SUBNORMAL SEMIGROUPS OF COMPOSITION OPERATORS

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Abstract. In this article we describe a model for subnormal semigroups of composition operators (with linear fractional symbol) acting on the Hardy space $H^2$. We also discuss cyclicity of such semigroups in the context of more general results studied by J. H. Shapiro and P. S. Bourdon.

1. Introduction

Let $A(\Delta)$ denote the space of all analytic functions in the unit disk $\Delta$ with the topology of uniform convergence on compact subsets of $\Delta$, and let $\mathcal{H}$ be a linear subspace of $A(\Delta)$. If $\psi$ is an analytic self-map of $\Delta$ such that $f \circ \psi$ belongs to $\mathcal{H}$ for all $f \in \mathcal{H}$, then $\psi$ induces a linear operator $C_\psi : \mathcal{H} \to \mathcal{H}$ defined as $C_\psi(f) := f \circ \psi$. $C_\psi$ is called the composition operator with symbol $\psi$.

As usual, we will denote by $H^2$ the Hilbert space $H^2(\Delta) := \{ f \equiv \sum_{i=0}^{\infty} a_i z^i \in A(\Delta) : \sum_{i=0}^{\infty} |a_i|^2 < \infty \}$, where the vector operations are the pointwise ones, and the inner product is given by $\langle \sum_{i=0}^{\infty} a_i z^i, \sum_{i=0}^{\infty} b_i z^i \rangle = \sum_{i=0}^{\infty} a_i \bar{b}_i$.

$H^2_0$ will denote the orthogonal complement of the constant functions in $H^2$; that is, $H^2_0 := \{ f \equiv \sum_{i=0}^{\infty} a_i z^i \in H^2 : a_0 = 0 \} = \{ f \in H^2 : f(0) = 0 \}$.

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space. An operator $S \in \mathcal{L}(\mathcal{H})$ is said to be subnormal if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator $N \in \mathcal{L}(\mathcal{K})$ such that $N|_\mathcal{H} = S$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be hyponormal if $T^* T \geq TT^*$.

In [10] the author pointed out that as a consequence of a result of C. C. Cowen (cf. [3]) if $\psi : \Delta \to \Delta$ is a linear fractional map, and the composition operator...
$C_\psi : H^2 \to H^2$ is hyponormal, then $C_\psi$ is unitarily equivalent to a composition operator with symbol $\phi$ of the form

\begin{equation}
\phi_a(z) := \frac{z}{az + (a + 1)}, \quad z \in \Delta, \quad a > 0.
\end{equation}

We also showed in [10] that if $\phi$ is a symbol of the form (1.1), then $\phi$ induces a strongly continuous one-parameter semigroup of subnormal composition operators. Moreover, if $\phi$ is a symbol of the form (1.2), then $\phi$ induces a strongly continuous semigroup of subnormal composition operators.

Let $(T_n)_{n=1}^\infty$ be a sequence of positive real numbers such that

\begin{equation}
\lim_{n \to \infty} \lambda_n = 0,
\end{equation}

and then put

\begin{equation}
\phi_{\lambda_n} := \phi((1+\lambda_n^{-1})), \quad \lambda_n := (1+\lambda_n^{-1}).
\end{equation}

Recall that if $\mathcal{H}$ is a Hilbert space, a function $A_t (t \in [0, \infty))$ from $[0, \infty)$ to the algebra of bounded operators on $\mathcal{H}$ is called a (one-parameter) semigroup of operators if

1. $A_t$ is a symbol of the form (1.1), then
2. $A_t = I$ (the identity operator on $\mathcal{H}$).

We will denote such a semigroup by $(\{A_t\}_{t \geq 0}, \mathcal{H})$. If it is clear from the context that $\{A_t\}_{t \geq 0} \subseteq \mathcal{L}(\mathcal{H})$, then we will refer to the semigroup simply as $\{A_t\}_{t \geq 0}$, or by a label such as $A = \{A_t\}_{t \geq 0}$. The semigroup $\{A_t\}_{t \geq 0}$ is said to be strongly continuous if $\lim_{t \to s} A_t = A_s$ (in the strong operator topology) $\forall s \in [0, \infty)$.

In fact, Theorem 2.5 in [10] can be restated as follows.

**Theorem 1.2.** Let $\Phi = \{\phi_s\}_{s \geq 0}$ be a semigroup of linear fractional self-maps of $\Delta$, such that no $\phi_s$ has $\infty$ as a fixed point. The following statements are equivalent:

1. $C_{\phi_s}$ is hyponormal for some $s \geq 0$.
2. $C_{\Phi}$ is a strongly continuous semigroup of subnormal composition operators.

Moreover, if (i) or (ii) hold, then there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ of positive real numbers such that

$C_{\Phi} \cong \{C_{\phi_{\lambda_n}}\}$ where $\psi(z) := \frac{1}{z + 2}$.

The purpose of these notes is to exhibit a model, as a semigroup of multiplication operators, of $\{C_{\phi_{\lambda_n}}\}$, as well as to describe certain cyclicity properties of it.

2. A MODEL FOR SUBNORMAL SEMIGROUPS OF OPERATORS

The following is a well-known result due to T. Itô [12].

**Theorem 2.1.** Every strongly continuous semigroup $(\{A_t\}_{t \geq 0}, \mathcal{H})$ of subnormal operators can be extended to a strongly continuous semigroup of normal operators acting on a Hilbert space $\mathcal{K}$ that contains $\mathcal{H}$.

**Definition 2.2.** A normal extension $(\{N_t\}_{t \geq 0}, \mathcal{K})$ of $(\{A_t\}_{t \geq 0}, \mathcal{H})$ is said to be minimal if $\mathcal{K}$ has no proper subspace that contains $\mathcal{H}$ and reduces each $N_t$. Equivalently, $(\{N_t\}_{t \geq 0}, \mathcal{K})$ is the m.n.e. of $(\{A_t\}_{t \geq 0}, \mathcal{H})$ if

$\mathcal{K} = \bigvee \{N_t^* \xi : t \in [0, \infty), \xi \in \mathcal{H}\}$.

**Theorem 2.3** (T. Itô). Any two minimal normal extensions (m.n.e.) of $(\{A_t\}_{t \geq 0}, \mathcal{H})$ are unitarily equivalent. If $(\{N_t\}_{t \geq 0}, \mathcal{K})$ is a minimal normal extension of $(\{A_t\}_{t \geq 0}, \mathcal{H})$, then $\|N_t\|_K = \|A_t\|_H, \forall t \in [0, \infty)$. 
Remark 2.4. Let \( \{ N_t \}_{t \geq 0} \) be a strongly continuous semigroup of normal operators. Since the Fuglede-Putnam Theorem implies that \( N_t \) commutes with \( N_t^* \) for all \( s, t \in [0, \infty) \), the selfadjoint operators \( U_t := N_t^* N_t \) also form a one-parameter semigroup.

**Proposition 2.5.** Let \( \{ N_t \}_{t \geq 0} \) be a strongly continuous semigroup of normal operators. Then the operators \( N_t \), \( t > 0 \), have one and the same null subspace.

**Proof.** It suffices to show that if \( N_s f = 0 \), then \( N_u f = 0 \) for \( u > s \), and \( N_u f = 0 \). The first of these assertions follows from the identity \( N_{u-s} N_s = N_u \). The second is a consequence of the relations

\[
\|N_s f\|^2 = \langle N_s^* N_s f, f \rangle = \langle U_s f, f \rangle
\]

and

\[
\langle U_s f, f \rangle = \langle U^2_s f, f \rangle = \langle U^2_s f, U^2_s f \rangle = \|U^2_s f\|^2.
\]

\( \square \)

**Definition 2.6.** A semigroup of operators \( T = \{ T_s \}_{s \geq 0} \) is said to be non-degenerate if the common null space \( N(T) \) of the operators \( T_s \) is \( \{ 0 \} \).

**Proposition 2.7.** If \( \{ N_t \}_{t \geq 0} \) is the minimal normal extension of a non-degenerate subnormal semigroup \( \{ S_t \}_{t \geq 0} \), then \( \{ N_t \}_{t \geq 0} \) itself is non-degenerate.

**Proof.** This follows from the definition of m.n.e., Theorem 2.3 and the fact that the kernel of a normal operator reduces it. \( \square \)

The following result also involves semigroups of normal operators (cf. [17]). A proof can be found in [14].

**Theorem 2.8.** Let \( \{ N_t \}_{t \geq 0} \) be a non-degenerate, normalized (i.e., \( \| N_t \| = 1 \), \( \forall t > 0 \)), strongly continuous semigroup of normal operators. Then \( \{ N_t \}_{t \geq 0} \) has a unique spectral representation of the form:

\[
N_t = \int_{\Pi^+} e^{-t z} dE(z)
\]

where \( \Pi^+ := \{ z \mid \Re(z) \geq 0 \} \) and \( E(z) \) is the spectral measure on \( \Pi^+ \) of the infinitesimal generator of the semigroup \( \{ N_t \}_{t \geq 0} \).

**Definition 2.9.** A strongly continuous semigroup \( T = \{ T(s) \}_{s \geq 0} \subset L(\mathcal{H}) \) is said to be cyclic if there is a vector \( x_0 \in \mathcal{H} \) such that \( \mathcal{V} \{ T(s)x_0 : s > 0 \} = \mathcal{H} \). The vector \( x_0 \) is said to be cyclic for \( T \). If there is family of vectors \( \{ x_s \}_{s \geq 0} \subseteq \mathcal{H} \) such that \( T(s)x_t = x_{s+t} \), \( \forall s, t > 0 \), and \( \mathcal{V} \{ x_s : s > 0 \} = \mathcal{H} \), then \( T \) is said to be quasicyclic, and the family \( \{ x_s \}_{s \geq 0} \) is called a quasicyclic family for \( T \).

Using Theorem 2.8, we can slightly modify the proof of a result by R. Frankfurt (9) to establish our next result. We will need first the following definition.

**Definition 2.10.** Let \( \Pi^+ := \{ z \mid \Re(z) \geq 0 \} \). Define a map \( \tau \) from \( \Pi^+ \) to the half-line \( \{ x \geq 0 \} \) by \( \tau(z) := \Re(z) \). If \( \mu \) is a given measure on \( \Pi^+ \) define a measure \( \nu \) on \( \{ x \geq 0 \} \) by \( d\nu(x) = d\mu(\tau^{-1}(x)) \). The measure \( \mu \) is said to have minimal exponential type if the Laplace-Stieltjes integral \( \int_0^\infty e^{-sz} d\nu(x) \) converges for all \( s > 0 \). In this case we have that \( e^{-sz} \in L^2(\mu) \) \( \forall s > 0 \). Denote by \( H^2(\mu) \) the \( L^2(d\mu) \)-closed linear span of these functions.
Theorem 2.11. Let \( \{T(s)\}_{s>0} \) be a quasicyclic subnormal semigroup of contractions on a Hilbert space \( \mathcal{H} \), with the property \( \|T(s)\| = e^{ks} \) for all \( s > 0 \), where \( k = \ln \|T(1)\| \). Then there is a measure \( \mu \) defined on the right halfplane \( \Pi^+ \) and having minimal exponential type such that the semigroup \( \{T(s)\} \) is unitarily equivalent to the semigroup of multiplication by \( e^{s(k-1)} \) on \( L^2(d\mu) \).

Proof. Let \( \{N(s)\}_{s>0} \) be the minimal normal semigroup extension of the (normalized) semigroup \( \{e^{-sk}T(s)\}_{s>0} \), acting on some Hilbert space \( \mathcal{K} \supset \mathcal{H} \). Let \( \{x_s\}_{s>0} \) be a quasicyclic family for \( T(s)_{s>0} \). Then the minimality of \( \{N(s)\}_{s>0} \) implies that it is non-degenerate, normalized (Theorem 2.3), and

\[ \mathcal{K} = \bigvee \{N(t)^* x_s : s, t > 0\}. \]

By Theorem 2.8, \( N(s) \) has a spectral representation of the form

\[ N(s) = \int_{\Pi^+} e^{-sz} dE(z) \]

where \( E(z) \) is a spectral measure on \( \Pi^+ \). For each \( s, t > 0 \), define a complex measure \( \mu_{s,t} \) on \( \Pi^+ \) by

\[ d\mu_{s,t}(z) = \langle dE(z)x_s, x_t \rangle. \]

If \( s = t \), we will write \( \mu_{s,t} = \mu_t \). Then we have, for any \( s, t, u, v > 0 \),

\[ d\mu_{s+u,t+v} = e^{(u+i\bar{v})z} d\mu_{s,t}(z). \]

Indeed, let \( f \) be any continuous function with compact support in \( \Pi^+ \). Then

\[ \int_{\Pi^+} f(z) d\mu_{s+u,t+v}(z) = \int_{\Pi^+} f(z) \langle dE(z)x_{s+u}, x_{t+v} \rangle \]

\[ = \int_{\Pi^+} f(z) \langle dE(z)N(v)^*N(u)x_s, x_t \rangle \]

\[ = \langle \int_{\Pi^+} f(z) dE(z) \int_{\Pi^+} e^{-(uw+\bar{v})} dE(w)x_s, x_t \rangle \]

\[ = \langle \int_{\Pi^+} f(z) e^{-(uz+\bar{v})} dE(z)x_s, x_t \rangle \]

\[ = \int_{\Pi^+} f(z) e^{-(uz+\bar{v})} d\mu_{s,t}, \]

which implies (2.3). In particular, for any \( s, t > 0 \),

\[ d\mu_{s+t}(z) = e^{-2sz} d\mu_t(z) \quad (x = \Re(z)). \]

Hence

\[ d\mu(z) := e^{2sz} d\mu_s(z) \quad (s > 0) \]

is a well-defined measure on \( \Pi^+ \). Moreover, since each \( d\mu_s \) is finite, it follows that \( d\mu \) has minimal exponential type. Let \( \mathcal{H}_0 = sp\{N(t)^* x_s\}_{s, t>0} \), and define a transformation \( U \) from \( \mathcal{H}_0 \) to \( L^2(d\mu) \) by

\[ U(N(t)^* x_s) = e^{-(sz+\bar{z})}. \]
for any $s, t > 0$, and extend it by linearity to $\mathcal{H}_0$. Thus, if $s, t, u, v > 0$,

\[
\langle N(t)x_s, N(v)x_u \rangle_{\mathcal{K}} = \langle N(t)^*N(v)x_s, x_u \rangle_{\mathcal{K}} = \int_{\Pi^+} e^{-(s\zeta + sz)}(dE(x_s, x_u)) = \int_{\Pi^+} e^{-(s\zeta + sz)}e^{-(s\zeta + sz)} d\mu(z) = \int_{\Pi^+} e^{-(s\zeta + sz)} e^{-(u\zeta + uz)} d\mu(z)
\]

(2.4)

which proves that $U$ is isometric. Hence $U$ extends to an isometric map from $\mathcal{K}$ to $L^2(d\mu)$. In particular, since $U(x_s) = e^{-sz}, \forall s > 0$, $U$ carries $\mathcal{H}$ isometrically onto $H^2(d\mu)$. It is readily seen that $U$ intertwines the semigroup $\{e^{-sT(s)}\}_{s>0}$, and the semigroup of multiplication by $e^{-sz}, s > 0$. The result now follows. □

3. CYCLICITY

P. S. Bourdon and J. H. Shapiro have exhaustively studied and described the cyclic behavior of linear fractional composition operators in their book Cyclic Phenomena for Composition Operators [3]. Recall that an operator $T$ in a Hilbert space $\mathcal{H}$ is said to be cyclic if there is a vector $x \in \mathcal{H}$ whose orbit

\[
\text{Orb}(T, x) := \{T^n x : n = 0, 1, \ldots \}
\]

has dense linear span. $T$ is called hypercyclic if there is $y \in \mathcal{H}$ such that $\overline{\text{Orb}(T, y)} = \mathcal{H}$.

As usual $x$ (resp. $y$) is called a cyclic (resp. hypercyclic) vector for $T$.

In [3] the reader can find a proof of the following fact: an operator has either no hypercyclic vector or a dense $G_δ$ set of them. As a consequence, Baire’s theorem implies: Every countable collection of hypercyclic operators has a common hypercyclic vector. On the other hand, [2, Theorem 2.8] implies that for any $\lambda > 0$, $C_{\phi_\lambda} |_{H^2}$ is not cyclic (as a single operator), and that, in fact, the closed linear span of any orbit of $C_{\phi_\lambda} |_{H^2}$ has infinite codimension.

Now, cyclicity (resp. hypercyclicity) deals with the fact that the set obtained by applying the integer powers of $T$ to a cyclic (resp. hypercyclic) vector, has dense linear span (resp. is dense); however, in our case we have more powers of $C_{\phi_\lambda} |_{H^2}$ in which to rely, and this might turn out to produce a better cyclic behavior in a certain sense.

Definition 3.1. A Hilbert space $\mathcal{H}$ of analytic functions defined on an open region $\Omega \subset \mathbb{C}$ is called a functional Hilbert space if for each $z \in \Omega$ the linear functional $f \mapsto f(z)$ is continuous. In this case, the Riesz representation theorem implies that for each $z \in \Omega$ there is a function $K_z \in \mathcal{H}$, called a reproducing kernel, such that $f(z) = (f, K_z)$.

One important step in establishing the hyponormality of $C_{\phi_\lambda} : H^2 \to H^2$ is to consider the restriction $C_{\phi_\lambda} |_{H^2_0}$. It is readily seen, on the other hand, that both $H^2$ and $H^2_0$ are functional Hilbert spaces, and since the set $\{z, z^2, \ldots, \}$ is an
orthonormal basis for $H^2_0$, it follows (cf. [11]) that the reproducing kernels of $H^2_0$ are the functions

$$K(z, w) := K_w(z) = \sum_{n=0}^{\infty} z^n w^n = \frac{\bar{w}z}{1 - \bar{w}z}, \quad z, w \in \Delta.$$ 

Notice that the kernels $K_w$ are linear fractional self-maps of $\Delta$. Moreover, the symbols $\{K_w\}$ are, in fact, scalar multiples of these kernels. Indeed,

$$\phi_a(z) = \frac{z}{az + 1 + a} = -\frac{1}{a} \left( \frac{-a}{1 + 1/a} z \right) = -\frac{1}{a} K(-\frac{1}{az + 1 + a}).$$

On the other hand, it is well known (cf. [5]) that the adjoint of any composition operator $C_\phi : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a functional Hilbert space contained in $A(\Delta)$, satisfies: $C_\phi^* K_w = K_{\phi(w)}$, for all $w \in \Delta$.

Notice that if $u_s := \phi_a^s$, $s \geq 0$, then $C_\phi^s u_s = u_{t+s}$. The following proposition shows that the set $\{u_s\}_{s>0}$ actually spans $H^2_0$.

**Proposition 3.2.** Let $T(s) := C_\phi^s |_{H^2_0}$, $s \geq 0$. For all $t \geq 0$, the function $u_t := \phi_a^t$ is a cyclic vector for the semigroup $\{T(s)\}$.

**Proof.** Fix $t \in [0, \infty)$. Suppose that $g \in H^2_0$ is orthogonal to $\sqrt{T(s)u_t : s \geq 0}$; then

$$0 = \langle g, T(s)u_t \rangle = \langle g, u_{s+t} \rangle = \frac{-1}{(1 + a)^{(s+t) - 1}} \langle g, K \{ -\frac{u_{s+t}}{1 + a^{(s+t) - 1}} \} \rangle = \frac{-1}{(1 + a)^{(s+t) - 1}} g \left( \frac{1}{(1 + a)^{(s+t) - 1}} \right).$$

Therefore, the analytic function $g$ vanishes along the curve $\{\frac{1}{(1 + a)^{(s+t) - 1}} - 1, s \geq 0\} \subset \Delta$, which means $g \equiv 0$. 

**Remark 3.3.** In general, for a strongly continuous semigroup $\{T(s)\}_{s \geq 0}$, there are constants $w \in \mathbb{R}$ and $M \geq 1$ such that $\|T(s)\| \leq Me^{ws}$ (cf. [13], Proposition I.5.5). The semigroup $\{C_\phi^s |_{H^2_0}\}_{s \geq 0}$, however, satisfies

$$\|C_\phi^s |_{H^2_0}\| = ((1 + a)^{-\frac{1}{t}})^s = e^{s \ln \|C_\phi^s |_{H^2_0}\|};$$

cf. [10, Theorem 2.14].

As a consequence of the preceding discussion we have

**Theorem 3.4.** The semigroup $\{T(s)\}_{s \geq 0}$ is unitarily equivalent to the semigroup of multiplication by $e^{s \ln \|T(1)\|^{1-z}}$ on $H^2(\mu)$, where $\mu$ is a measure defined on $\Pi^+$, having minimal exponential type.

Now we establish a general result on quasicyclicity of the adjoint of a semigroup of composition operators.

**Theorem 3.5.** Let $\Phi := \{\phi(s)\}_{s \geq 0}$ be a semigroup of analytic self-maps of $\Delta$, such that $\phi(s)$ has no fixed points in $\Delta$, for $s \neq 0$. Then the adjoint, $\{C_\phi^*\}_{s \geq 0} \subseteq \mathcal{L}(H^2)$, of the semigroup of composition operators induced by $\Phi$, is quasicyclic.
Moreover, for any non constant convergent sequence \( \{ s_k \} \subseteq [0, \infty) \), there are infinitely many vectors \( f \in H^2 \) such that
\[
\bigvee \{ C_{\phi(s)_k}^* f : k > 0 \} = H^2.
\]

Proof. Pick any \( w \in \Delta \), and let \( x_s := K_{\phi(s)(w)} = C_{\phi(s)}^* K_w \). Then
\[
C_{\phi(t)}^* x_s = C_{\phi(t)}^* C_{\phi(s)}^* K_w = C_{\phi(t+s)}^* K_w = K_{\phi(t+s)}(w) = x_{t+s}.
\]

The hypothesis on \( \Phi \) guarantees that the relation
\[
s \rightarrow \phi(s)(w)
\]
defines a simple smooth curve contained in \( \Delta \); indeed, suppose \( t > s \) and that \( \phi(t)(w) = \phi(s)(w) \). Then \( \phi(s) \circ \phi(t-s)(w) = \phi(s)(w) \), which implies (since \( \phi(t) \) is univalent) that \( \phi(t-s)(w) = w \) and therefore (since \( \phi(t-s) \) has no fixed points in \( \Delta \)) that \( t = s \). Thus if \( g \in H^2 \), then
\[
\langle g, x_s \rangle = 0 \quad \forall s \quad \iff \quad \langle g, C_{\phi(s)}^* K_w \rangle = 0 \quad \forall s
\]
\[
\iff \quad g(\phi(s)(w)) = 0 \quad \forall s \quad \iff \quad g \equiv 0,
\]
by the uniqueness principle for analytic functions. This argument shows that \( \{ C_{\phi(s)}^* \}_{s > 0} \) is quasicyclic. The last assertion can be proved similarly (by putting \( f := K_w \)). \( \square \)

Remarks 3.6.

\( \bullet \) Actually, Proposition 3.2 can be deduced from Theorem 3.5 due to the fact that \( C_{\phi_a}|_{H_2^2} = s M_z C_{\sigma_a}^* M_{\bar{z}} \), where \( \sigma_a(z) = sz + s - 1 \) with \( s = \frac{1}{1 + a} \), and \( M_z \) and \( M_{\bar{z}} \) denote the operators of multiplication by \( z \) on \( H^2 \) and by \( \frac{1}{z} \) on \( H_2^2 \), respectively; see [4].

Thus, the semigroup \( \{ C_{\phi_a}^* |_{H_2^2} \}_{a > 0} \) is unitarily equivalent to a semigroup of the form \( \{ s^t C_{\sigma(t)}^* \}_{t > 0} \), where \( \{ \sigma(t) \} \) is a semigroup of analytic self-maps of \( \Delta \) such as the one described in Theorem 3.5.

Note also that if \( \{ S_t \} \) is a quasicyclic subnormal semigroup, \( k \) is a positive constant, and we define
\[
T_t := k^t S_t \quad (t > 0),
\]
then \( \{ T_t \}_{t > 0} \) is also a quasicyclic subnormal semigroup.

\( \bullet \) Applying Theorem 3.5 to the semigroup \( \{ \phi_a^* \} \), equation (3.2) turns into
\[
\bigvee \{ (C_{\phi_a}^*)^{*k} f : k > 0 \} = H^2,
\]
which resembles cyclicity.

In [3] R. Frankfurt points out the fact that if the measure \( \mu \), given in Theorem 2.11 is finite, then \( U^*(1) \) is a cyclic vector for \( \{ T(s) \}_{s > 0} \). He also gives an example that the converse, in general, is not true. In the present situation, however, we have

Corollary 3.7. The measure \( \mu \), corresponding to \( \{ e^{-sk} C_{\phi_a}^* |_{H_2^2} \}_{s > 0} \) by Theorem 2.11 is a probability measure.
Proof. It is readily seen that if \( w = 0 \) in the proof of Theorem 3.3, then for all \( \lambda, \mu > 0 \), \[
\langle x^{\lambda}, x^{\mu} \rangle_{H^2_0} = \frac{1}{1 - (s^{\lambda} - 1)(s^{\mu} - 1)}.
\]
Hence
\[
\int_{\Pi^+} d\mu = \langle 1, 1 \rangle_{H^2(d\mu)} = \langle x_0, x_0 \rangle_{H^2_0} = 1.
\]
\(\square\)

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References


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