \in F(x) \text{ and } \forall \ y \in D \cap B(x, \delta),$$

then following [3], we say $F$ is lower Hausdorff semi-continuous at $x$. The set-valued map $F$ is upper Hausdorff semi-continuous at $x$ in $D$ if given $\epsilon > 0$, there exists
\( \delta > 0 \) such that \( F(y) \subseteq F(x) + \epsilon B_X \), for all \( y \) in \( D \cap B(x, \delta) \). The map \( F \) is said to be lower Hausdorff (upper Hausdorff) semi-continuous on the domain \( D \) if \( F \) is lower Hausdorff (upper Hausdorff) semi-continuous at each point \( x \) in \( D \).

If \( F(x) \) belongs to \( C(Y) \) for all \( x \) in \( D \subseteq X \) and \( x \) is in \( D \), we say \( F \) is Hausdorff metric continuous at \( x \) in \( D \) if the single-valued map \( F \) from \( D \) into the metric space \( (C(Y), h) \) is continuous. We say \( F \) is Hausdorff metric continuous on \( D \) if \( F \) is Hausdorff metric continuous at all \( x \) in \( D \).

All subspaces are assumed to be closed. Let \( Y \) be a subspace of \( X \). For \( x \in X \), let

\[
P_Y(x) = \{ y \in Y : \| x - y \| = d(x, Y) \}.
\]

The subspace \( Y \) is said to be proximinal in \( X \), if for each \( x \in X \), the set \( P_Y(x) \) is non-empty. It is easily verified that if \( Y \) is a proximinal subspace of \( X \), then the set \( P_Y(x) \) is bounded, closed and convex. The set-valued map \( P_Y : X \to 2^Y \) is called the metric projection from \( X \) onto \( Y \). A usual compactness argument shows that all finite-dimensional subspaces are proximinal.

We also need the notion of strong proximinality as defined in \([7]\).

**Definition 1.1.** A proximinal subspace \( Y \) of a Banach space is called strongly proximinal if for each \( x \) in \( X \) and \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
s(x, \delta) = \sup \{ d(z, P_Y(x)) : z \in Y \text{ and } \| x - z \| < d(x, Y) + \delta \} < \epsilon.
\]

**Remark 1.2.** It is easily verified that if \( Y \) is a strongly proximinal subspace of a Banach space \( X \), then the metric projection \( P_Y \) is upper Hausdorff semi-continuous on \( X \). However, a proximinal subspace \( Y \), with \( P_Y \) upper Hausdorff semi-continuous, need not be strongly proximinal. For example, there exist proximinal hyperplanes that are not strongly proximinal (see Remark 1.2 of \([7]\)). But the metric projection onto any proximinal hyperplane is upper Hausdorff semi-continuous.

A subspace \( Y \) of a Banach space \( X \) is called an L-summand of \( X \) if there is a subspace \( Z \) of \( X \) such that

\[
X = Y \oplus Z
\]

and for any \( x \) in \( X \) with \( x = y + z \), where \( y \) is in \( Y \) and \( z \) is in \( Z \), we have

\[
\| x \| = \| y \| + \| z \|.
\]

A subspace \( E \) of a Banach space \( X \) is said to be an M-ideal of \( X \) if \( E^\perp \) is an L-summand of the dual space \( X^* \). A Banach space that is an M-ideal in its second dual is called an M-embedded space.

A finite-dimensional normed linear space \( X \) is called polyhedral if \( B_X \) has only a finite number of extreme points. A Banach space \( X \) is called polyhedral if every finite-dimensional subspace of \( X \) is polyhedral. A well-known example of an infinite-dimensional polyhedral space is the sequence space \( c_0 \).

2. **List of Known Results Needed**

We require a few known results about approximative properties of M-ideals and finite-dimensional polyhedral spaces. We quote them below with the appropriate references. All the results on M-ideals, which we list below, can be found in \([9]\). The following proposition couples Proposition 1.1 and Proposition 1.8 of Chapter II in \([9]\).
Proposition 2.1. Let $Y$ be an $M$-ideal of a Banach space $X$. Then $Y$ is proximinal in $X$ and the metric projection $P_Y$ from $X$ onto $Y$ is Hausdorff metric continuous on $X$.

Proposition 2.2 (See Example 1.4 of Chapter III of [2]). The sequence space $c_0$ is an $M$-ideal in its second dual space $\ell_\infty$ or equivalently, $c_0$ is an $M$-embedded space.

Remark 2.3. M-ideals are strongly proximinal. In fact, they have a stronger proximinality property. M-ideals are known to have the 3-ball property (Theorem I.2.2, [2]). It was shown in [8] and [10] that if a subspace $Y$ has the 3-ball property in $X$, then $Y$ is L-proximinal. That is, for each $x$ in $X$, we have

$$\|x\| = d(x, Y) + d(0, P_Y(x)).$$

Thus if $Y$ is an M-ideal in $X$, then $Y$ is L-proximinal. It is easily verified that L-proximinality implies strong proximinality.

We now move on to a few facts about finite-dimensional spaces. We first observe that the metric projection, onto even one-dimensional subspaces, need not be lower semi-continuous [2]. However, the following result of A. L. Brown, from [1], has an affirmative assertion in the polyhedral case.

Proposition 2.4. Let $X$ be a finite-dimensional polyhedral space and $Y$ be a subspace of $X$. Then the metric projection $P_Y$ from $X$ onto $Y$ is lower semi-continuous on $X$.

We also need some standard facts about finite-dimensional subspaces, which can be derived using the usual compactness arguments. We prove one below.

Fact 2.5. Let $Y$ be a finite-dimensional subspace of a Banach space $X$, and assume that the metric projection $P_Y$ is lower semi-continuous at some $x$ in $X$. Then $P_Y$ is lower Hausdorff semi-continuous at $x$.

Proof. The set $P_Y(x)$ is compact since it is closed and bounded. Let $\epsilon > 0$ be given. Using the lower semi-continuity of $P_Y$ at $x$, select for each $z$ in $P_Y(x)$, a positive number $\delta_z$ such that for every $y$ in $B_X(x, \delta_z)$, the set $P_Y(y)$ intersects the open ball $B_X(z, \epsilon/2)$. Select a finite subcover, say, $\{B_X(z_i, \epsilon/2) \cap P_Y(x) : 1 \leq i \leq k\}$, of the open cover $\{B_X(z, \epsilon/2) \cap P_Y(x) : z \in P_Y(x)\}$ of $P_Y(x)$. Set $\delta = \min\{\delta_z : 1 \leq i \leq k\}$. Choose any $z$ in $P_Y(x)$ and $i$ such that $z$ is in $B_X(z_i, \epsilon/2)$. Now for any $y$ in $B_X(x, \delta)$, we have $P_Y(y) \cap B_X(z, \epsilon/2)$ is non-empty and so $P_Y(y) \cap B_X(z, \epsilon)$ is non-empty.

An easy compactness argument again proves the following statement.

Fact 2.6. Any finite-dimensional subspace of a Banach space is strongly proximinal.

The fact below now follows from Remark 1.2.

Fact 2.7. If $Y$ is a finite-dimensional subspace of a Banach space $X$, then the metric projection $P_Y$ is upper Hausdorff semi-continuous on $X$.

Finally, we make an easy observation connecting the three semi-continuity concepts we mentioned earlier.

Remark 2.8. Let $X$ and $Y$ be Banach spaces, and let $F$ be a set-valued map from $X$ into $Y$ with $F(x)$ in $C(Y)$ for all $x$ in $X$. Then $F$ is Hausdorff metric continuous at $x$ in $X$ if and only if $F$ is both lower Hausdorff semi-continuous and upper Hausdorff semi-continuous at $x$. 

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This remark follows from the fact that if $E$ and $G$ are in $C(Y)$, then

$$h(E, G) < \epsilon \iff G \subseteq E + \epsilon B_Y \text{ and } G \cap B_Y(z, \epsilon) \neq \emptyset \forall z \in E.$$ 

The fact below now follows from the above observations and results of this section.

**Fact 2.9.** Let $X$ be a finite-dimensional polyhedral space and $Y$ be a subspace of $X$ or $X$ be a Banach space and $Y$ be an $M$-ideal in $X$. In either case, $Y$ is strongly proximinal in $X$ and the metric projection $P_Y$ from $X$ onto $Y$ is Hausdorff metric continuous.

3. **Semi-continuity in direct sum spaces**

In this section, we consider the $\ell_\infty$- direct sum, $X = X_1 \oplus X_2$, of two Banach spaces $X_1$ and $X_2$. If $Y_1$ and $Y_2$ are subspaces of $X_1$ and $X_2$ respectively, we set $Y = Y_1 \oplus Y_2$. For any $x$ in $X$, we denote by $x_i$ the unique elements of $X_i$, for $i \in \{1, 2\}$, satisfying $x = x_1 + x_2$. Clearly,

$$\|x\| = \max\{|\|x_1\|, \|x_2\|\}.$$ 

We set

$$d_i(x) = d(x, Y_i), \text{ for } i \in \{1, 2\}.$$ 

We note that

$$d(x, Y) = \max\{d_1(x), d_2(x)\}$$

and if $z$ is in $X$, then

$$|d_i(x) - d_i(z)| \leq \|x_i - z_i\| \text{ for } i \in \{1, 2\}.$$ 

The following remark, with $X$ and $Y$ as above, is easy to verify.

**Remark 3.1.** Let $Y_1$ and $Y_2$ be proximinal subspaces of $X_1$ and $X_2$ respectively. Then $Y$ is proximinal in $X$ and

$$P_Y(x) = \begin{cases} 
P_{Y_1}(x_1) + P_{Y_2}(x_2) & \text{if } d_1(x) = d_2(x), \\
B_{X_1}[x_1, d_2(x)] \cap Y_1 + P_{Y_2}(x_2) & \text{if } d_1(x) < d_2(x), \\
P_{Y_1}(x_1) + B_{X_2}[x_2, d_1(x)] \cap Y_2 & \text{if } d_1(x) > d_2(x). 
\end{cases}$$

Note that in all the above three cases, we have

$$P_Y(x) \supseteq P_{Y_1}(x_1) + P_{Y_2}(x_2).$$

We need the following fact in the sequel.

**Fact 3.2.** Let $E$ be a Banach space, $F$ be a proximinal subspace of $E$ and $x$ be in $E \setminus F$. Let $\alpha > d(x, F) = d_x$. Then given $\epsilon > 0$, there exists $\delta > 0$ such that for any $y$ in $B_E(x, \delta)$ and $\beta$ satisfying $|\beta - \alpha| < \delta$, we have

$$h(B_E[x, \alpha] \cap F, B_E[y, \beta] \cap F) \leq \epsilon.$$ 

**Proof.** Let $2\gamma = \alpha - d_x$, $K = \alpha + d_x + 2$ and $\delta = \min\{1, \gamma/2, \gamma \epsilon/(2K)\}$. Let $y$ be in $B_E(x, \delta)$. If $d_y = d(y, F)$ and $\beta$ is a scalar such that $|\alpha - \beta| < \delta$, then it is easily verified, using (1), that

$$|d_x - d_y| < \delta \text{ and } \beta - d_y > \gamma.$$
Select any $t$ in $B_E[x, \alpha] \cap F$. We will construct an element $v$ in $B_E[y, \beta] \cap F$ satisfying $\|t - v\| < \epsilon$. We have
\[
\|y - t\| \leq \|y - x\| + \|x - t\| \leq \delta + \alpha \leq \beta + 2\delta.
\]
Now select any $w$ in $P_F(y)$, and let
\[
v = \lambda t + (1 - \lambda)w, \text{ where } \lambda = \frac{\beta - d_y}{\beta - d_y + 2\delta}.
\]
Then $v$ is in $F$ and
\[
\|y - v\| \leq \lambda \|t - y\| + (1 - \lambda)d_y \\
\leq \lambda (\beta + 2\delta) + (1 - \lambda)d_y \\
= \lambda (\beta - d_y + 2\delta) + d_y = \beta.
\]
Now
\[
\|t - v\| = (1 - \lambda)\|t - w\| = \frac{2\delta}{\beta - d_y + 2\delta}\|t - w\| \\
< \frac{2\delta}{\gamma} (\|t - x\| + \|x - y\| + \|y - w\|) \\
\leq \frac{2\delta}{\gamma} (\alpha + \delta + d_y) \\
\leq \frac{2\delta}{\gamma} (\alpha + d_x + 2\delta) \\
\leq \frac{2\delta}{\gamma} K < \epsilon.
\]
Similarly, for any $s$ in $B_E[y, \beta] \cap F$, we can get $v'$ in $B_E[x, \alpha] \cap F$ satisfying $\|s - v'\| < \epsilon$, and this completes the proof of the fact.

Now we can prove the main result of this section.

**Theorem 3.3.** Let $Y_i$ be a proximinal subspace of the normed linear space $X_i$ for $i \in \{1, 2\}$, and let $Y = Y_1 \oplus_\infty Y_2$. If $P_{Y_i}$ is lower Hausdorff semi-continuous on $X_i$ for $i \in \{1, 2\}$, then $P_Y$ is lower Hausdorff semi-continuous on $X = X_1 \oplus_\infty X_2$.

**Proof.** By Remark 3.1, $Y$ is proximinal in $X$. Fix $x$ in $X$ and let $\epsilon > 0$ be given. Using the lower Hausdorff semi-continuity of the maps $P_{Y_i}$ at $x_i$, we can get $\delta > 0$ such that
\[
(2) \quad z \in X, \ \|x - z\| < \delta \Rightarrow B_{X_i}(p_i, \epsilon) \cap P_{Y_i}(z_i) \neq \emptyset, \quad \text{for } i \in \{1, 2\}.
\]

**Case 1.** $d_1(x) = d_2(x)$.

In this case, we have $P_Y(x) = P_{Y_1}(x_1) \oplus_\infty P_{Y_2}(x_2)$. Select any $p_i \in P_{Y_i}(x_i)$ for $i \in \{1, 2\}$ and $z$ in $X$ with $\|x - z\| < \delta$. Using (2), we can pick $r_i$ from $B_{X_i}(p_i, \epsilon) \cap P_{Y_i}(z_i)$ for $i \in \{1, 2\}$. By Remark 3.1, $r_1 + r_2$ is in $P_Y(z)$. Since $\|(p_1 + p_2) - (r_1 + r_2)\| < \epsilon$, it follows that $P_Y$ is lower Hausdorff semi-continuous at $x$.

**Case 2.** $d_1(x) \neq d_2(x)$.


We discuss only the case $d_1(x) < d_2(x)$, the proof for the other case being similar. Let $2\delta = d_2(x) - d_1(x)$. Replacing $x$ by $x_1$ and $a$ by $d_2(x)$ in Fact 3.2, we can get $\delta > 0$ such that if $\|x - z\| < \delta$, then

$$d_2(z) - d_1(z) > \gamma$$

and

$$h(B_{X_1}[x_1, d_2(x)] \cap Y_1, B_{X_1}[z_1, d_2(z)] \cap Y_1) < \epsilon.$$ 

Without loss of generality, we assume that $\delta$ is so chosen that (2) is also satisfied.

We have, by Remark 3.1,

$$P_Y(w) = B_{X_1}[w_1, d_2(w)] \cap Y_1 + P_{Y_1}(w_2)$$

for all $w$ in $X$ with $\|x - w\| < \delta$. Choose any $z$ in $X$ with $\|x - z\| < \delta$. If $t$ is in $B_{X_1}[x_1, d_2(x)] \cap Y_1$ and $s$ in $P_{Y_1}(x_2)$, using the above inequality and (2), we select $r$ in $B_{X_1}[z_1, d_2(z)] \cap Y_1$ and $p$ in $P_{Y_1}(z_2)$ satisfying $\|t - r\| < \epsilon$ and $\|s - p\| < \epsilon$. Clearly $r + p$ is in $P_Y(z)$, and this completes the proof for this case.

We now prove a similar result for upper Hausdorff semi-continuity.

**Theorem 3.4.** Let $X_i$ be a Banach space, $Y_i$ a strongly proximinal subspace of $X_i$, for $i \in \{1, 2\}$. If $X = X_1 \oplus \infty X_2$ and $Y = Y_1 \oplus \infty Y_2$, then the metric projection $P_Y$, from $X$ onto $Y$, is upper Hausdorff semi-continuous.

**Proof.** By Remark 3.1, $Y$ is proximinal in $X$, and by Remark 1.2, the metric projection from $X_i$ onto $Y_i$ is upper Hausdorff semi-continuous, for $i \in \{1, 2\}$. Fix $x$ in $X$ and let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$\|x - z\| < \delta \Rightarrow P_{Y_i}(z_i) \subseteq P_{Y_i}(x_i) + \epsilon B_{X_i},$$

for $i \in \{1, 2\}$.

**Case 1.** $d_1(x) = d_2(x)$.

In this case, we have $P_Y(x) = P_{Y_1}(x_1) \oplus \infty P_{Y_2}(x_2)$. Since $Y_i$ is strongly proximinal in $X_i$, we can select $\eta > 0$ such that

$$s(x_i, \eta) < \epsilon \quad \text{for} \quad i \in \{1, 2\},$$

where $s(x_i, \eta)$ is given by Definition 1.1. We now choose $0 < \delta < \eta/4$ so that (3) holds and consider any $z$ with $\|x - z\| < \delta$. If $d_1(z) = d_2(z)$, then $P_Y(z) = P_{Y_1}(z_1) + P_{Y_2}(z_2)$ and clearly by (3),

$$P_Y(z) \subseteq P_Y(x) + \epsilon B_X$$

in this case.

Now assume that $d_1(z) < d_2(z)$. Since

$$|d_1(x) - d_1(z)| \leq \|x - z\| < \eta/4, \quad \text{for} \quad i \in \{1, 2\},$$

we have

$$|d_2(z) - d_1(x)| \leq |d_2(z) - d_2(x)| + |d_2(x) - d_1(x)| = |d_2(z) - d_2(x)| < \eta/4. \quad \text{(5)}$$

Now, by Remark 3.1, $P_Y(z) = B_{X_1}[z_1, d_2(z)] \cap Y_1 + P_{Y_2}(z_2)$. Select any $t$ in $B_{X_1}[z_1, d_2(z)] \cap Y_1$. Then, using (5), we have

$$\|t - x_1\| \leq \|t - z_1\| + \|z_1 - x_1\| \leq d_2(z) + \eta/4 \leq d_1(x) + \eta/2.$$

By (4), $s(x_1, \eta) < \epsilon$ and so we have $d(t, P_{Y_1}(x_1)) < \epsilon$. Thus there exists $r$ in $P_{Y_1}(x_1)$ satisfying $\|t - r\| < \epsilon$ and

$$B_{X_1}[z_1, d_2(z)] \cap Y_1 \subseteq P_{Y_1}(x_1) + \epsilon B_{X_1}.$$
then the annihilator and is zero for all $n$ from $[4]$

Since, by (3),

$$P_{Y_2}(z_2) \subseteq P_{Y_2}(x_2) + \epsilon B_{X_2},$$

we conclude that

$$P_Y(z) \subseteq P_Y(x) + \epsilon B_X.$$  

If $d_2(z) < d_1(z)$, we argue just as above to conclude that $P_Y$ is upper Hausdorff semi-continuous.

**Case 2.** $d_1(x) \neq d_2(x)$.

We discuss only the case $d_1(x) < d_2(x)$, the proof for the other case being similar. Let $2\gamma = d_2(x) - d_1(x)$. Replacing $x$ by $x_1$ and $\alpha$ by $d_2(x)$ in Fact 3.2, we can get $\delta > 0$ such that if $\|x - z\| < \delta$, then

$$d_2(z) - d_1(z) > \gamma$$

and

$$h(B_{X_1}[z_1, d_2(x)] \cap Y_1, B_{X_1}[z_1, d_2(z)] \cap Y_1) < \epsilon.$$  

Without loss of generality, we assume that $\delta$ is so chosen that (3) is also satisfied.

We have

$$P_Y(w) = B_{X_1}[w_1, d_2(w)] \cap Y_1 + P_{Y_2}(w_2)$$

for all $w$ in $X$ with $\|x - w\| < \delta$. Select any $z$ in $X$ with $\|x - z\| < \delta$. If $t$ is in $B_{X_1}[z_1, d_2(z)] \cap Y_1$ and $s$ in $P_{Y_2}(z_2)$, using the above inequality and (3), we select $r$ in $B_{X_1}[x_1, d_2(x)] \cap Y_1$ and $p$ in $P_{Y_2}(x_2)$ satisfying $\|t - r\| < \epsilon$ and $\|s - p\| < \epsilon$. Clearly $r + p$ is in $P_Y(x)$ and $P_Y(z) \subseteq P_Y(x) + \epsilon B_X$. $\square$

**Remark 3.5.** Let $X$ be an $\ell_\infty$-direct sum of two non-zero Banach spaces $X_1$ and $X_2$ and $Y_i$ be a proximinal, proper subspace of $X_i$, for $i \in \{1, 2\}$. It was recently shown in [4] that if $P_{Y_i}$ is upper Hausdorff semi-continuous on $X$, where $Y = Y_1 \oplus_\infty Y_2$, then $Y_i$ must be strongly proximinal in $X_i$, for $i \in \{1, 2\}$. This clearly implies that Theorem 3.4 does not hold if, for any one of the two values of $i$, strong proximinality of $Y_i$ is replaced by the strictly weaker assumption that $Y_i$ is proximinal and $P_{Y_i}$ is upper Hausdorff semi-continuous.

The following theorem now follows from Remark 2.8 and Theorems 3.3 and 3.4.

**Theorem 3.6.** Let $X_i$ be a Banach space, $Y_i$ a strongly proximinal subspace of $X_i$ with the metric projection from $X_i$ onto $Y_i$ Hausdorff metric continuous, for $i \in \{1, 2\}$. If $X = X_1 \oplus_\infty X_2$ and $Y = Y_1 \oplus_\infty Y_2$, then the metric projection $P_Y$ from $X$ onto $Y$ is Hausdorff metric continuous.

4. **Proximinal subspaces of finite codimension of $c_0$**

If $Y$ is a proximinal subspace of finite codimension in a normed linear space $X$, then the annihilator $Y^\perp$ of $Y$ is contained in $NA(X)$, the class of norm attaining functionals on $X$ (see [5] and [6]). Let $Y'$ be a proximinal subspace of finite codimension in $c_0$. Since $NA(c_0)$ is the set of finite sequences in $\ell_1$ and $Y^\perp$ is finite dimensional, there exists a positive integer $k$ such that for any $f = (f_n)$ in $Y^\perp$, $f_n$ is zero for all $n \geq k$. In the rest of this section, the subspace $Y$ and positive integer $k$ are fixed as above.
Let \( \{ e_n : n \geq 1 \} \) denote the natural basis of \( c_0 \). For any sequence \( x = (x_n) \) of scalars, we set \( \hat{x} = \sum_{n=1}^{k} x_n e_n \). Also, we set
\[
X_1 = \text{span} \{ e_1, e_2, \cdots, e_k \},
\]
\[
X_2 = \{ (x_n) \in \ell_{\infty} : x_n = 0 \text{ for } 1 \leq n \leq k \},
\]
\[
Y_1 = \{ \hat{x} : x \in Y \}
\]
and finally
\[
Y_2 = \{ (x_n) \in c_0 : x_n = 0 \text{ for } 1 \leq n \leq k \}.
\]
Then clearly \( Y_i \) is a subspace of \( X_i \) for \( i = 1, 2 \) and
\[
X = X_1 \oplus_{\infty} X_2.
\]
Also, note that if \( x \) is in \( c_0 \), then
\[
x \in Y \iff \hat{x} \in Y \iff \hat{x} \in Y_1.
\]
It is now clear that \( Y = Y_1 \oplus_{\infty} Y_2 \).

Now, following the same proof for \( c_0 \) an M-ideal in \( \ell_{\infty} \), we get \( Y_2 \) to be an M-ideal in \( X_2 \). Since \( X_1 \) is a finite-dimensional subspace of \( c_0 \), it is a polyhedral space. By Fact 2.9, \( Y_i \) is a strongly proximinal subspace of \( X_i \) with the metric projection \( P_{Y_i} \) from \( X_i \) onto \( Y_i \) Hausdorff metric continuous for \( i \in \{1, 2\} \). It is now clear that the main theorem of this article, given below, follows immediately from Theorem 3.6.

**Theorem 4.1.** Let \( Y \) be a proximinal subspace of finite codimension in \( c_0 \). Then \( Y \) is proximinal in \( \ell_{\infty} \) and the metric projection from \( \ell_{\infty} \) onto \( Y \) is Hausdorff metric continuous.

**Remark 4.2.** Let \( Y \) be a subspace of codimension \( k \) in \( c_0 \), and assume \( Y^\perp \) is the span of a linearly independent set \( \{ f_1, f_2, \cdots, f_k \} \). Then it follows from Example 1.4 (a) of \([9]\) that \( Y \) is an M-ideal in \( \ell_{\infty} \) if and only if \( Y \) is an M-ideal in \( c_0 \) if and only if \( f_i \) belongs to \( \{ e_n : n \geq 1 \} \) for each \( i, 1 \leq i \leq k \). We recall, from \([9]\), that \( Y \) is proximinal in \( c_0 \) (and hence in \( \ell_{\infty} \)) if and only if \( Y^\perp \) is contained in \( NA(c_0) \) or equivalently, every element of \( Y^\perp \) is a sequence of \( \ell_1 \) with only a finite number of nonzero entries. Thus, there are plenty of proximinal subspaces of finite codimension of \( c_0 \) that are not M-ideals in \( \ell_{\infty} \) and for these, Theorem 4.1 cannot be derived from Proposition 2.1.

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**References**


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