

ON THE RESOLVABILITY OF LOCALLY CONNECTED SPACES

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ABSTRACT. We solve a problem of Padmavally about resolvability of locally connected spaces, in the case where the space under consideration is regular.

0. INTRODUCTION

As is well known, a space X is said to be *resolvable* (more generally, ν -*resolvable* for some cardinal $\nu \geq 2$) if it may be written as the union of two (respectively, ν -many) disjoint dense subspaces. The introduction of this property was essentially motivated by the discovery of crowded (i.e., without isolated points) topological spaces that fail to have it, even if this kind of space seems to be quite rare “in nature”. As a matter of fact, many topological spaces with “reasonable” properties (such as, for example, locally compact spaces, or k -spaces—hence also metrizable, first-countable, sequential and Čech-complete spaces) turn out to be resolvable, whenever they contain no isolated point. From this point of view, the notion of irresolvable space is to be considered, in the realm of crowded spaces, by the side of other stronger (and more “unnatural”) topological properties, like *maximality*, *submaximality*, and the notions of *nodec* and *door* space (see [AC] for an overview).

In this vein, it is natural to investigate if other well-known topological properties could imply resolvability. For example, the question of whether connectedness together with some suitable separation axiom implies non-submaximality (or non-maximality, or resolvability) is implicitly raised by Hewitt in 1943, since Theorem 16 of [He] shows that for every infinite cardinal number ν there is a connected submaximal T_1 space of cardinality ν (actually, the argument given by Hewitt is needlessly complicated, because taking on any infinite set X a non-principal ultrafilter as a topology for it, makes X a connected maximal T_1 space). Later on, Padmavally [Pad] and Simon [Si] gave (nontrivial) examples, respectively, of a connected submaximal Urysohn space and of a connected maximal Hausdorff space. However, the question of the existence of a regular connected irresolvable (or submaximal, maximal, nodec, etc.) infinite space is still open (cf. [AC, Problem 4.1]), and stands as one of the main problems in this area.

In [Pad, Theorem 2], Padmavally also proves that every Hausdorff submaximal space cannot be locally connected at any point, and in his reflections before such

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a result raises the question of the existence of Hausdorff or Urysohn locally connected irresolvable spaces. A first result in this vein is attributed to Yashenko in [Pav, considerations after Corollary 3.11], where it is stated that every Tychonoff connected space is resolvable—actually, \mathfrak{c} -resolvable. In the present paper we show that the hypothesis of complete regularity in Yashenko's result may be relaxed to regularity, even if in this case we can prove only ω -resolvability. Actually, this fact is obtained as a corollary of Theorem 2, which states (in one of its possible formulations) that every regular and non-trivial connected space, with a π -base consisting of connected sets, is ω -resolvable. As far as we know, our Theorem 2 and [Pav, Corollary 3.11] are the only results in the literature that show how connectedness and regularity, together with some reasonable supplementary assumptions, may entail resolvability.

1. THE MAIN RESULT

Lemma 1. *Let X be a regular space without isolated points. Then for every open subset Ω of X there exists a collection \mathcal{A} of pairwise disjoint nonempty open subsets of Ω , such that:*

- (1) $\forall A \in \mathcal{A}: \overline{A} \not\subseteq \Omega$;
- (2) $\bigcup \mathcal{A}$ is dense in Ω .

Proof. If $\Omega = \emptyset$, take $\mathcal{A} = \emptyset$. Otherwise, fix two nonempty disjoint open subsets A_1, A_2 of Ω , with $\overline{A_1}, \overline{A_2} \subseteq \Omega$. Then, automatically, strict inclusion also holds. Thus the collection Γ of all collections \mathcal{A}' of pairwise disjoint subsets of Ω , having property (1) and containing both A_1 and A_2 , is nonempty and inductive. By regularity, it is clear that a maximal element \mathcal{A} of Γ has both property (1) and property (2). \square

Theorem 2. *If X is a regular space without isolated points (in particular, if X is a connected regular space with more than one point), for which the nonempty connected open subspaces form a π -base, then X is ω -resolvable.*

Proof. By Lemma 1, it is possible to associate to every nonempty open subset Ω of X a collection $\mathcal{A}(\Omega)$ of nonempty open subsets of Ω , such that conditions (1) and (2) are satisfied. Put $\nu = |X|$ and define by transfinite induction, for every $\alpha \in \nu^+$, a collection \mathcal{A}_α of pairwise disjoint nonempty open subsets of X in the following way:

- a) $\mathcal{A}_0 = \{X\}$;
- b) $\forall \alpha \in \nu^+ : \mathcal{A}_{\alpha+1} = \bigcup_{\Omega \in \mathcal{A}_\alpha} \mathcal{A}(\Omega)$;
- c) for $\lambda \in \nu^+$ with λ limit and λ greater than 0, we consider the collection Δ_λ of all λ -sequences $\{\Omega_\alpha\}_{\alpha \in \lambda}$ such that $\Omega_\alpha \in \mathcal{A}_\alpha$ for every $\alpha \in \lambda$ and $\text{Int}(\bigcap_{\alpha \in \lambda} \Omega_\alpha) \neq \emptyset$; then we put $\mathcal{A}_\lambda = \{\text{Int}(\bigcap_{\alpha \in \lambda} \Omega_\alpha) \mid \{\Omega_\alpha\}_{\alpha \in \lambda} \in \Delta_\lambda\}$.

The fact that every \mathcal{A}_α actually consists of pairwise disjoint sets is easily proved by transfinite induction on α . Observe that each collection \mathcal{A}_α with $\alpha \in \omega$ is such that $\bigcup \mathcal{A}_\alpha$ is dense in X ; but this is not true any more, in general, when α is an arbitrary element of ν^+ . In particular, it is perfectly possible that some \mathcal{A}_α be empty (and in this case, of course, we will also have that $\mathcal{A}_{\alpha'} = \emptyset$ for $\alpha' > \alpha$).

Let us point out now some other properties of the collections \mathcal{A}_α .

- I) $\forall \alpha, \beta \in \nu^+, \alpha < \beta : \forall \Omega \in \mathcal{A}_\beta : \exists \Omega' \in \mathcal{A}_\alpha : \overline{\Omega} \not\subseteq \Omega'$.

Given $\hat{\alpha} \in \nu^+$, we prove the property by transfinite induction on $\beta > \hat{\alpha}$. Thus, suppose (I) holds for every β' with $\hat{\alpha} < \beta' < \beta$. If β is a successor ordinal, then

let $\beta = \beta^* + 1$: Given any $\Omega \in \mathcal{A}_\beta = \mathcal{A}_{\beta^*+1}$, Ω must belong to some $\mathcal{A}(\Omega')$ with $\Omega' \in \mathcal{A}_{\beta^*}$; hence $\overline{\Omega} \subsetneq \Omega'$ by the definition of $\mathcal{A}(\Omega')$. If $\beta^* = \hat{\alpha}$, then we are done; otherwise, we have $\hat{\alpha} < \beta^* < \beta$. Hence by the inductive hypothesis there exists $\Omega'' \in \mathcal{A}_{\hat{\alpha}}$ with $\overline{\Omega'} \subsetneq \Omega''$, so that $\overline{\Omega} \subsetneq \Omega' \subseteq \overline{\Omega'} \subsetneq \Omega''$. Suppose now that β is limit: then there is $\{\Omega_\gamma\}_{\gamma \in \beta} \in \Delta_\beta$ such that $\Omega = \text{Int}(\bigcap_{\gamma \in \beta} \Omega_\gamma)$. Taking β^* with $\hat{\alpha} < \beta^* < \beta$, we have on the one hand that $\Omega \subseteq \Omega_{\beta^*}$, and on the other hand that there exists (by the inductive hypothesis) an $\Omega' \in \mathcal{A}_{\hat{\alpha}}$ with $\overline{\Omega_{\beta^*}} \subsetneq \Omega'$. Therefore $\overline{\Omega} \subseteq \overline{\Omega_{\beta^*}} \subsetneq \Omega'$.

II) If $\Omega' \in \mathcal{A}_{\alpha'}$ and $\Omega'' \in \mathcal{A}_{\alpha''}$, with $\alpha' < \alpha''$, then either $\overline{\Omega''} \subsetneq \Omega'$ or $\overline{\Omega'} \cap \overline{\Omega''} = \emptyset$.

This is an easy consequence of (I) and of the fact that the elements of each \mathcal{A}_α are open and pairwise disjoint.

III) If A is an open connected subset of X , and $\alpha \in \nu^+$ is such that there are two distinct $\Omega', \Omega'' \in \mathcal{A}_\alpha$ that both meet A , then for every $n \in \omega$ there exists $\Omega \in \mathcal{A}_{\alpha+n}$ such that $A \cap \text{Fr } \Omega \neq \emptyset$.

The proof is by induction on n . For $n = 0$, we have that $A \cap \text{Fr } \Omega' \neq \emptyset$ —since otherwise $A \cap \Omega'$ and $A \setminus \overline{\Omega'}$ would be nonempty open complementary subsets of A (which is connected). Suppose now that $A \cap \text{Fr } \Omega \neq \emptyset$, with Ω belonging to some $\mathcal{A}_{\alpha+\bar{n}}$. Then A meets Ω , and at the same time it cannot be entirely contained in $\overline{\Omega}$ —otherwise, it would miss Ω' or Ω'' (use (II) if $\bar{n} > 0$, while for $\bar{n} = 0$ just consider that $\overline{\Omega}$ is disjoint from each other element of \mathcal{A}_α). By definition of the \mathcal{A}_α 's, we have that $\mathcal{A}(\Omega) \subseteq \mathcal{A}_{\alpha+\bar{n}+1}$; and since $\bigcup \mathcal{A}(\Omega)$ is dense in Ω , there exists $\Omega^\# \in \mathcal{A}(\Omega) \subseteq \mathcal{A}_{\alpha+\bar{n}+1}$ such that $(A \cap \Omega) \cap \Omega^\# = A \cap \Omega^\# \neq \emptyset$. At the same time, A is not included in $\overline{\Omega^\#}$ (because $\overline{\Omega^\#} \subseteq \Omega$); therefore, we may deduce as before that $A \cap \text{Fr } \Omega^\# \neq \emptyset$.

Now we are going to define ω many pairwise disjoint subsets of X , each of which will turn out to be dense. Let $\{N_m \mid m \in \omega\}$ be a partition of ω , consisting of infinite subsets and indexed in a one-to-one way. For every $m \in \omega$ we put

$$D_m = \bigcup \{ \text{Fr } \Omega \mid \exists \lambda \in \nu^+ : \exists n \in N_m : (\lambda \text{ is limit} \wedge \Omega \in \mathcal{A}_{\lambda+n}) \}.$$

The fact that $D_m \cap D_{m'} = \emptyset$ for $m \neq m'$ is an easy consequence of (II). To prove that each D_m is dense in X , first notice the following fact:

IV) If A is an open connected subset of X , and for some $\alpha \in \nu^+$ there are two distinct $\Omega', \Omega'' \in \mathcal{A}_\alpha$ both of which meet A , then $D_m \cap A \neq \emptyset$ for every $m \in \omega$.

Write α as $\lambda + \bar{n}$, with λ limit and $\bar{n} \in \omega$. By (III), for every $n \in \omega$ there is $\Omega \in \mathcal{A}_{\lambda+\bar{n}+n}$ such that $A \cap \text{Fr } \Omega \neq \emptyset$ —hence $A \cap D_m \neq \emptyset$, where $m \in \omega$ is such that $\bar{n} + n \in N_m$. Since for each $m \in \omega$ there is an $n \in \omega$ with $\bar{n} + n \in N_m$ (because N_m is infinite), we have the desired property.

Suppose now that B is any nonempty open subset of X , and let A be a nonempty open connected subset of B . Put $S = \{ \alpha \in \nu^+ \mid \exists \Omega \in \mathcal{A}_\alpha : A \subseteq \Omega \}$: Due to (I), S is an initial segment of ν^+ . Let, for every $\alpha \in S$, $\Omega_\alpha \in \mathcal{A}_\alpha$ be such that $A \subseteq \Omega_\alpha$: By (II) we have that $\overline{\Omega_{\alpha'}} \subsetneq \Omega_{\alpha'}$ for $\alpha', \alpha'' \in S$ with $\alpha' < \alpha''$ —in particular, each $\Omega_{\alpha+1}$ is strictly smaller than Ω_α . Since $\nu = |X|$, S cannot coincide with the whole of ν^+ , and it turns out to be an ordinal $\alpha^\# < \nu^+$. However, by the definition of the \mathcal{A}_λ 's for λ limit, we clearly have that $\alpha^\#$ must be a successor ordinal, i.e. $\alpha^\# = \alpha^* + 1$. Then $\mathcal{A}(\Omega_{\alpha^*}) \subseteq \mathcal{A}_{\alpha^*+1} = \mathcal{A}_{\alpha^\#}$, and since $\bigcup \mathcal{A}(\Omega_{\alpha^*})$ is dense in Ω_{α^*} and $A \subseteq \Omega_{\alpha^*}$, there exists $\Omega^\# \in \mathcal{A}_{\alpha^\#}$ such that $A \cap \Omega^\# \neq \emptyset$. Now, if $A \setminus \overline{\Omega^\#} \neq \emptyset$, then using again the density of $\bigcup \mathcal{A}(\Omega_{\alpha^*})$ in Ω_{α^*} we get another $\tilde{\Omega} \in \mathcal{A}_{\alpha^\#}$ with $(A \setminus \overline{\Omega^\#}) \cap \tilde{\Omega} \neq \emptyset$, so that by (IV) we have that $A \cap D_m \neq \emptyset$ (hence also $B \cap D_m \neq \emptyset$) for every

$m \in \omega$. Thus, suppose $A \subseteq \overline{\Omega^\#}$: Clearly A cannot be included in $\Omega^\#$ (because $\Omega^\# \in \mathcal{A}_{\alpha^\#}$ and $\alpha^\# \notin \alpha^\# = S$), so that $A \cap \text{Fr } \Omega^\# \neq \emptyset$. Since $A \cap \Omega^\# = A \setminus \text{Fr } \Omega^\#$ is a nonempty open subset of $\Omega^\#$, the density of $\bigcup \mathcal{A}(\Omega^\#)$ in $\Omega^\#$ implies that there is $\Omega^* \in \mathcal{A}(\Omega^\#) \subseteq \mathcal{A}_{\alpha^\#+1}$ with $(A \setminus \text{Fr } \Omega^\#) \cap \Omega^* \neq \emptyset$. Then $A \cap \text{Fr } \Omega^\#$ and $A \cap \overline{\Omega^*}$ turn out to be nonempty disjoint closed subsets of A , which is connected, so that $A \setminus (\overline{\Omega^*} \cup \text{Fr } \Omega^\#)$ must be nonempty. Thus, again, there exists $\hat{\Omega} \in \mathcal{A}(\Omega^\#) \subseteq \mathcal{A}_{\alpha^\#+1}$ such that $(A \setminus (\overline{\Omega^*} \cup \text{Fr } \Omega^\#)) \cap \hat{\Omega} \neq \emptyset$; hence we may apply (IV) with $\alpha = \alpha^\# + 1$, $\Omega' = \Omega^*$ and $\Omega'' = \hat{\Omega}$, to conclude that A must meet every D_m . \square

Corollary 3. *Every regular locally connected space without isolated points is ω -resolvable.*

Proof. Let X be such a space: As is well known, local connectedness implies that all connected components are open (cf. [En, Exercise 6.3.3(a)]). In particular, every connected component of X has more than one point. Of course, if we prove that every connected component of X is ω -resolvable, the whole of X will be ω -resolvable, too. Therefore, the result follows from Theorem 2. \square

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