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AN EXTENSION OF BIRAN'S LAGRANGIAN BARRIER THEOREM

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Abstract. We use the Gromov-Witten invariants and a nonsqueezing theorem by the author to affirm a conjecture by P. Biran on the Lagrangian barriers.

1. Main results

A Kähler manifold is a triple consisting of a symplectic manifold (M, ω) and an integrable complex structure J compatible with ω on M. If $[\omega] \in H^2(M^{2n}, \mathbb{Z})$, it follows from Kodaira's embedding theorem that there exists a smooth and reduced complex hypersurface $\Sigma \subset M$ such that its homology class $[\Sigma] \in H_{2n-2}(M)$ represents the Poincaré dual $k[\omega] \in H^2(M)$ for some $k \in \mathbb{N}$. Following [1] $\mathcal{P} = (M, \omega, J; \Sigma)$ is called a *smoothly polarized Kähler manifold*. Under the conditions that either $\dim_{\mathbb{R}} M \leq 6$ or $\omega|_{\pi_2(M)} = 0$ the following two theorems were proved in Theorem 1.D and Theorem 4.A of [1] respectively.

Theorem 1.1. *If (M, ω) is a Kähler manifold with $[\omega] \in H^2(M, \mathbb{Q})$, then for every $\epsilon > 0$ there exists a Lagrangian CW-complex $\triangle_\epsilon \subset (M, \omega)$ such that every symplectic embedding $\varphi : B(\epsilon) \to (M, \omega)$ must satisfy $\varphi(B(\epsilon)) \cap \triangle_\epsilon \neq \emptyset$.*

Theorem 1.2. *If $\mathcal{P} = (M, \omega, J; \Sigma)$ is an n-dimensional polarized Kähler manifold of degree k, then every symplectic embedding $\varphi : B^{2n}(\lambda) := \{x \in \mathbb{R}^{2n} \mid |x| \leq \lambda^2\} \to (M, \omega)$ with $\lambda^2 \geq \frac{1}{k\pi}$ must intersect the skeleton $\triangle_{\mathcal{P}}$ associated to the polarization $\mathcal{P}$.*

For the definitions of $\triangle_\epsilon$ and $\triangle_{\mathcal{P}}$ the reader may refer to [1]. Actually, such generalizations were conjectured in Remark 4.B of [1]. As in [1] Theorem 1.1 may be derived from Theorem 1.2.

Following [2] a Stein manifold is said to be *subcritical* if it admits a plurisubharmonic function that has only critical points of index less than half the real dimension. A polarization $\mathcal{P} = (M, \omega, J; \Sigma)$ is called *subcritical* if the complement $(M \setminus \Sigma, \omega)$ is a subcritical Stein manifold. From the proof of Theorem 1.2 we easily get the following generalizations of Theorem F and Theorem G in [2].

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Theorem 1.3. *If a closed Kähler manifold (M, ω, J) admits a subcritical polarization $\mathcal{P} = (M, \omega, J; \Sigma)$ of degree k, then $\mathcal{W}_G(M, \omega) \leq \frac{1}{k}$ and $k \leq \dim_{\mathbb{C}} M$. Moreover, if the linear system of holomorphic sections of the normal bundle to Σ, $N_{\Sigma/M} \to \Sigma$, is base point free, then $\mathcal{W}_G(M, \omega) = \frac{1}{k} \geq 1/\dim_{\mathbb{C}} M$. Here $\mathcal{W}_G(M, \omega)$ stands for the Gromov width of (M, ω).*

2. Proof of the theorems

Our purpose is to prove Theorem 1.2. The ideas are similar to those of Biran. However, we use the theory of virtual cycles and some techniques in [4] to compute the desired the Gromov-Witten invariant without additional assumptions as in [1]. Then we directly use the author's previous work on pseudo-symplectic capacities and avoid Gromov's arguments as used in [1]. For convenience of the reader we need to recall some related notions in [1]. A subset $\triangle$ of a symplectic manifold (M, ω) is called an *embedded CW-complex* if there exists an abstract finite and path-connected CW-complex K and a homeomorphism $i : K \to \triangle \subset M$ such that for every cell $C \subset K$ the restriction $i|_{\mathrm{Int}(C)} : \mathrm{Int}(C) \to M$ is a smooth embedding. When the image $i(\mathrm{Int}(C))$ by i of each cell C in K is an isotropic submanifold of (M, ω) the above embedded CW-complex $\triangle$ is called *isotropic*. The dimension $\dim \triangle$ of $\triangle$ is defined as the maximum of those of cells in K. An embedded isotropic CW-complex of dimension $\frac{1}{2} \dim_{\mathbb{R}} M$ in (M, ω) is called a *Lagrangian* CW-complex.

Another key role in [1] is the *standard symplectic disc bundle*. For a closed symplectic manifold (S, σ) with $[\sigma] \in H^2(S; \mathbb{Z})$ there exists a Hermitian line bundle $p : L \to S$ with $c_1(L) = [\sigma]$ and a compatible connection ∇ on L with curvature $R^{\nabla} = 2\pi i \sigma$ (cf. [5, Prop. 8.3.1]). Denote by $\| \cdot \|$ the Hermitian metric and by $E_L = \{v \in L \mid \|v\| < 1\}$ the open unit disc bundle of L. Let α^{∇} be the associated *transgression* 1*-form* on $L \setminus 0$ with $d\alpha^{\nabla} = -p^*\sigma$, and r the radial coordinate along the fibres induced by $\| \cdot \|$. It was shown that

$$\omega_{\mathrm{can}} := p^*\sigma + d(r^2 \alpha^{\nabla})$$

is a symplectic form on E_L and that the symplectic type of $(E_L, \omega_{\mathrm{can}})$ depends only on the symplectic type of (S, σ) and the topological type of the complex line bundle $p : L \to S$. Moreover, $(E_L, \omega_{\mathrm{can}})$ is uniquely characterized (up to symplectomorphism) by the following three properties:

- All fibres of $p : E_L \to S$ are symplectic with respect to ω_{can} and have area 1.
- The restriction of ω_{can} to the zero section $S \subset E_L$ equals σ.
- ω_{can} is S^1-invariant with respect to the obvious circle action on E_L.

Following [1] we call

$$p : (E_L, \omega_{\mathrm{can}}) \to (S, \sigma)$$

a *standard symplectic disc bundle over* (S, σ) *modelled on* L and $p : (E_L, c\omega_{\mathrm{can}}) \to (S, \sigma)$ a *standard symplectic disc bundle with fibres of area* c for each $c > 0$.

As was done in [1] this standard symplectic disc bundle can be compactified into a $\mathbb{C}P^1$-bundle over (S, σ). Indeed, let $\mathbb{C}$ stand for the trivial complex line bundle over S. Then the direct sum $L \oplus \mathbb{C}$ is a complex vector bundle of rank 2 over S. Denote by $p_X : X_L = P(L \oplus \mathbb{C}) \to S$ its projective bundle, which is a $\mathbb{C}P^1$-bundle over S. It has two distinguished sections: the zero section $Z_0 = P(0 \oplus \mathbb{C})$ and the section at infinity $Z_\infty = P(L \oplus 0)$. Clearly, the open manifold $X_L \setminus Z_\infty$ is diffeomorphic to the disc bundle E_L. Let F_s stand for the fibre of X_L at $s \in S$. It

was shown in [1] that a given $J_S \in \mathcal{J}(S,\sigma)$ and the connection ∇ determine a unique almost complex structure J_L on the total space of L. This J_L induces the almost complex structures J_E on E and J_X on X_L again. In particular, $J_E \in \mathcal{J}(E_L, \omega_{\mathrm{can}})$ and Z_0, Z_∞ and all fibres F_s of $p_X : X_L \to S$ are holomorphic with respect to J_X. These show that $p_X : (X_L, J_X) \to (S, J_S)$ is an almost complex fibration with fibre $(\mathbb{C}P^1, i)$ in the following sense.

Definition 2.1 ([1, Def. 6.3.A])**.** Let (F, J_F), (B, J_B) and (X, J) be three almost complex manifolds. A holomorphic map $p : (X, J) \to (B, J_B)$ is called an almost complex fibration with fibre (F, J_F) if every $b \in B$ has a neighborhood U and a trivialization $\varphi : X|_U \to U \times F$ such that for every $a \in U$ the map $\varphi|_{F_a} : (F_a, J|_{F_a}) \to (a \times F, J_F)$ is holomorphic.

For every $0 < \rho < 1$ both $E_L(\rho) := \{v \in E_L \mid \|v\| < \rho\}$ and $\overline{E_L(\rho)} := \{v \in E_L \mid \|v\| \le \rho\}$ are subbundles of E_L.

Lemma 2.2 ([1, Lem. 5.2.A])**.** *Let $p : (E_L, \omega_{\mathrm{can}}) \to (S, \sigma)$ be a standard symplectic disc bundle modeled on a Hermitian line bundle $p : L \to S$ as above. Then there exists a diffeomorphism $f : E_L \to X_L \setminus Z_\infty$ and a family of symplectic forms $\{\eta_\rho\}_{0<\rho<1}$ on X_L such that:*

1. *$f^*\eta_\rho = \omega_{\mathrm{can}}$ on $\overline{E_L(\rho)}$ for every $0 < \rho < 1$.*
2. *f sends the fibres of $E_L \to S$ to the fibres of $X_L \setminus Z_\infty \to S$.*
3. *If S is identified with the zero-section of E_L, then $f(S) = Z_0$ and $p_X \circ f|_S : S \to S$ is the identity map.*
4. *Z_0, Z_∞ and all fibres $F_s = p_X^{-1}(s)\,(s \in S)$ of $p_X : X_L \to S$ are not only symplectic with respect to η_ρ for every $0 < \rho < 1$, but also holomorphic with respect to J_X. Moreover, for every $0 < \rho < 1$ it follows that $\eta_\rho|_{TZ_0} = (p_X^*\sigma)|_{TZ_0}$ and $\eta_\rho|_{TZ_\infty} = c_\rho(p_X^*\sigma)|_{TZ_\infty}$ for some $0 < c_\rho < 1 - \rho^2$.*
5. *The area of the fibres F_s satisfies $\rho^2 < \int_{F_s} \eta_\rho < 1$ for every $0 < \rho < 1$.*

Moreover, if $J_S \in \mathcal{J}(S,\sigma)$, then $J_X \in \mathcal{J}(X_L, \eta_\rho)$ for every $0 < \rho < 1$. In particular, if (S, σ, J_S) is Kähler and $L \to S$ is a holomorphic line bundle, then (X_L, η_ρ, J_X) is Kähler for every $0 < \rho < 1$.

It was proved in Lemma 6.A of [1] that the almost complex structure J_X is regular for the class of the fibre $F \in H_2(X_L, \mathbb{Z})$ and that the space of J_X-holomorphic spheres in the class F is made up of exactly the fibres $F_s = p_X^{-1}(s)$, $s \in S$. So the space $\overline{M}_{0,3}(X_L, J_X, F)$ of all J_X-holomorphic stable maps in class F of genus 0 and with 3 marked points is nonempty. As expected we have:

Lemma 2.3. *Let $F \in H_2(X_L; \mathbb{Z})$ denote the homology class of a fibre of $X_L \to S$. Then the Gromov-Witten invariant of (X_L, η_ρ),*

$$\Psi_{F,0,3}^{(X_L,\eta_\rho)}(pt; [Z_0], [Z_\infty], pt) = 1.$$

That is, (X_L, η_ρ) is a strong 0-symplectic uniruled manifold in the sense of Definition 1.13 in [3].

Proof. Let $A = [\mathbb{C}P^1] \in H_2(\mathbb{C}P^1; \mathbb{Z})$ and $i : \mathbb{C}P^1 \to X_L$ be the inclusion. Then $F = i_*(A)$. Since the Gromov-Witten invariants are symplectic deformation invariants, we may fix an $\eta \equiv \eta_\rho$. By Lemma 2.2 all fibres $F_s = \pi_X^{-1}(s)$ $(s \in S)$ are not only symplectic with respect to η but also holomorphic with respect to J_X. Thus for each $s \in S$ and $x \in F_s$ the tangent space T_xF_s is a symplectic and $J_X(x)$-invariant

subspace of (T_xX_L, η_x). Let H_x denote the η_x-orthogonal complement of T_xF_s. Then $T_xX_L = H_x \oplus T_xF_s$ and $Dp_X(x)|_{H_x} : H_x \to T_sS$ is an isomorphism. Note that J_X is also compatible with η. We have that $\eta(J_X\xi, J_X\eta) = \eta(\xi, \eta)$ for all $\xi, \eta \in TX_L$. This implies that H_x is also a $J_X(x)$-invariant subspace of T_xX_L. Consequently, $J_X(x)$ preserves the splitting $T_xX_L = H_x \oplus T_xF_s$. Moreover, the projection $p_X : (X_L, J_X) \to (S, J_S)$ is holomorphic (cf. [1, §6.3]). Hence the almost complex structure J_X on X_L is *fibered* in the sense of Definition 2.2 in [4].

Since any stable map has connected image set it is easily checked that each stable map in $\overline{M}_{0,3}(X_L, J_X, F)$ must also entirely lie in a fibre of X_L. It follows from this and Lemma 4.3 in [4] that for any representative $\tilde{\tau} = (\Sigma, \tilde{h})$ of $\tau \in \overline{M}_{0,3}(X_L, J_X, F)$ the cokernel of $D\bar{\partial}_{J_X}(\tilde{h})$ can be spanned by elements of the space

$$\mathcal{L}^V_{\tilde{h}} = L^{1,p}\Big(\wedge^{0,1} (\tilde{h}^*(T_{vert}X_L))\Big),$$

where $V = T_{vert}X_L$ is the vertical tangent bundle of TX_L. By Proposition 4.3 in [4] one can choose R and the embeddings e in the construction of the virtual moduli cycle $\overline{M}^{\nu}_{0,3}(X_L, J_X, F)$ of $\overline{M}_{0,3}(X_L, J_X, F)$ such that $e_{\tilde{\tau}}(\nu) \in \mathcal{L}^V_{\tilde{h}}$ for all $\nu \in R$ and all $\tilde{\tau}$. That is, we can choose a fibered pair (J_X, ν) on X_L in the sense of Definition 4.4 in [4]. As proved in Proposition 4.4 of [4] such a virtual moduli cycle $\overline{M}^{\nu}_{0,3}(X_L, J_X, F)$ has the following property. For each element of it, a parameterized stable map $(\Sigma, \tilde{h})$, each component $\tilde{h}_k$ of $\tilde{h}$ satisfies an equation of the form $\bar{\partial}_{J_X}\tilde{h}_k = \nu_k$ for some section ν_k of $\wedge^{0,1}(\tilde{h}^*_k(T_{vert}X_L))$. This implies that each $p_X \circ \tilde{h}_k : S^2 \to S$ is (i, J_S)-holomorphic, and therefore that each element in $\overline{M}^{\nu}_{0,3}(X_L, J_X, F)$ has the image contained in a single fibre of X_L. For any $s \in S$ we identify $(F_s, J_X|_{F_s})$ with $(\mathbb{C}P^1, i)$. Then the subset $\overline{M}^{\nu}_{0,3}(X_L, J_X, F)_s$ of $\overline{M}^{\nu}_{0,3}(X_L, J_X, F)$ consisting of stable maps with image in F_s regularizes the space $\overline{M}_{0,3}(\mathbb{C}P^1, i, A)$ in the sense of Definition 4.3 in [4]. Hence

$$\Psi^{(X_L,\eta)}_{F,0,3}(pt; [Z_0], [Z_\infty], pt) = \Psi^{(\mathbb{C}P^1,\omega_{FS})}_{A,0,3}(pt; pt, pt, pt) = 1.$$

Here we have used pt to stand for the point classes in the different spaces. Lemma 2.3 is proved. □

Recall that for a closed symplectic manifold (M, ω) and homology classes $\alpha_0, \alpha_\infty \in H_*(M, \mathbb{Q})$ we, in [3, Def. 1.8], defined a number

$$\mathrm{GW}_g(M, \omega; \alpha_0, \alpha_\infty) \in (0, +\infty]$$

by the infimum of the ω-areas $\omega(A)$ of the homology classes $A \in H_2(M; \mathbb{Z})$ for which the Gromov-Witten invariant $\Psi_{A,g,m+2}(C; \alpha_0, \alpha_\infty, \beta_1, \cdots, \beta_m) \neq 0$ for some homology classes $\beta_1, \cdots, \beta_m \in H_*(M; \mathbb{Q})$ and $C \in H_*(\overline{\mathcal{M}}_{g,m+2}; \mathbb{Q})$ and integer $m > 0$. Moreover, in Definition 1.25 of [3] we also defined another number $\mathrm{GW}(M, \omega) \in (0, +\infty]$ by

$$\mathrm{GW}(M, \omega) = \inf \mathrm{GW}_g(M, \omega; pt, \alpha),$$

where the infimum is taken over all nonnegative integers g and all homology classes $\alpha \in H_*(M; \mathbb{Q}) \setminus \{0\}$ of degree $\deg \alpha \leq \dim M - 1$. Theorem 1.26 in [3] claimed that

$$\mathcal{W}_G(M, \omega) \leq \mathrm{GW}(M, \omega) \tag{1}$$

for any symplectic uniruled manifold (M, ω) of dimension at least 4.

Proof of Theorem 1.2. By the decomposition Theorem 2.6.A in [1], $(M \setminus \triangle_{\mathcal{P}}, \omega)$ is symplectomorphic to the standard symplectic disc bundle $(E_{N_\Sigma}, \frac{1}{k}\omega_{\text{can}}) \to (\Sigma, \sigma = k\omega|_\Sigma)$ over Σ, which is modeled on the normal bundle N_Σ and with fibres of area $1/k$. Assume that $\varphi(B(\lambda)) \cap \triangle_{\mathcal{P}} = \emptyset$ for some symplectic embedding $\varphi : B^{2n}(\lambda) \to (M, \omega)$ with $\lambda^2 \geq \frac{1}{k\pi}$. Then there exists a symplectic embedding $\psi : B(\lambda) \to (E_{N_\Sigma}, \frac{1}{k}\omega_{\text{can}})$. Since $\psi(B(\lambda))$ is compact there exists a positive number $\rho \in (0,1)$ such that $\psi(B(\lambda)) \subset E_{N_\Sigma}(\rho)$. Lemma 2.2 gives a symplectic embedding from $\psi(B(\lambda))$ into $(X_{N_\Sigma}, \frac{1}{k}\eta_\rho)$. Hence

$$\mathcal{W}_G(X_{N_\Sigma}, \frac{1}{k}\eta_\rho) \geq \pi\lambda^2 \geq 1/k. \tag{2}$$

On the other hand, by the definition of $\text{GW}(M, \omega)$ and Lemmas 2.2, 2.3 we have

$$\text{GW}(X_{N_\Sigma}, \eta_\rho) \leq \int_{F_s} \eta_\rho < 1$$

for this ρ. This and (1) together give

$$\mathcal{W}_G(X_{N_\Sigma}, \frac{1}{k}\eta_\rho) \leq \text{GW}(X_{N_\Sigma}, \frac{1}{k}\eta_\rho) \leq \frac{1}{k}\int_{F_s} \eta_\rho < \frac{1}{k},$$

which contradicts (2). Theorem 1.2 is proved. □

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