PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 6, Pages 1587–1591
S 0002-9939(04)07673-7
Article electronically published on December 6, 2004

SOLUTIONS TO TWO QUESTIONS ABOUT THE WEYL ALGEBRAS

V. BAVULA

(Communicated by Martin Lorenz)

ABSTRACT. Affirmative answers are given to the following two questions about the Weyl algebras: a question of J. Alev: *Does the first Weyl algebra contain a non-noetherian subalgebra?*, and a question of T. Lenagan: *Is there a uniserial module M of length 2 over the Weyl algebra A_n with a holonomic submodule U such that $V = M/U$ is non-holonomic?*

1. THE FIRST WEYL ALGEBRA CONTAINS A LEFT AND RIGHT NON-NOETHERIAN SUBALGEBRA

Let K be a field of characteristic zero. The *(first) Weyl algebra* $A_1 = \langle x, \partial \,|\, \partial x - x\partial = 1\rangle$ is a simple noetherian domain. *Any commutative subalgebra of the Weyl algebra A_1 is a finitely generated algebra (hence noetherian)* [1, 7]. An old problem of Dixmier [7] which is still open (problem 4, p. 242) asks *whether for each K-derivation δ of the Weyl algebra A_1 the commutative graded algebra* $\mathrm{gr}(N(\delta)) := \bigoplus_{i\geq 0} \ker(\delta^i)/\ker(\delta^{i-1})$ *is finitely generated* (an affirmative answer to this problem would imply that the filtered subalgebra $N(\delta) = \bigcup_{i\geq 0} \ker(\delta^i)$ of the Weyl algebra A_1 is a finitely generated noetherian algebra). Many ring theorists (including the author) believe that the answer is affirmative. Numerous 'experimental facts' support this belief.

For quite some time the following question of J. Alev was open: *Does the first Weyl algebra contain a non-noetherian subalgebra?* Lemma 1.1 gives an example of such a subalgebra.

Let $B := K[h][x;\sigma]$ be a skew polynomial algebra with coefficients from a polynomial algebra $K[h]$ in one variable h where the K-algebra automorphism σ of $K[h]$ is given by the rule $\sigma(h) = h - 1$. So, $B = \bigoplus_{i\geq 0} K[h]x^i$ and

$$a(h)x^i\, b(h)x^j = a(h)\sigma^i(b(h))x^{i+j} = a(h)b(h-i)x^{i+j} \quad (1)$$

for all $i, j \geq 0$, where $a(h), b(h) \in K[h]$.

The algebra B is a subalgebra of the Weyl algebra A_1 via the algebra monomorphism

$$B \to A_1,\ x \mapsto x,\ h \mapsto \partial x.$$

Received by the editors December 20, 2003 and, in revised form, February 13, 2004.
2000 *Mathematics Subject Classification.* Primary 16S32, 16P40, 16W50, 16W70.

For each $i \geq 1$, let $K[h]_{<i} := \{a \in K[h] \mid \deg(a) < i\}$ and $K[h]_{<0} := K$. Obviously, $\sigma(K[h]_{<i}) = K[h]_{<i}$ for all $i \geq 0$. It follows that $C := \bigoplus_{i\geq 0} K[h]_{<i}x^i$ is a subalgebra of B.

Lemma 1.1. *The subalgebra C of the Weyl algebra A_1 is not a finitely generated algebra and is not a left and right noetherian algebra.*

Proof. The algebra $C = \bigoplus_{i\geq 0} C_i$ is a positively graded algebra where $C_i := K[h]_{<i}x^i$ with

$$C_iC_j = K[h]_{<i}\sigma^i(K[h]_{<j})x^{i+j} \subseteq K[h]_{<i}K[h]_{<j}x^{i+j} \subseteq K[h]_{<i+j-1}x^{i+j} \subset K[h]_{<i+j}x^{i+j} = C_{i+j},$$

a strict inclusion for all $i, j \geq 1$. It follows that the algebra C is not finitely generated and that the ideal $I := \bigoplus_{i\geq 1} C_i$ of the algebra C is not a finitely generated left (or right) C-module. This means that C is not a left (or right) noetherian algebra. □

2. Non-holonomic modules of length two over the Weyl algebra A_n

Let K be a field of characteristic zero. The *Weyl algebra* $A_n = A_n(K)$ is a K-algebra generated over the field K by $2n$ generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ subject to the defining relations:

$$\partial_i x_j - x_j\partial_i = \delta_{ij} \text{ (the Kronecker delta)}, \quad x_ix_j = x_jx_i, \quad \partial_i\partial_j = \partial_j\partial_i.$$

The Weyl algebra A_n is canonically isomorphic to the ring of differential operators with polynomial coefficients $K[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$ ($x_i \leftrightarrow x_i$, $\partial_i \leftrightarrow \frac{\partial}{\partial x_i}$).

In this section, $K = \mathbb{C}$ is the field of complex numbers. The Weyl algebra $A_n = A_n(\mathbb{C})$ is a simple Noetherian domain of Gelfand-Kirillov dimension $\mathrm{GK}\,(A_n) = 2n$, $A_n = A_1 \otimes \cdots \otimes A_1$, n times. J. Bernstein proved that $\mathrm{GK}\,(M) \geq n$ for any nonzero finitely generated A_n-module ([4] or [5], Chap. 1).

A finitely generated A_n-module M is called *holonomic* if $\mathrm{GK}\,(M) = n$. In the case of the first Weyl algebra A_1 all *simple* modules were classified by R. Block, [6] (a different approach is given in [3]) and all of them are holonomic A_1-modules. Let $M_1, \ldots, M_n$ be nonzero simple A_1-modules. Then their tensor product $\bigotimes_{i=1}^n M_i$ is a simple holonomic A_n-module. It was believed that all simple A_n-modules ($n \geq 2$) are holonomic, but T. Stafford gave an example of a simple non-holonomic A_n-module, [10].

Theorem 2.1 (Theorem 1.1, [10])**.** *For $2 \leq i \leq n$, pick $\lambda_i \in \mathbb{C}$ that are linearly independent over the field of rational numbers $\mathbb{Q}$. Then the element*

$$\alpha = x_1 + (\sum_2^n \lambda_i\partial_ix_i)\partial_1 + \sum_2^n (x_i + \partial_i)$$

generates a maximal left ideal of A_n. In particular, the simple A_n-module $A_n/A_n\alpha$ has Gelfand-Kirillov dimension $2n - 1$.

Denote by $St_n(\alpha)$ the Stafford's module $A_n/A_n\alpha$ from Theorem 2.1.

Question of Lenagan ([8])**.** *Is there a uniserial module M of length 2 over the Weyl algebra A_n with a holonomic submodule U such that $V = M/U$ is non-holonomic.*

Theorem 2.2 gives a positive answer to this question.

Given $\mu \in \mathbb{C}\backslash\mathbb{Z}$, then the A_1-module

$$S(\mu) = A_1/A_1(\partial_1 x_1 - \mu)$$

is simple, and two such modules are isomorphic, i.e., $S(\mu) \simeq S(\mu')$, iff $\mu - \mu' \in \mathbb{Z}$ ([3], Theorem 3.2).

Then the tensor product of the modules above

$$S(\mu_1, \dots, \mu_n) = \bigotimes_1^n S(\mu_i) \simeq A_n/A_n(\partial_1 x_1 - \mu_1, \dots, \partial_n x_n - \mu_n)$$

is a *simple holonomic* A_n-module and two such modules are isomorphic if and only if the corresponding tensor multiples are isomorphic as A_1-modules (that is, $\mu_i - \mu_j \in \mathbb{Z}$ for all i, j).

Theorem 2.2. *Let the A_n-modules $St_n(\alpha)$ and $S(\mu_1, \dots, \mu_n)$ be as above* ($n \geq 2$)*. Then*

$$\dim \operatorname{Ext}^1_{A_n}(St_n(\alpha), S(\mu_1, \dots, \mu_n)) = \infty.$$

Remark. G. Perets [9] gave an example of a uniserial module M of length 2 over the Weyl algebra A_n with a *non-holonomic* submodule U such that $V = M/U$ is *holonomic*.

2.1. **Proof of Theorem 2.2.** Let D be a ring with an automorphism σ and a central element a. The *generalized Weyl algebra* $A = D(\sigma, a)$ of degree 1 is the ring generated by D and two indeterminates X and Y subject to the defining relations [2]:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y, \ \forall \alpha \in D, \ YX = a \text{ and } XY = \sigma(a).$$

The algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a $\mathbb{Z}$-graded algebra where $A_n = Dv_n$, $v_n = X^n$ ($n > 0$), $v_n = Y^{-n}$ ($n < 0$), and $v_0 = 1$. It follows from the defining relations that

$$v_n v_m = (n, m) v_{n+m}$$

for some $(n, m) \in D$.

The first Weyl algebra is a generalized Weyl algebra

$$A_1 \simeq K[H](\sigma, a = H), \ \ x \leftrightarrow X, \ \ \partial \leftrightarrow Y, \ \ \partial x \leftrightarrow H,$$

with $D = K[H]$, the polynomial algebra in one variable, and $\sigma : H \to H - 1$. We say that an element

$$u = d_i v_i + d_{i+1} v_{i+1} + \dots + d_j v_j \in A_1, \ \text{ all } \ d_k \in D, \ \ d_i \neq 0, \ \ d_j \neq 0,$$

has *length* $l(u) = j - i$. Clearly, $l(uv) = l(u) + l(v)$ for $u, v \in A_1$.

For $\mathbf{k} := (k_1, ..., k_n) \in \mathbb{Z}^n$, let $v_{\mathbf{k}} = v_{k_1}(1) \cdots v_{k_n}(n)$ where for $1 \leq i \leq n$ and $m \geq 0$: $v_m(i) = (x_i)^m$, $v_{-m}(i) = (\partial_i)^m$, $v_0(i) = 1$. The Weyl algebra A_n is the tensor product of the first Weyl algebras $A_1^{(i)} = \mathbb{C}\langle x_i, \partial_i\rangle = \mathbb{C}[H_i](\sigma_i, H_i)$. It follows from the definition of generalized Weyl algebras that

$$A_n = \bigoplus_{\mathbf{k}\in\mathbb{Z}^n} A_{\mathbf{k}}$$

is a $\mathbb{Z}^n$-graded algebra ($A_{\mathbf{k}} A_{\mathbf{e}} \subset A_{\mathbf{k}+\mathbf{e}}, \forall \mathbf{k}, \mathbf{e} \in \mathbb{Z}^n$) where $A_{\mathbf{k}} = K[H_1, \dots, H_n] v_{\mathbf{k}}$ and $K[H_1, \dots, H_n]$ is the polynomial algebra in n variables. The $\mathbb{Z}^n$-grading above of the Weyl algebra A_n is the tensor product of the $\mathbb{Z}$-gradings of the tensor multiples $A_1^{(i)}$.

The A_1-module $S(\mu)$ $(\mu \notin \mathbb{Z})$ is a $\mathbb{Z}$-graded module:

$$S(\mu) = \bigoplus_{i\in\mathbb{Z}} S(\mu)_i, \quad S(\mu)_i = \mathbb{C}e_i, \quad e_i = v_i + A_1(H-\mu),$$
$$He_i = (\mu+i)e_i, \quad Xe_k = e_{k+1}, \quad Ye_{-k} = e_{-k-1} \ (k \geq 0),$$
$$Xe_{-m} = (\mu - m)e_{-m+1}, \quad Ye_m = (\mu + m - 1)e_{m-1} \ (m > 0).$$

We say that a nonzero element

$$s = s_ie_i + s_{i+1}e_{i+1} + \cdots + s_je_j \in S(\mu), \ \text{ all } \ s_k \in \mathbb{C}, \ \ s_i \neq 0, \ \ s_j \neq 0,$$

has *length* $l(s) = j - i$. Then $l(us) = l(u) + l(s)$ for $s \in S(\mu)$ and $u = d_mv_m + \cdots + d_nv_n \in A_1$ such that none of the scalars from the set $\mu + \mathbb{Z}$ is a root of the polynomial $d_md_n \neq 0$.

Consider the tensor product of the A_1-modules $S(\mu_i)$, $i = 1, \ldots, n$, above $(\mu_i \notin \mathbb{Z})$:

$$S = S(\mu_1, \ldots, \mu_n) = \bigotimes_{i=1}^{n} S(\mu_i).$$

It is a $\mathbb{Z}^n$-graded A_n-module

$$S = \bigoplus_{\mathbf{k}\in\mathbb{Z}^n} S_{\mathbf{k}}, \quad S_{\mathbf{k}} = S(\mu_1)_{k_1} \otimes \cdots \otimes S(\mu_n)_{k_n}, \quad \mathbf{k} = (k_1, \ldots, k_n),$$

and a $\mathbb{Z}$-graded A_n-module with respect to the *total* $\mathbb{Z}$-grading

$$S = \bigoplus_{j\in\mathbb{Z}} S_j, \quad S_j = \bigoplus \{\bigotimes_{i=1}^{n} S(\mu_i)_{m_i} \mid m_1 + \cdots + m_n = j\}.$$

Fix $k \in \{1, \ldots, n\}$. A nonzero element $s \in S$ can be uniquely written as a sum

$$s = s_ie_i(k) + s_{i+1}e_{i+1}(k) + \cdots + s_je_j(k), \quad s_i \neq 0, \quad s_j \neq 0,$$

where the coefficients s_t are from the tensor product $\bigotimes_{m\neq k} S(\mu_m)$ and $\{e_i(k)\}$ is the basis of the A_1-module $S(\mu_k)$. The k-*length* of the element s is defined as $l_k(s) = j - i$.

Let α be from Theorem 2.1 and $0 \neq s \in S$. Now it is easy to see that

$$l_k(\alpha s) = l_k(\alpha) + l_k(s) = 2 + l_k(s) \geq 2, \ \text{ for any } k > 1.$$

So, the linear map

$$\alpha_S : S \to S, \quad s \to \alpha s,$$

is injective and the cokernel $\operatorname{coker}(\alpha_S) \equiv S/\alpha S$ of the map α_S has

$$\dim\ \operatorname{coker}(\alpha_S) = \infty \tag{2}$$

since $\operatorname{im}(\alpha_S) \cap W = 0$ where $W = S(\mu_1) \otimes \cdots \otimes S(\mu_{k-1}) \otimes \bar{1} \otimes S(\mu_{k+1}) \otimes \cdots \otimes S(\mu_n)$, $\bar{1} = 1 + A_1(H_k - \mu_k)$, and $\dim(W) = \infty$.

Using the short exact sequence

$$0 \to A_n \to^{\cdot\alpha} A_n \to St(\alpha) \to 0,$$

we get $\operatorname{Ext}^1_{A_n}(St_n(\alpha), S) = \operatorname{coker}(\alpha_S)$; hence, by (2), $\dim\ \operatorname{Ext}^1_{A_n}(St_n(\alpha), S) = \infty$. This finishes the proof of Theorem 2.2. □

For a polynomial $f \in K[H_1, \ldots, H_n]$ let $V(f)$ be the set

$$\{(\lambda_1, \ldots \lambda_n) \in \mathbb{C}^n \ : \ f(\lambda_1, \ldots \lambda_n)\} = 0.$$

Corollary 2.3. *Suppose that* $u = \sum_{\mathbf{i}\in\mathbb{Z}^n} u_{\mathbf{i}} v_{\mathbf{i}} \in A_n$ *is not a homogeneous element where all* $u_{\mathbf{i}} \in K[H_1, \ldots, H_n]$*, the* A_n*-module* $S = \bigotimes_{i=1}^n S(\mu_i)$ *is as above (i.e. all* $\mu_i \in \mathbb{C}\backslash\mathbb{Z}$*) and*

$$\{\mathbb{Z}^n \cup ((\mu_1, \ldots, \mu_n) + \mathbb{Z}^n)\} \cap V(u_{\mathbf{i}}) = \emptyset, \ \textit{for all}\ \mathbf{i}. \tag{3}$$

Then the linear map $u_S : S \to S$*,* $s \to us$*, is injective and* $\dim \operatorname{coker}(u_S) = \infty$.

Proof. The element u is not homogeneous, so there exists k such that $l_k(u) \geq 1$. Now, by (3),

$$l_k(us) = l_k(u) + l_k(s) > 0 \ \text{ for any } 0 \neq s \in S.$$

So, the linear map u_S is injective and $\operatorname{im}(u_S) \cap W = 0$ where W is as above. Hence $\dim \operatorname{coker}(u_S) = \infty$. □

References

[1] S. A. Amitsur, Commutative linear differential operators, *Pacific J. Math.* **8** (1958), 1–10. MR0095305 (20:1808)

[2] V. V. Bavula, The finite-dimensionality of Ext^n's and Tor_n's of simple modules over a class of algebras, *Funct. Anal. and Appl.* **25** (1991), no. 3, 80–82. MR1139880 (92k:16019)

[3] V. V. Bavula, Generalized Weyl algebras and their representations, *Algebra i Analiz*, **4** (1992), no. 1, 75-97; English trans. in *St.Petersburg Math. J.* **4** (1993), no. 1, 71–92. MR1171955 (93h:16043)

[4] I. N. Bernstein, Modules over a ring of differential operators. An investigation of the fundamental solutions of equations with constant coefficients. (Russian) *Funkcional. Anal. i Prilozen.* **5** (1971) no. 2, 1–16. MR0290097 (44:7282)

[5] J.-E. Björk, *Rings of differential operators*. North Holland, Amsterdam, 1979. MR0549189 (82g:32013)

[6] R. E. Block, The irreducible representations of the Lie algebra $sl(2)$ and of the Weyl algebra, *Adv. Math.* **39** (1981), 69–110. MR0605353 (83c:17010)

[7] J. Dixmier, Sur les algèbres de Weyl, *Bull. Soc. Math. France* **96** (1968), 209–242. MR0242897 (39:4224)

[8] T. H. Lenagan, Stability of Gelfand-Kirillov dimension for rings with the strong second layer condition, *Proc. Edinburgh Math. Soc.* **37** (1994), 347–354. MR1280687 (95g:16019)

[9] G. S. Perets, d-critical modules of length 2 over Weyl algebras, *Israel J. of Math.* **83** (1993), 361–368. MR1239069 (94m:17007)

[10] J. T. Stafford, Nonholonomic modules over Weyl algebras and enveloping algebras, *Invent. Math.* **79** (1985), 619–638. MR0782240 (86h:17009)

Department of Pure Mathematics, University of Sheffield, Hicks Building, Sheffield S3 7RH, England

E-mail address: v.bavula@sheffield.ac.uk