S-IN Variant SUBSPACES OF L^p(T)

D. A. REDETT

(Communicated by Joseph A. Ball)

Abstract. In this note, we give a new proof of the characterization of the S-invariant subspaces of L^p(T) for p in \( P \equiv \{ p : 1 < p < \infty, p \neq 2 \} \) using ideas from approximation theory.

In this short note we give a new proof of the characterization of the S-invariant subspaces of L^p(T) = L^p(T, m) for p in \( P \equiv \{ p : 1 < p < \infty, p \neq 2 \} \) where S denotes the operator of multiplication by the coordinate function. Here, T denotes the unit circle in the complex plane and m denotes Lebesgue measure on T normalized so that m(T) = 1. The proof is based on well-known facts from approximation theory.

Henceforth, any time we make reference to L^p(T) or H^p(T) we assume p is in \( P \). We point out first that L^p(T) and all of the subspaces of L^p(T) are uniformly convex Banach spaces. Hence we may employ the following result found in [5].

Lemma 1. Let \( X \) be a uniformly convex Banach space and \( K \) a subspace of \( X \). Then for all \( x \) in \( X \) there corresponds a unique \( y \) in \( K \) satisfying \( \| x - y \| = \inf_{z \in K} \| x - z \| \). We call such a \( y \) the best approximate of \( x \) in \( K \).

We say \( w \) is orthogonal to \( K \) and write \( w \perp K \) if \( \| w \| \leq \| w + k \| \) for all \( k \) in \( K \). For \( K \) a subspace of L^p(T), then \( f \perp K \) if

\[
\int_T g |f|^{p-1} \text{sgn} f \, dm = 0
\]

for all \( g \) in \( K \), where \( \text{sgn} f \) is a complex measurable function of modulus 1 such that \( f = \text{sgn} f |f| \). A proof of this can be found in [5].

If \( f \) is in L^p(T) and \( f^* \) is the best approximate of \( f \) in \( K \), then \( g = f - f^* \) is orthogonal to \( K \). This remark suggests the following lemma, which is a corollary of our first lemma.

Lemma 2. Let \( X \) be a uniformly convex Banach space and \( K \) a subspace of \( X \). Then there exists an \( x \) in \( X \) such that \( x \perp K \). If \( K \) is a proper subspace, then \( x \) may be chosen such that \( x \neq 0 \).

We say an S-invariant subspace of L^p(T) is S-simply invariant if S(M) is a proper subspace of M. We also recall that H^p(T) = \{ f \in L^p(T) : \int_T z^n f \, dm = 0 \forall n > 0 \}.
Theorem 1. \( \mathcal{M} \) is an \( S \)-simply invariant subspace of \( L^p(T) \) if and only if \( \mathcal{M} = \phi H^p(T) \) with \( \phi \) unimodular.

Proof. If \( \mathcal{M} = \phi H^p(T) \) with \( \phi \) unimodular, then it is clear that \( \mathcal{M} \) is an \( S \)-simply invariant subspace of \( L^p(T) \). It remains to show the converse. Since \( S(\mathcal{M}) \) is a proper subspace of \( \mathcal{M} \) by Lemma [2] there exists a nonzero \( \phi \) in \( \mathcal{M} \) such that \( \phi \perp S(\mathcal{M}) \). There is no loss of generality if we choose \( \phi \) such that \( \|\phi\|_p = 1 \). So in particular, \( \phi \perp z^n \phi \) for all \( n > 0 \). That is,

\[
\int_T z^n |\phi|^p \, dm = 0
\]

for all \( n > 0 \). Taking complex conjugates we get

\[
\int_T z^n |\phi|^p \, dm = 0
\]

for all \( n \neq 0 \). So, \( |\phi| = 1 \) a.e. on \( T \). That is, \( \phi \) is unimodular. Since \( \phi \) is in \( \mathcal{M} \), so is \( z^n \phi \) for all \( n \geq 0 \). Therefore, \( \phi P \) is in \( \mathcal{M} \) for every polynomial \( P \). Since polynomials are dense in \( H^p(T) \) and \( |\phi| = 1 \) we get that \( \phi H^p(T) \subseteq \mathcal{M} \). It remains to show that \( \phi H^p(T) \) is all of \( \mathcal{M} \). Let \( \psi \) be an element of \( \mathcal{M} \) orthogonal to \( \phi H^p(T) \). Since \( \phi \) is unimodular, we get that \( \psi \overline{\phi} \) is in \( L^p(T) \). By the way we chose \( \phi \) we get that \( \phi \perp z^n \psi \) for all \( n > 0 \). That is,

\[
\int_T z^n \psi \overline{\phi} \, dm = 0
\]

for all \( n > 0 \). These two facts together give us that \( \psi \overline{\phi} \) is in \( H^p(T) \). That is, \( \psi \) is in \( \phi H^p(T) \). This can only happen if \( \psi = 0 \). Therefore, \( \mathcal{M} = \phi H^p(T) \) as desired. \( \Box \)

Corollary 1. \( \mathcal{M} \) is an \( S \)-invariant subspace of \( H^p(T) \) if and only if \( \mathcal{M} = \phi H^p(T) \) with \( \phi \) inner.

This is easy to see since every \( S \)-invariant subspace of \( H^p(T) \) is \( S \)-simply invariant and since unimodular plus analytic implies inner.

We say an \( S \)-invariant subspace of \( L^p(T) \) is \( S \)-doubly invariant if it is \( S \)-invariant but not \( S \)-simply invariant. That is, \( S(\mathcal{M}) = \mathcal{M} \). So in particular, \( \mathcal{M} \) is invariant under both \( S \) and \( S^{-1} \).

Theorem 2. \( \mathcal{M} \) is an \( S \)-doubly invariant subspace of \( L^p(T) \) if and only if \( \mathcal{M} = 1_E L^p(T) \) where \( E \) is a measurable subset of \( T \).

By \( 1_E \) we mean a function that takes the value 1 on \( E \) and 0 on \( E^c \).

Proof. If \( \mathcal{M} = 1_E L^p(T) \), then it is clear that \( \mathcal{M} \) is an \( S \)-doubly invariant subspace of \( L^p(T) \). It remains to show the converse. If \( 1_T \) is in \( \mathcal{M} \), then \( \mathcal{M} = L^2(T) \) and we are done. So we may assume \( 1_T \) is not in \( \mathcal{M} \), and let \( q \) denote the best approximate of \( 1_T \) in \( \mathcal{M} \). Then \( 1_T - q \) is orthogonal to \( \mathcal{M} \). So in particular, \( (1_T - q) \perp z^n q \) for all \( n \in \mathbb{Z} \). That is,

\[
\int_T z^n q |1_T - q|^{p-1} \text{sgn} (1_T - q) \, dm = 0
\]

for all \( n \in \mathbb{Z} \). So, \( q |1_T - q|^{p-1} \text{sgn} (1_T - q) = 0 \) a.e. Let \( E = \{ z \in T : q(z) = 1 \} \). Then on \( E^c \), \( q = 0 \) a.e. That is, \( q = 1_E \). \( 1_E L^p(T) \) is the smallest \( S \)-doubly invariant subspace of \( L^p(T) \) containing \( 1_E \). Therefore, \( 1_E L^p(T) \subseteq \mathcal{M} \). It remains
to show that $1_E L^p(T) = M$. Let $g$ be an element of $M$ orthogonal to $1_E L^p(T)$. So in particular, $g \perp 1_E z^n$ for all $n \in \mathbb{Z}$. That is,

$$\int_T z^n 1_E |g|^{p-1} \text{sgn} g \, dm = 0$$

for all $n \in \mathbb{Z}$. So, $1_E |g|^{p-1} \text{sgn} g = 0$ a.e. Therefore, $g = 0$ on $E$. It remains to show that $g = 0$ on $E^c$. Since $g$ is in $M$, so are $z^n g$ for all $n \in \mathbb{Z}$. So, $1_T - 1_E \perp z^n g$ for all $n \in \mathbb{Z}$. That is,

$$\int_T z^n g |1_T - 1_E|^{p-1} \text{sgn} (1_T - 1_E) \, dm = 0$$

for all $n \in \mathbb{Z}$. So, $g |1_T - 1_E|^{p-1} \text{sgn} (1_T - 1_E) = 0$ a.e. On $E^c$ we see that $g = 0$. Therefore, $g = 0$ a.e. So, $M = 1_E L^p(T)$ as desired. □

Remark 1. The above proofs apply equally well for the case $p = 2$, in which case the best approximation operator is then just a Hilbert space orthogonal projection. However, this proof is well known (see [3]).

Remark 2. Another well-known proof for the case $p = 2$ involves the so-called “Wold Decomposition”. This approach is very useful. It applies equally well to Hilbert space shift operators of arbitrary multiplicity (see [2]) as well as for proving the characterization of certain invariant sub-Hilbert spaces of $H^p(T)$ and other Banach spaces of analytic functions (see [3]). These sub-Hilbert spaces are referred to as de Branges subspaces and have received some recent attention. The referee pointed out a generalized “Wold Decomposition” (see [1]) that applies to certain Banach spaces. Such a decomposition would be a useful tool both here and in possibly generalizing the de Branges subspace idea. However, it is unclear to us that such a decomposition may be employed here.

Remark 3. Finally, we point out that the above results are well known and the proofs known to us are found in [3]. It is shown in [3] that the $L^p$ results follow from the case $p = 2$. For $p < 2$ they use a density argument and for the case when $p > 2$ they employ a duality argument utilizing their result for $p < 2$.

References


Department of Mathematics, Texas A&M University, College Station, Texas 77843-3368

E-mail address: redett@math.tamu.edu