AN EXTENSION OF BIRAN’S LAGRANGIAN BARRIER THEOREM

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Abstract. We use the Gromov-Witten invariants and a nonsqueezing theorem by the author to affirm a conjecture by P. Biran on the Lagrangian barriers.

1. Main results

A Kähler manifold is a triple consisting of a symplectic manifold \((M, \omega)\) and an integrable complex structure \(J\) compatible with \(\omega\) on \(M\). If \([\omega] \in H^2(M, \mathbb{Z})\), it follows from Kodaira’s embedding theorem that there exists a smooth and reduced complex hypersurface \(\Sigma \subset M\) such that its homology class \([\Sigma] \in H_{2n-2}(M)\) represents the Poincaré dual \(k[\omega] \in H^2(M)\) for some \(k \in \mathbb{N}\). Following \([1]\) \(\mathcal{P} = (M, \omega, J; \Sigma)\) is called a smoothly polarized Kähler manifold. Under the conditions that either \(\dim_{\mathbb{R}} M \leq 6\) or \(\omega|_{\pi_{2}}(M) = 0\) the following two theorems were proved in Theorem 1.D and Theorem 4.A of \([1]\) respectively.

**Theorem 1.1.** If \((M, \omega)\) is a Kähler manifold with \([\omega] \in H^2(M, \mathbb{Q})\), then for every \(\epsilon > 0\) there exists a Lagrangian CW-complex \(\Delta_\epsilon \subset (M, \omega)\) such that every symplectic embedding \(\varphi : B(\epsilon) \to (M, \omega)\) must satisfy \(\varphi(B(\epsilon)) \cap \Delta_\epsilon \neq \emptyset\).

**Theorem 1.2.** If \(\mathcal{P} = (M, \omega, J; \Sigma)\) is an \(n\)-dimensional polarized Kähler manifold of degree \(k\), then every symplectic embedding \(\varphi : B^{2n}(\lambda) := \{x \in \mathbb{R}^{2n} \mid |x| \leq \lambda^2\} \to (M, \omega)\) with \(\lambda^2 \geq \frac{1}{k\pi}\) must intersect the skeleton \(\Delta_{\mathcal{P}}\) associated to the polarization \(\mathcal{P}\).

For the definitions of \(\Delta_\epsilon\) and \(\Delta_{\mathcal{P}}\) the reader may refer to \([1]\). Actually, such generalizations were conjectured in Remark 4.B of \([1]\). As in \([1]\) Theorem 1.1 may be derived from Theorem 1.2.

Following \([2]\) a Stein manifold is said to be subcritical if it admits a plurisubharmonic function that has only critical points of index less than half the real dimension. A polarization \(\mathcal{P} = (M, \omega, J; \Sigma)\) is called subcritical if the complement \((M \setminus \Sigma, \omega)\) is a subcritical Stein manifold. From the proof of Theorem 1.2 we easily get the following generalizations of Theorem F and Theorem G in \([2]\).

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**Theorem 1.3.** If a closed Kähler manifold \((M, \omega, J)\) admits a subcritical polarization \(P = (M, \omega, J; \Sigma)\) of degree \(k\), then \(W_G(M, \omega) \leq \frac{1}{k}\) and \(k \leq \dim \mathbb{C} M\). Moreover, if the linear system of holomorphic sections of the normal bundle to \(\Sigma\), \(N_{\Sigma/M} \to \Sigma\), is base point free, then \(W_G(M, \omega) = \frac{1}{k} \geq 1/\dim \mathbb{C} M\). Here \(W_G(M, \omega)\) stands for the Gromov width of \((M, \omega)\).

2. **Proof of the theorems**

Our purpose is to prove Theorem 1.2. The ideas are similar to those of Biran. However, we use the theory of virtual cycles and some techniques in [4] to compute the desired the Gromov-Witten invariant without additional assumptions as in [1]. Then we directly use the author’s previous work on pseudo-symplectic capacities and avoid Gromov’s arguments as used in [1]. For convenience of the reader we need to recall some related notions in [1]. A subset \(\Delta\) of a symplectic manifold \((M, \omega)\) is called an embedded CW-complex if there exists an abstract finite and path-connected CW-complex \(K\) and a homeomorphism \(i : K \to \Delta \subset M\) such that for every cell \(C \subset K\) the restriction \(i_{\text{Int}(C)} : \text{Int}(C) \to M\) is a smooth embedding. When the image \(i(\text{Int}(C))\) by \(i\) of each cell \(C\) in \(K\) is an isotropic submanifold of \((M, \omega)\) the above embedded CW-complex \(\Delta\) is called isotropic. The dimension \(\dim \Delta\) of \(\Delta\) is defined as the maximum of those of cells in \(K\). An embedded isotropic CW-complex of dimension \(\frac{1}{2} \dim \mathbb{R} M\) in \((M, \omega)\) is called a Lagrangian CW-complex.

Another key role in [1] is the standard symplectic disc bundle. For a closed symplectic manifold \((S, \sigma)\) with \([\sigma] \in H^2(S; \mathbb{Z})\) there exists a Hermitian line bundle \(p : L \to S\) with \(c_1(L) = [\sigma]\) and a compatible connection \(\nabla\) on \(L\) with curvature \(R^\nabla = 2\pi i \sigma\) (cf. [5] Prop. 8.3.1]). Denote by \(\| \cdot \|\) the Hermitian metric and by \(E_L = \{ v \in L \mid \| v \| < 1 \}\) the open unit disc bundle of \(L\). Let \(\alpha^\nabla\) be the associated transgression 1-form on \(L \setminus 0\) with \(d \alpha^\nabla = -p^* \sigma\), and \(r\) the radial coordinate along the fibres induced by \(\| \cdot \|\). It was shown that

\[
\omega_{\text{can}} := p^* \sigma + \langle r^2 \alpha^\nabla \rangle
\]

is a symplectic form on \(E_L\) and that the symplectic type of \((E_L, \omega_{\text{can}})\) depends only on the symplectic type of \((S, \sigma)\) and the topological type of the complex line bundle \(p : L \to S\). Moreover, \((E_L, \omega_{\text{can}})\) is uniquely characterized (up to symplectomorphism) by the following three properties:

- All fibres of \(p : E_L \to S\) are symplectic with respect to \(\omega_{\text{can}}\) and have area 1.
- The restriction of \(\omega_{\text{can}}\) to the zero section \(S \subset E_L\) equals \(\sigma\).
- \(\omega_{\text{can}}\) is \(S^1\)-invariant with respect to the obvious circle action on \(E_L\).

Following [4] we call

\[p : (E_L, \omega_{\text{can}}) \to (S, \sigma)\]

a standard symplectic disc bundle over \((S, \sigma)\) modelled on \(L\) and \(p : (E_L, c \omega_{\text{can}}) \to (S, \sigma)\) a standard symplectic disc bundle with fibres of area \(c\) for each \(c > 0\).

As was done in [4] this standard symplectic disc bundle can be compactified into a \(\mathbb{C}P^1\)-bundle over \((S, \sigma)\). Indeed, let \(\mathbb{C}\) stand for the trivial complex line bundle over \(S\). Then the direct sum \(L \oplus \mathbb{C}\) is a complex vector bundle of rank 2 over \(S\). Denote by \(p_X : X_L = P(L \oplus \mathbb{C}) \to S\) its projective bundle, which is a \(\mathbb{C}P^1\)-bundle over \(S\). It has two distinguished sections: the zero section \(Z_0 = P(0 \oplus \mathbb{C})\) and the section at infinity \(Z_{\infty} = P(L \oplus 0)\). Clearly, the open manifold \(X_L \setminus Z_{\infty}\) is diffeomorphic to the disc bundle \(E_L\). Let \(F_s\) stand for the fibre of \(X_L\) at \(s \in S\). It
was shown in [1] that a given \( J_S \in \mathcal{F}(S, \sigma) \) and the connection \( \nabla \) determine a unique almost complex structure \( J_L \) on the total space of \( L \). This \( J_L \) induces the almost complex structures \( J_E \) on \( E \) and \( J_X \) on \( X_L \) again. In particular, \( J_E \in \mathcal{F}(E, \omega) \) and \( Z_0, Z_\infty \) and all fibres \( F_s \) of \( p_X : X_L \to S \) are holomorphic with respect to \( J_X \). These show that \( p_X : (X_L, J_X) \to (S, J_S) \) is an almost complex fibration with fibre \((\mathbb{C}P^1, i)\) in the following sense.

**Definition 2.1** ([1] Def. 6.3.A). Let \((F, J_F), (B, J_B)\) and \((X, J)\) be three almost complex manifolds. A holomorphic map \( p : (X, J) \to (B, J_B) \) is called an almost complex fibration with fibre \((F, J_F)\) if every \( b \in B \) has a neighborhood \( U \) and a trivialization \( \varphi : X|_U \to U \times F \) such that for every \( a \in U \) the map \( \varphi|_{F_a} : (F_a, J|_{F_a}) \to (a \times F, J_F) \) is holomorphic.

For every \( 0 < \rho < 1 \) both \( E_L(\rho) := \{ v \in E_L \| v\| < \rho \} \) and \( E_L(\rho) := \{ v \in E_L \| v\| \leq \rho \} \) are subbundles of \( E_L \).

**Lemma 2.2** ([1] Lem. 5.2.A). Let \( p : (E_L, \omega_{\text{can}}) \to (S, \sigma) \) be a standard symplectic disc bundle modeled on a Hermitian line bundle \( p : L \to S \) as above. Then there exists a diffeomorphism \( f : E_L \to X_L \setminus Z_\infty \) and a family of symplectic forms \( \{\eta_\rho\}_{0 < \rho < 1} \) on \( X_L \) such that:

1. \( f^* \eta_\rho = \omega_{\text{can}} \) on \( E_L(\rho) \) for every \( 0 < \rho < 1 \).
2. \( f \) sends the fibres of \( E_L \to S \) to the fibres of \( X_L \setminus Z_\infty \to S \).
3. If \( S \) is identified with the zero-section of \( E_L \), then \( f(S) = Z_0 \) and \( p_X \circ f|_S : S \to S \) is the identity map.
4. \( Z_0, Z_\infty \) and all fibres \( F_s = p_X^{-1}(s) \) of \( p_X : X_L \to S \) are not only symplectic with respect to \( \eta_\rho \) for every \( 0 < \rho < 1 \), but also holomorphic with respect to \( J_X \). Moreover, for every \( 0 < \rho < 1 \) it follows that \( \eta_\rho|_{TZ_0} = (p_{X|_{Z_0}}^{-1})|_{TZ_0} \) and \( \eta_\rho|_{TZ_\infty} = c_\rho(p_{X|_{Z_\infty}}^{-1})|_{TZ_\infty} \) for some \( 0 < c_\rho < 1 - \rho^2 \).
5. The area of the fibres \( F_s \) satisfies \( \rho^2 < \int_{F_s} \eta_\rho < 1 \) for every \( 0 < \rho < 1 \).

Moreover, if \( J_S \in \mathcal{F}(S, \sigma) \), then \( J_X \in \mathcal{F}(X_L, \eta_\rho) \) for every \( 0 < \rho < 1 \). In particular, \( (S, \sigma, J_S) \) is Kähler and \( L \to S \) is a holomorphic line bundle, then \((X_L, \eta_\rho, J_X)\) is Kähler for every \( 0 < \rho < 1 \).

It was proved in Lemma 6.A of [1] that the almost complex structure \( J_X \) is regular for the class of the fibre \( F \in H_2(X_L, \mathbb{Z}) \) and that the space of \( J_X \)-holomorphic spheres in the class \( F \) is made up of exactly the fibres \( F_s = p_X^{-1}(s) \), \( s \in S \). So the space \( \overline{M}_{0,3}(X_L, J_X, F) \) of all \( J_X \)-holomorphic stable maps in class \( F \) of genus 0 and with 3 marked points is nonempty. As expected we have:

**Lemma 2.3.** Let \( F \in H_2(X_L; \mathbb{Z}) \) denote the homology class of a fibre of \( X_L \to S \). Then the Gromov-Witten invariant of \((X_L, \eta_\rho)\),

\[
\Psi_{F,0,3}^{(X_L, \eta_\rho)}(pt; [Z_0], [Z_\infty], pt) = 1.
\]

That is, \((X_L, \eta_\rho)\) is a strong 0-symplectic uniruled manifold in the sense of Definition 1.13 in [3].

**Proof.** Let \( A = [\mathbb{C}P^1] \in H_2(\mathbb{C}P^1; \mathbb{Z}) \) and \( i : \mathbb{C}P^1 \to X_L \) be the inclusion. Then \( F = i_* (A) \). Since the Gromov-Witten invariants are symplectic deformation invariants, we may fix an \( \eta \equiv \eta_\rho \). By Lemma 2.2 all fibres \( F_s = \pi_X^{-1}(s) \) \( s \in S \) are not only symplectic with respect to \( \eta \) but also holomorphic with respect to \( J_X \). Thus for each \( s \in S \) and \( x \in F_s \) the tangent space \( T_x F_s \) is a symplectic and \( J_X(x) \)-invariant
subspace of \((T_xX_L, \eta_x)\). Let \(H_x\) denote the \(\eta_x\)-orthogonal complement of \(T_xF_s\). Then \(T_xX_L = H_x \oplus T_xF_s\) and \(D_{\partial}X(x)|_{H_x} : H_x \to T_xS\) is an isomorphism. Note that \(J_X\) is also compatible with \(\eta\). We have that \(\eta(J_X \xi, J_X \eta) = \eta(\xi, \eta)\) for all \(\xi, \eta \in TX_L\). This implies that \(H_x\) is also a \(J_X(x)\)-invariant subspace of \(T_xX_L\). Consequently, \(J_X(x)\) preserves the splitting \(T_xX_L = H_x \oplus T_xF_s\). Moreover, the projection \(p_X : (X_L, J^2_X) \to (S, J_0)\) is holomorphic (cf. [1, §6.3]). Hence the almost complex structure \(J_X\) on \(X_L\) is fibered in the sense of Definition 2.2 in [2].

Since any stable map has connected image set it is easily checked that each stable map in \(\overline{M}_{0,3}(X_L, J_X, F)\) must also entirely lie in a fibre of \(X_L\). It follows from this and Lemma 4.3 in [4] that for any representative \(\hat{\tau} = (\Sigma, \hat{h})\) of \(\tau \in \overline{M}_{0,3}(X_L, J_X, F)\) the cokernel of \(D\hat{\partial}J_X(\hat{h})\) can be spanned by elements of the space

\[
\mathcal{L}^V = L^{1, p}\left(\wedge^{0, 1}(\hat{h}^*TX_L)\right),
\]

where \(V = T_{vert}X_L\) is the vertical tangent bundle of \(TX_L\). By Proposition 4.3 in [4] one can choose \(R\) and the embeddings \(e\) in the construction of the virtual moduli cycle \(\overline{M}_{0,3}(X_L, J_X, F)\) of \(\overline{M}_{0,3}(X_L, J_X, F)\) such that \(e_\pm(\nu) \in \mathcal{L}^V\) for all \(\nu \in R\) and all \(\hat{\tau}\). That is, we can choose a fibered pair \((J_X, \nu)\) on \(X_L\) in the sense of Definition 4.4 in [4]. As proved in Proposition 4.4 of [4] such a virtual moduli cycle \(\overline{M}_{0,3}(X_L, J_X, F)\) has the following property. For each element of it, a parameterized stable map \((\Sigma, \hat{h})\), each component \(\hat{h}_k\) of \(\hat{h}\) satisfies an equation of the form \(\hat{\partial}J_X \hat{h}_k = \nu_k\) for some section \(\nu_k\) of \(\wedge^{0, 1}(\hat{h}^*TX_L)\). This implies that each \(p_X \circ \hat{h}_k : S^2 \to S\) is \((i, J_S)\)-holomorphic, and therefore that each element in \(\overline{M}_{0,3}(X_L, J_X, F)\) has the image contained in a single fibre of \(X_L\). For any \(s \in S\) we identify \((F_s, J_X|_{F_s})\) with \((CP^1, i)\). Then the subset \(\overline{M}_{0,3}(X_L, J_X, F)\) of \(\overline{M}_{0,3}(X_L, J_X, F)\) consisting of stable maps with image in \(F_s\) regularizes the space \(\overline{M}_{0,3}(CP^1, i, A)\) in the sense of Definition 4.3 in [4]. Hence

\[
\Psi^{(X_L, \eta)}(pt; [Z_0], [Z_\infty], pt) = \Psi^{(CP^1, \omega_{FS})}(pt; pt, pt, pt) = 1.
\]

Here we have used \(pt\) to stand for the point classes in the different spaces. Lemma 2.3 is proved.

Recall that for a closed symplectic manifold \((M, \omega)\) and homology classes \(\alpha_0, \alpha_{\infty} \in H_*(M, \mathbb{Q})\) we, in [3, Def. 1.8], defined a number

\[
GW_g(M, \omega; \alpha_0, \alpha_{\infty}) \in (0, +\infty]
\]

by the infimum of the \(\omega\)-areas \(\omega(A)\) of the homology classes \(A \in H_2(M; \mathbb{Z})\) for which the Gromov-Witten invariant \(\Psi_{A,g,m+2}(C; \alpha_0, \alpha_{\infty}, \beta_1, \cdots, \beta_m) \neq 0\) for some homology classes \(\beta_1, \cdots, \beta_m \in H_* (M; \mathbb{Q})\) and \(C \in H_*(\overline{M}_{g,m+2}; \mathbb{Q})\) and integer \(m \geq 0\). Moreover, in Definition 1.25 of [3] we also defined another number \(GW(M, \omega) \in (0, +\infty]\) by

\[
GW(M, \omega) = \inf GW_g(M, \omega; pt, \alpha),
\]

where the infimum is taken over all nonnegative integers \(g\) and all homology classes \(\alpha \in H_* (M; \mathbb{Q}) \setminus \{0\}\) of degree \(\deg \alpha \leq \dim M - 1\). Theorem 1.26 in [3] claimed that

\[
W_C(M, \omega) \leq GW(M, \omega)
\]

for any symplectic uniruled manifold \((M, \omega)\) of dimension at least 4.
**Proof of Theorem 1.2.** By the decomposition Theorem 2.6.A in [1], $(M \setminus \Delta_P, \omega)$ is symplectomorphic to the standard symplectic disc bundle $(E_{N_{\Sigma}}, \frac{1}{k} \omega_{can}) \to (\Sigma, \sigma = k\omega|_{\Sigma})$ over $\Sigma$, which is modeled on the normal bundle $N_{\Sigma}$ and with fibres of area $1/k$. Assume that $\varphi(B(\lambda)) \cap \Delta_P = \emptyset$ for some symplectic embedding $\varphi : B^{2n}(\lambda) \to (M, \omega)$ with $\lambda^2 \geq \frac{1}{k\pi}$. Then there exists a symplectic embedding $\psi : B(\lambda) \to (E_{N_{\Sigma}}, \frac{1}{k} \omega_{can})$. Since $\psi(B(\lambda))$ is compact there exists a positive number $\rho \in (0, 1)$ such that $\psi(B(\lambda)) \subset E_{N_{\Sigma}}(\rho)$. Lemma 2.2 gives a symplectic embedding from $\psi(B(\lambda))$ into $(X_{N_{\Sigma}}, \eta_\rho)$. Hence

\begin{equation}
W_G(X_{N_{\Sigma}}, \frac{1}{k} \eta_\rho) \geq \pi \lambda^2 \geq \frac{1}{k}.
\end{equation}

On the other hand, by the definition of $GW(M, \omega)$ and Lemmas 2.2, 2.3 we have

$$GW(X_{N_{\Sigma}}, \eta_\rho) \leq \int_{F_\rho} \eta_\rho < 1$$

for this $\rho$. This and (1) together give

$$W_G(X_{N_{\Sigma}}, \frac{1}{k} \eta_\rho) \leq GW(X_{N_{\Sigma}}, \frac{1}{k} \eta_\rho) \leq \frac{1}{k} \int_{F_\rho} \eta_\rho < \frac{1}{k},$$

which contradicts (2). Theorem 1.2 is proved. \hfill \Box

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**References**


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