

ESSENTIAL NUMERICAL RANGE OF ELEMENTARY OPERATORS

M. BARRAA

(Communicated by Joseph A. Ball)

ABSTRACT. Let $A = (A_1, \dots, A_p)$ and $B = (B_1, \dots, B_p)$ denote two p -tuples of operators with $A_i \in \mathcal{L}(H)$ and $B_i \in \mathcal{L}(K)$. Let $R_{2,A,B}$ denote the elementary operators defined on the Hilbert-Schmidt class $\mathcal{C}^2(H, K)$ by $R_{2,A,B}(X) = A_1XB_1 + \dots + A_pXB_p$. We show that

$$\text{co}[(W_e(A) \circ W(B)) \cup (W(A) \circ W_e(B))] \subseteq V_e(R_{2,A,B}).$$

Here $V_e(\cdot)$ is the essential numerical range, $W(\cdot)$ is the joint numerical range and $W_e(\cdot)$ is the joint essential numerical range.

1. INTRODUCTION

Let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on a separable infinite-dimensional Hilbert space H . Let $A = (A_1, \dots, A_p)$ and $B = (B_1, \dots, B_p)$ denote two p -tuples of operators with $A_i \in \mathcal{L}(H)$ and $B_i \in \mathcal{L}(K)$. Let $R_{A,B} : \mathcal{L}(H) \rightarrow \mathcal{L}(H)$ denote the elementary operators defined by

$$R_{A,B}(X) = A_1XB_1 + \dots + A_pXB_p.$$

The class of Hilbert-Schmidt operators from a Hilbert space H to a Hilbert space K will be denoted by $\mathcal{C}^2(H, K)$ and, of course, $\mathcal{C}^2(H) = \mathcal{C}^2(H, H)$; see [8]. Recall that $\mathcal{C}^2(H, K)$ is a Hilbert space and that $A_iXB_i \in \mathcal{C}^2(H, K)$ for every $A_i \in \mathcal{L}(H)$, $X \in \mathcal{C}^2(H, K)$ and $B_i \in \mathcal{L}(K)$. So the elementary operator $R_{A,B}$ is a bounded endomorphism of $\mathcal{C}^2(H, K)$. We denote by $R_{2,A,B}$ the restriction of $R_{A,B}$ to $\mathcal{C}^2(H, K)$.

If \mathcal{A} is a Banach algebra with unit e , the algebraic numerical range of an arbitrary element $a \in \mathcal{A}$ is defined by

$$V(a) = \{f(a); f \in \mathcal{A}', \|f\| = f(e) = 1\}.$$

Here, of course, \mathcal{A}' denotes the space of all continuous linear functionals on \mathcal{A} . Recall that $V(a)$ is a compact convex set.

For $T \in \mathcal{L}(H)$, the numerical range of T is defined as

$$W(T) = \{\langle Tx, x \rangle : \|x\| = 1\}.$$

Received by the editors November 14, 2003 and, in revised form, February 13, 2004.

2000 *Mathematics Subject Classification*. Primary 47B47; Secondary 47A12.

Key words and phrases. Elementary operators, essential numerical range, Hilbert-Schmidt class.

©2004 American Mathematical Society
Reverts to public domain 28 years from publication

The essential numerical range, $V_e(T)$, is (by definition) the numerical range of the coset $T + K(H)$ in the Calkin algebra $\mathcal{L}(H)/K(H)$ where $K(H)$ is the ideal of all compact operators on H ; see [1, 2] and [9].

It is known that $V_e(T) \subseteq W(T)^-$, the closure of $W(T)$.

For a p -tuple $A = (A_1, \dots, A_p)$ of operators on a Hilbert space H we define:

- i) the joint numerical range of A by

$$W(A) = \{(\langle A_1 x, x \rangle, \dots, \langle A_p x, x \rangle); \quad x \in H, \|x\| = 1\};$$

- ii) the joint essential numerical range of A by
 $\lambda \in W_e(A)$ if $\lambda = (\lambda_1, \dots, \lambda_p) \in C^p$ and there exists an orthonormal sequence (x_n) in H such that $\lambda_i = \text{Lim} \langle A_i x_n, x_n \rangle$, $i = 1, \dots, p$.

To simplify the statements, we shall use the following notation: for $\alpha, \beta \in C^n$, we let $\alpha \circ \beta = \sum_{i=1}^p \alpha_i \beta_i$ and for $K, L \subseteq C^n$,

$$K \circ L = \{\alpha \circ \beta, \quad \alpha \in K, \beta \in L\}.$$

For vectors $x, y \in H$, the notation $x \otimes y$ will refer to the operator in $L(H)$ defined by $x \otimes y(z) = \langle z, y \rangle .x$.

In the past, elementary operators and their restrictions to norm ideals in $L(H)$ have been studied by many authors. Up to now, their spectra and their essential spectra have been characterized; see [4, 5, 3]. In [7], B. Magajna has determined the essential numerical range of the restriction of a generalized derivation to the class of Hilbert-Schmidt.

In this paper, we give some results about the essential numerical range of the restriction of an elementary operator to the class of Hilbert-Schmidt. More precisely, we prove that

$$\text{co}[(W_e(A) \circ W(B)) \cup (W(A) \circ W_e(B))] \subseteq V_e(R_{2,A,B}),$$

and we give some consequences of this inclusion.

2. THE ESSENTIAL NUMERICAL RANGE

We need the following characterization of the essential numerical range, obtained by Fillmore, Stampfli, and Williams in [9].

Lemma 2.1. *Let $T \in \mathcal{L}(H)$. Each of the following conditions is necessary and sufficient in order that $\lambda \in V_e(T)$:*

- (1) $\langle T x_n, x_n \rangle \rightarrow \lambda$ for some sequence (x_n) of unit vectors such that $x_n \rightarrow 0$ weakly.
- (2) $\langle T e_n, e_n \rangle \rightarrow \lambda$ for some orthonormal sequence (e_n) .

The main result of this paper is the following.

Theorem 2.2. *Let H, K be two separable Hilbert spaces and $A = (A_1, \dots, A_p)$, $B = (B_1, \dots, B_p)$ two p -tuples with $A_i \in \mathcal{L}(H)$ and $B_i \in \mathcal{L}(K)$ for $i = 1, \dots, p$. Then*

$$\text{co}[(W_e(A) \circ W(B)) \cup (W(A) \circ W_e(B))] \subseteq V_e(R_{2,A,B}).$$

Proof. Let $\lambda \in W_e(A)$. There exists an orthonormal sequence (x_n) in H such that

$$\lambda_i = \text{Lim} \langle A_i x_n, x_n \rangle \quad \text{for each } i = 1, \dots, p.$$

Let $\mu \in W(B)$. There exists a unit vector y in K such that $\mu_i = \langle B_i y, y \rangle$. It is easily verified that $(x_n \otimes y)$ is an orthonormal sequence in $C^2(H, K)$ and

$$\langle A_i(x_n \otimes y) B_i, x_n \otimes y \rangle = \text{tr}(A_i(x_n \otimes y) B_i(y \otimes x_n)) = \langle A_i x_n, x_n \rangle \cdot \langle B_i y, y \rangle.$$

Hence,

$$\langle R_{2,A,B}(x_n \otimes y), x_n \otimes y \rangle = \sum_{i=1}^p \langle A_i x_n, x_n \rangle \cdot \langle B_i y, y \rangle.$$

That is, $\lambda \circ \mu \in V_e(R_{2,A,B})$. □

The essential numerical range of the restriction of a generalized derivation to the class of Hilbert-Schmidt has been computed in [7], by B. Magajna. He has shown that

$$V_e(\delta_{2,A,B}) = co[(V_e(A) - W(B)^-) \cup (W(A)^- - V_e(B))].$$

Here we give only the easiest inclusion.

Corollary 2.3. For $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$,

$$co[(V_e(A) - W(B)^-) \cup (W(A)^- - V_e(B))] \subseteq V_e(\delta_{2,A,B}).$$

If, in addition, $V_e(A) = W(A)^-$ or $V_e(B) = W(B)^-$, then we have equality.

Corollary 2.4. For $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$,

$$\begin{aligned} co[(V_e(A).W(B)^-) \cup (W(A)^-.V_e(B))] &\subseteq V_e(M_{2,A,B}), \\ V_e(L_{2,A}) = W(A)^- \quad \text{and} \quad V_e(R_{2,B}) &= W(B)^-. \end{aligned}$$

Proof. We have $W(A)^- \subseteq V_e(L_{2,A}) \subseteq W(L_{2,A})^- = W(A)^-$. □

3. NONNEGATIVE OPERATORS AND THE ESSENTIAL NUMERICAL RANGE

Lemma 3.1. Let A be a nonnegative, selfadjoint operator and $AB = BA$. Then

$$(1) \quad V_e(AB) \subseteq V_e(A)V_e(B).$$

Proof. Let $\lambda \in V_e(AB)$. There exists a sequence (x_n) of unit vectors in H such that $x_n \rightarrow 0$ weakly and

$$\lambda = \text{Lim} \langle AB(x_n), x_n \rangle.$$

Let $y_n = A^{\frac{1}{2}}x_n$. If $y_{n_k} = 0$ for some subsequence, then 0 is in both sides of (1). If not and by passing to a subsequence if necessary, we can assume that $y_n \neq 0 \ \forall n$. Put $z_n = \frac{y_n}{\|y_n\|}$. Then (z_n) is a sequence of unit vectors with $z_n \rightarrow 0$ weakly and

$$\lambda = \text{Lim} \langle Bz_n, z_n \rangle \cdot \langle Ax_n, x_n \rangle.$$

But $\text{Lim} \langle Bz_n, z_n \rangle \in V_e(B)$. So $\lambda \in V_e(A)V_e(B)$. □

Corollary 3.2. Let $A \in \mathcal{L}(H)$ be a nonnegative, selfadjoint operator and $B \in \mathcal{L}(K)$. Then

$$V_e(M_{2,A,B}) \subseteq W(A)^-W(B)^-.$$

Proof. Recall that $L_{2,A}R_{2,B} = R_{2,B}L_{2,A}$, $V_e(L_{2,A}) = W(A)^-$ and $V_e(R_{2,B}) = W(B)^-$. The rest is from Lemma 3.1. □

ACKNOWLEDGEMENT

The author wishes to express his thanks to the referee for several helpful comments concerning this paper.

REFERENCES

1. F.F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and Elements of Normed Algebras*, London Math. Soc. Lecture Note Series 2, Cambridge, 1971. MR0288583 (44:5779)
2. F.F. Bonsall and J. Duncan, *Numerical ranges, vol II*, Cambridge University Press (1973). MR0442682 (56:1063)
3. R. Curto, *The spectra of elementary operators*, Indiana Univ. Math.J., **32**(1983), 193–197. MR0690184 (84e:47005)
4. L.A. Fialkow, *A note on norm ideals and the operator $X \rightarrow AX - XB$* , Israel J. Math., **32**(1979), 331–348. MR0571087 (81g:47046)
5. L. A. Fialkow, *Spectral properties of elementary operators*, Acta Sci. Math., **46**(1983), 269–282. MR0739043 (85h:47003)
6. C. K. Fong, *Normal operators on Banach space*, Glasgow Math. J. **20**(1979), 163–168. MR0536389 (80e:47020)
7. B. Magajna, *On the essential numerical range of a generalized derivation*, Proc. Amer. Math. Soc., **99**(1987), 86–92. MR0866435 (88e:47065)
8. R. Schatten, *Norm ideals of completely continuous operators*, Ergebnisse der Math. und ihrer Grenzgebiete, **27**, 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York, 1970. MR0257800 (41:2449)
9. J. Stampfli, P. Fillmore and J. Williams, *On the essential numerical range, the essential spectrum and a problem of Halmos*, Acta Sci. Math., **33**(1973), 172–192. MR0322534 (48:896)

DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES SEMLALIA, MARRAKECH, MAROC
E-mail address: `barraa@ucam.ac.ma`