SOME REMARKS ON AN EXISTENCE PROBLEM FOR DEGENERATE ELLIPTIC SYSTEMS

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ABSTRACT. The system
\[ au_x + bu_y = v_y, \quad cu_x + du_y = -v_x, \]
which yields Beltrami’s system if \( b = c \), is considered. Under a condition for the coefficients \( a, b, c, d \) a non-existence theorem is proved.

1. Main results

Let \( D, \ D \subset \mathbb{R}^2 \) be simply connected domains of the \( z = (x, y) \) and the \( w = (u, v) \) planes, respectively. Below we consider the problem of existence of homeomorphic \( ACL_{loc}^2 \) mappings \( w = w(z) \) from \( D \) onto \( D \) satisfying the system
\[ (1.1) \quad a u_x + b u_y = v_y, \quad c u_x + d u_y = -v_x. \]
A function \( h \) is, by definition, in the class \( ACL_{loc}^2 \) iff \( h \) is absolutely continuous along a.e. horizontal and vertical lines and its partial derivatives \( h_x, h_y \) belong to \( L_{loc}^2 \). Here the coefficients \( a, b, c, d \) are measurable functions in \( D \). We set
\[ \delta \equiv ad - (b + c)^2 / 4. \]
In what follows we assume that \( a > 0 \) and
\[ (1.2) \quad \text{ess inf}_{D'} (\delta) > 0 \quad \text{for every} \quad D' \subset \subset D. \]

In the particular case \( a = d = 1 \) and \( b = c = 0 \), (1.1) is the classic Cauchy-Riemann system and the solution of the existence problem is given by the Riemann mapping theorem. In the case \( b = c \), we have the well-known Beltrami system. For more on the existence theorems given here see, for example, G. David [5], M.A. Brakalova and J.A. Jenkins [2], U. Srebro and E. Yakubov [14], V. Gutiyauskii, O. Martio, T. Sugawa and M. Vuorinen [8], O. Martio and V. M. Miklyukov [10].

In the general case, the global mapping problem is much more complicated than in the aforementioned particular cases. Under the assumption
\[ (1.3) \quad \text{ess inf}_{D} \delta > 0, \]
the existence problem was solved by B. Bojarski [1]. If this condition is not fulfilled, then there are isolated results only. The case in which (1.3) is violated at a finite number of boundary points was considered by A. Džuraev [6] and E.A. Chicherbakov [4]. On the other hand, I.S. Ovchinnikov [13] and A.P. Mikhailov [12] proved some results pertaining to the solvability problem of the system.
(1.1) with (1.2) under some special conditions, which allow degeneration close to a boundary arc.

Here we give a condition for the coefficients of (1.1) under which the coordinate function $u(z)$ of $w(z)$ is monotone close to the boundary [11]. We use this condition for a non-existence theorem.

Let $D \subset \mathbb{R}^2$ be a domain. By $\partial D$ we denote the boundary of $D$ in the extended plane $\mathbb{R}^2 = \mathbb{R}^2 \cup \{\infty\}$. For an arbitrary subdomain $\Delta \subset D$, we set

$$\partial'\Delta = \partial \Delta \setminus \partial D \quad \text{and} \quad \partial''\Delta = \partial \Delta \cap \partial D.$$  

Let $\Gamma$ be a subset of $\partial D$. A continuous function $f : D \to \mathbb{R}$ is called monotone close to $\Gamma$ if for every subdomain $\Delta \subset D$ with $\partial''\Delta \subset \Gamma$,

$$\text{osc}(f, \Delta) \leq \text{osc}(f, \partial'\Delta).$$

Here the symbol $\text{osc}(f, E)$ means the oscillation of $f$ along the set $E$.

Every function $f$ monotone close to $\Gamma$ is monotone in the classical sense of Lebesgue since for domains $\Delta \subset D$, (1.3) reduces to $\text{osc}(f, \Delta) \leq \text{osc}(f, \partial\Delta)$. On the other hand, if $\Gamma = \partial D$, then every function monotone close to $\Gamma$ is a constant function. This is obvious since choosing $\Delta = D \setminus \{z_0\}$, where $z_0 \in D$ is an arbitrary point, by (1.4) we obtain

$$\text{osc}(f, \Delta) \leq \text{osc}(f, \{z_0\}) = 0.$$  

We shall describe the behaviour of the coefficients of the system close to a set of degeneracy by means of a special exhaustion function $h$. Fix a set $\Gamma \subset \partial D$ and a positive locally Lipschitz function $h : D \to \mathbb{R}$ such that $\lim_{z \to \Gamma} h(z) = 0$ and

$$0 < h_0 \leq \text{ess inf}_D |\nabla h(z)| \leq \text{ess sup}_D |\nabla h(z)| \leq h_{1} < \infty,$$

where $h_0$, $h_1$ are some constants.

Below we let $E_t = \{z \in D : h(z) = t\}$ denote the level curve of $h$.

**Theorem 1.6.** Let $\Gamma \subset \partial D$ be an arbitrary set, and let $w = (u, v)$ be an $A^{CL}_{loc}^2$ homeomorphic solution of (1.1) from $D$ onto $\mathcal{D}$ satisfying (1.2) such that $|u| < M$ in $D$. If

$$\int_0^t dt \left( \int_{E_t \cap D} (a + d) \frac{ad - bc}{t} d\mathcal{H}^1(E_t) \right)^{-1} = \infty,$$

then $u$ is monotone close to $\Gamma$.

For a homeomorphism $w : D \to \mathcal{D}$ and for an arbitrary $\Gamma \subset \partial D$ we set

$$w(\Gamma) = \{y \in \partial \mathcal{D} : \exists a \text{ sequence } z_n \in D, z_n \to \Gamma, \text{ such that } w(z_n) \to y\}.$$  

Clearly, if $\Gamma \subset \partial D$ is connected and $\partial D$ is a simple Jordan curve, then $w(\Gamma)$ is also connected.

A set $L \subset \partial D$ is called $u$-forked if there are at least two points $w' = (u', v')$, $w'' = (u', v'') \in L$ such that the segment

$$l = \{(u, v) \in \mathbb{R}^2 : u = u', v' < v < v''\}$$  

lies in $\mathcal{D}$ and separates from $\partial D \setminus L$ some subdomain $U \subset \mathcal{D}$ with $\partial U \subset \Gamma \cup L$.

For example, let $\mathcal{D}$ be a disk and let $\partial D$ be its boundary circle. Then the right and left semicircles are $u$-forked; however, the upper and lower semicircles are not $u$-forked.
Theorem 1.8. Let $M > 0$ be a constant. Suppose that $D$ is a subdomain of \{(u, v) : |u| < M\}, the coefficients of (1.1) satisfy (1.7) and $L \subset \partial D$ is $u$-forked. Then there is no $ACL^2_{\text{loc}}$ homeomorphic solution $w = w(z)$ of (1.1) from $D$ onto $\mathcal{D}$ such that $w(\Gamma) \supset L$.

2. Proof of Theorem 1.6

Let $\Delta$ be a subdomain of $D$ with $\partial'' \Delta \subset \Gamma$. We shall prove that

(2.1) $\sup_{\Delta} u(z) = \sup_{\partial' \Delta} u(z)$.

Assume the contrary, that is, there exists a point $z_0 \in \Delta$ such that

$u(z_0) > \sup_{\partial' \Delta} u(z) = A$.

Choose $\epsilon > A$ such that $u(z_0) > \epsilon$. Fix the connected component $U$ of the set $\{z \in \Delta : u(z) > \epsilon\}$ containing $z_0$. By [15, Theorem 5.4.4] for almost all $\epsilon > A$, the sets $\{z \in \Delta : u(z) = \epsilon\}$ are locally rectifiable. Therefore, without loss of generality, we may assume that $\partial' \Delta$ is locally rectifiable.

Fix numbers $0 < \delta' < \delta'' < h(z_0)$ and an absolutely continuous function $\psi_0 : [\delta', \delta''] \rightarrow [0, 1]$ such that $\psi_0(\delta'') = 1$ and $\psi_0(\delta') = 0$. We shall specify $\psi_0$ later. Define $\psi : (0, \infty) \rightarrow \mathbb{R}$ as

\[
\psi(\tau) = \begin{cases} 
1 & \text{for } \delta'' < \tau < \infty, \\
\psi_0(\tau) & \text{for } \delta' \leq \tau \leq \delta'', \\
0 & \text{for } 0 < \tau < \delta'.
\end{cases}
\]

Then $\psi$ is an absolutely continuous function. Write $\phi(z) = \psi(h(z))^2 (u(z) - \epsilon)$ for $z \in U$ and $\phi \equiv 0$ for $z \in D \setminus U$. Using [9, Theorem 1.20] we conclude that $\phi \in ACL^2_{\text{loc}}(D)$. Because $\text{supp} \phi \subset \subset D$ we have by Green’s formula

\[
\int_U d(\phi dv) = 0
\]

and hence

\[
\int_U d\phi \wedge dv = 0.
\]

Because

\[
d\phi \wedge dv = \psi^2 du \wedge dv + 2\psi(u - \epsilon) d\psi \wedge dv,
\]

we obtain

\[
\int_U \psi^2(u_x v_y - u_y v_x) dx dy = -2 \int_U \psi(u - \epsilon) (\psi_x v_y - \psi_y v_x) dx dy.
\]

This is clear for $u, v \in C^2(D)$. In the general case we can easily prove it if we use a standard approximation procedure (see, for example, [9, Lemma 14.15]).

From here

\[
\int_U \psi^2(u_x v_y - u_y v_x) dx dy \leq 2 \int_U |\psi(u - \epsilon)| |\nabla \psi| |\nabla v| dx dy.
\]
Using (1.1) we find
\[
\int_U \psi^2 (au^2_x + (b + c)u_xu_y + du^2_y) \, dxdy \\
\leq 4M \int_U \psi |\nabla \psi| \sqrt{(au_x + bu_y)^2 + (cu_x + du_y)^2} \, dxdy \\
= 4M \int_U \psi |\nabla \psi| \sqrt{(a^2 + c^2)^2u^2_x + 2(ab + cd)u_xu_y + (b^2 + d^2)u^2_y} \, dxdy \\
\leq 4M \left( \int_U |\nabla \psi|^2 \frac{(a^2 + c^2)^2u^2_x + 2(ab + cd)u_xu_y + (b^2 + d^2)u^2_y}{au^2_x + (b + c)u_xu_y + du^2_y} \, dxdy \right)^{1/2} \\
\times \left( \int_U \psi^2 (au^2_x + (b + c)u_xu_y + du^2_y) \, dxdy \right)^{1/2}.
\]

Thus we obtain
\[
\begin{align*}
\int_U \psi^2 (au^2_x + (b + c)u_xu_y + du^2_y) \, dxdy \\
\leq 16M^2 \int_U |\nabla \psi|^2 \frac{(a^2 + c^2)^2u^2_x + 2(ab + cd)u_xu_y + (b^2 + d^2)u^2_y}{au^2_x + (b + c)u_xu_y + du^2_y} \, dxdy.
\end{align*}
\]

We write
\[
I(\xi, \eta) = (a^2 + c^2)\xi^2 + 2(ab + cd)\xi\eta + (b^2 + d^2)\eta^2
\]
and
\[
II(\xi, \eta) = (a\xi^2 + (b + c)\xi\eta + d\eta^2).
\]

From (1.2) it follows that the quadratic form $I(\xi, \eta)$ is positive definite, and hence the bundle of quadratic forms
\[
H(\xi, \eta; \lambda) = I(\xi, \eta) - \lambda II(\xi, \eta)
\]
is regular in the sense of [7, Chapter X].

Let $\lambda_{\text{max}}$ be a maximal eigenvalue of $H(\xi, \eta; \lambda)$. Then using well-known properties of quadratic forms (see [7, Theorem 13, Chapter X]) for every $\xi, \eta \neq 0$, we have
\[
\frac{I(\xi, \eta)}{II(\xi, \eta)} \leq \lambda_{\text{max}}.
\]

The characteristic equation of $H(\xi, \eta; \lambda)$ has the form
\[
\begin{vmatrix}
a^2 + c^2 - \lambda a & ab + cd - \lambda(b + c)/2 \\
ab + cd - \lambda(b + c)/2 & b^2 + d^2 - \lambda d
\end{vmatrix} = 0,
\]
i.e.,
\[
\delta \lambda^2 - (a + d)(ad - bc) \lambda + (ad - bc)^2 = 0.
\]
Solving this equation for \( \lambda \) and using (1.2) we find that
\[
\lambda_{\text{max}} \leq \mu = \frac{(ad - bc)(a + d)}{\delta}.
\]
Clearly, \( ad - bc > 0 \) since from (1.2) it follows that
\[
0 < 4ad - (b + c)^2 \leq 4ad - 4bc.
\]
By (2.3) for every \((x, y) \in U\), we obtain
\[
(2.4) \quad \frac{(a^2 + c^2)u_x^2 + 2(ab + cd)u_xu_y + (b^2 + d^2)u_y^2}{au_x^2 + (b + c)u_xu_y + du_y^2} \leq \mu.
\]
Thus from (2.2) we arrive at the estimate
\[
(2.5) \quad \int_U \psi^2 (au_x^2 + (b + c)u_xu_y + du_y^2) \, dx \, dy \leq 16M^2 \int_U \mu |\nabla \phi|^2 \, dx \, dy.
\]
Now we shall estimate the integral in the right side of (2.5). We have
\[
I \equiv \int_U \mu |\nabla \phi|^2 \, dx \, dy = \int_{\nu < h(z) < \nu''} \mu \psi_0^2 (h(z)) |\nabla h|^2 \, dx \, dy
\]
and next, by the well-known Kronrod-Federer coarea formula [3, Chapter 3, §2.4],
\[
I = \int_{\delta'}^{\delta''} \psi_0^2 (t) \, dt \int_{E_t \cap U} \mu |\nabla h| \, d\mathcal{H}^1 (E_t).
\]
Using (1.5) we find
\[
(2.6) \quad I \leq h_1 \int_{\delta'}^{\delta''} \psi_0^2 (t) \, dt \int_{E_t \cap U} \mu \, d\mathcal{H}^1 (E_t).
\]
Since \( \psi_0 (\delta') = 1 \) and \( \psi_0 (\delta'') = 0 \), we have
\[
1 \leq \left( \int_{\delta'}^{\delta''} |\psi_0'| \, dt \right)^2 \leq \int_{\delta'}^{\delta''} |\psi_0'|^2 \, dt \int_{E_t \cap U} \mu \, d\mathcal{H}^1 (E_t)
\times \int_{\delta'}^{\delta''} \, dt \left( \int_{E_t \cap U} \mu \, d\mathcal{H}^1 (E_t) \right)^{-1}.
\]
Therefore, for every absolutely continuous function \( \psi_0 : [\delta', \delta''] \rightarrow [0, 1] \) such that \( \psi_0 (\delta'') = 1 \) and \( \psi_0 (\delta') = 0 \), we have
\[
\left( \int_{\delta'}^{\delta''} \, dt \left( \int_{E_t \cap U} \mu \, d\mathcal{H}^1 (E_t) \right)^{-1} \right)^{-1} \leq \int_{\delta'}^{\delta''} |\psi_0'|^2 \, dt \int_{E_t \cap U} \mu \, d\mathcal{H}^1 (E_t).
\]
Choosing for \( \delta' \leq \tau \leq \delta'' \),

\[
\psi_0(\tau) = \left( \int_{E_t \cap U} \mu \, d\mathcal{H}^1(E_t) \right)^{-1} \int_{\delta'}^{\delta''} dt \left( \int_{E_t \cap U} \mu \, d\mathcal{H}^1(E_t) \right)^{-1},
\]

we have

\[
\min_{\psi_0} \int_{\delta'}^{\delta''} |\psi_0|^2 dt \int_{E_t \cap U} \mu \, d\mathcal{H}^1(E_t) = \left( \int_{\delta'}^{\delta''} dt \left( \int_{E_t \cap U} \mu \, d\mathcal{H}^1(E_t) \right)^{-1} \right)^{-1}.
\]

Thus from (2.5) and (2.6) we see that

\[
\int_{\delta''}^{\delta'} |h(z)| \, dz \leq 16 M^2 h_1 \left( \int_{\delta'}^{\delta''} dt \left( \int_{E_t \cap U} \mu \, d\mathcal{H}^1(E_t) \right)^{-1} \right)^{-1}.
\]

Setting here \( \delta' \to 0 \) we obtain

\[
\int_{\delta''}^{\delta'} |h(z)| \, dz \leq 16 M^2 h_1 \left( \int_{0}^{\delta''} dt \left( \int_{E_t \cap U} \mu \, d\mathcal{H}^1(E_t) \right)^{-1} \right)^{-1}.
\]

Previously we chose \( \delta'' < h(z_0) \) where \( z_0 \in U \). Thus the open set \( U_1 = \{ z \in U : \delta'' < h(z) \} \) is not empty. Now from (1.7) we conclude that

\[
\int_{U_1} (au_x^2 + (b + c)u_xu_y + du_y^2) \, dxdy = 0,
\]

that is,

\[
a u_x^2 + (b + c)u_xu_y + du_y^2 = 0 \quad \text{a.e. in } U_1.
\]

The assumption (1.2) implies that \( \nabla u = 0 \) a.e. in \( U_1 \) and hence \( u \equiv \text{const} \) in \( U_1 \). But \( \delta'' > \delta' > 0 \) can be arbitrarily small, and hence \( \nabla u = 0 \) a.e. in \( U \). There is a contradiction with the definition of the connected component \( U \) and (2.1) is true.

Analogously, we obtain

\[
\inf_{\Delta} u(z) = \inf_{\partial \Delta} u(z)
\]

and next, we arrive at (1.4). \( \square \)

3. PROOF OF THEOREM 1.5

Suppose that there exists a homeomorphic solution \( w(z) \) of \( 1.1 \), \( 1.2 \) such that \( w(D) = D \) and \( L \subset w(\Gamma) \). Since \( L \) is \( w \)-forked, it follows that there is a vertical segment \( l \subset D \) with the end points in \( L \) and moreover, there exists a subdomain
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U of D with the boundary \( \partial \bar{U} \subset T \cup L \). Consider its preimage \( \Delta = w^{-1}(U) \). By
Theorem 1.4 the function \( u \) is monotone close to \( \Gamma \). Hence,

\[
\text{osc} (u, \Delta) \leq \text{osc} (u, \partial \bar{\Delta}) = \text{osc} (u, w^{-1}(l)) = 0.
\]

Thus, \( u \equiv \text{const} \) in \( \Delta \), which is impossible. \( \square \)

4. AN EXAMPLE

Let \( \gamma \) be a simple open Jordan arc lying in the upper half-plane with end points (0,0) and (1,0) on the x-axis. We set

\[
\Gamma = \{(x, y) : 0 \leq x \leq 1, y = 0\},
\]

and denote by \( D \) the subdomain of \( \mathbb{R}^2 \) enclosed by \( \gamma \cup \Gamma \).

Choose in Theorem 1.8 the function \( h(z) = y \). Evidently, (1.5) is fulfilled.

Consider the case of (1.1) in which the functions \( a = a(y) \), \( d = d(y) \) are positive and \( b = c = 0 \). Then (1.1) admits the form

\[
(4.1) \quad a(y) u_x = v_y, \quad d(y) u_y = -v_x.
\]

The assumption (1.2) is fulfilled too.

The assumption (1.7) takes the form

\[
(4.2) \quad \int_0^1 \frac{dy}{a(y) + d(y)} = \infty.
\]

We obtain:

Corollary 4.3. Let \( D \) be a simply connected subdomain of \( \{(u, v) : |u| < M\} \), where \( 0 < M < \infty \) is a constant. Assume that (1.2) is fulfilled and \( L \subset \partial D \) is \( u \)-forked. Then there are no ACL\(_{\text{loc}} \) solutions \( w = w(z) \) of (4.1) mapping \( D \) homeomorphically onto \( D \) such that \( L \subset w(\Gamma) \).

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