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SUBNORMAL SEMIGROUPS OF COMPOSITION OPERATORS

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ABSTRACT. In this article we describe a model for subnormal semigroups of composition operators (with linear fractional symbol) acting on the Hardy space H^2. We also discuss cyclicity of such semigroups in the context of more general results studied by J. H. Shapiro and P. S. Bourdon.

1. INTRODUCTION

Let $A(\Delta)$ denote the space of all analytic functions in the unit disk Δ with the topology of uniform convergence on compact subsets of Δ, and let $\mathcal{H}$ be a linear subspace of $A(\Delta)$. If ψ is an analytic self-map of Δ such that $f \circ \psi$ belongs to $\mathcal{H}$ for all $f \in \mathcal{H}$, then ψ induces a linear operator $C_\psi : \mathcal{H} \to \mathcal{H}$ defined as $C_\psi(f) := f \circ \psi$. C_ψ is called the *composition operator* with *symbol* ψ.

As usual, we will denote by H^2 the Hilbert space

$$H^2(\Delta) := \{f \equiv \sum_{i=0}^{\infty} a_i z^i \in A(\Delta) : \sum_{i=0}^{\infty} |a_i|^2 < \infty\},$$

where the vector operations are the pointwise ones, and the inner product is given by

$$\langle \sum_{i=0}^{\infty} a_i z^i, \sum_{i=0}^{\infty} b_i z^i \rangle = \sum_{i=0}^{\infty} a_i \bar{b_i}.$$

H_0^2 will denote the orthogonal complement of the constant functions in H^2; that is,

$$H_0^2 := \{f \equiv \sum_{i=0}^{\infty} a_i z^i \in H^2 : a_0 = 0\} = \{f \in H^2 : f(0) = 0\}.$$

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space. An operator $S \in \mathcal{L}(\mathcal{H})$ is said to be subnormal if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator $N \in \mathcal{L}(\mathcal{K})$ such that $N|_{\mathcal{H}} = S$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be hyponormal if $T^*T \geq TT^*$.

In [10] the author pointed out that as a consequence of a result of C. C. Cowen (cf. [4]) if $\psi : \Delta \to \Delta$ is a linear fractional map, and the composition operator

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$C_\psi : H^2 \to H^2$ is hyponormal, then C_ψ is unitarily equivalent to a composition operator with symbol ϕ of the form

$$\phi_a(z) := \frac{z}{az+(a+1)}, \quad z \in \Delta, \quad a > 0. \tag{1.1}$$

We also showed in [10] that if ϕ is a symbol of the form (1.1), then ϕ induces a strongly continuous one-parameter semigroup of subnormal composition operators. More precisely, we may define non-negative powers of ϕ_a by the formula

$$\phi_a^\lambda := \phi_{(1+a)^\lambda - 1},$$

and then put

$$C_{\phi_a}^\lambda := C_{\phi_{(1+a)^\lambda-1}}. \tag{1.2}$$

Recall that if $\mathcal{H}$ is a Hilbert space, a function A_t $(t \in [0,\infty))$ from $[0,\infty)$ to the algebra of bounded operators on $\mathcal{H}$ is called a (one-parameter) semigroup of operators if

(i) $A_t A_s = A_{t+s}$, $\forall\, t,s \in [0,\infty)$, and

(ii) $A_0 = I$ (the identity operator on $\mathcal{H}$).

We will denote such a semigroup by $(\{A_t\}_{t\geq 0}, \mathcal{H})$. If it is clear from the context that $\{A_t\}_{t\geq 0} \subset \mathcal{L}(\mathcal{H})$, then we will refer to the semigroup simply as $\{A_t\}_{t\geq 0}$, or by a label such as $\mathcal{A} = \{A_t\}_{t\geq 0}$. The semigroup $\{A_t\}_{t\geq 0}$ is said to be strongly continuous if $\lim_{t\to s} A_t = A_s$ (in the strong operator topology) $\forall\, s \in [0,\infty)$.

In fact, Theorem 2.5 in [10] can be restated as follows.

Theorem 1.2. *Let $\mathbf{\Phi} = \{\phi_s\}_{s\geq 0}$ be a semigroup of linear fractional self-maps of Δ, such that no ϕ_s has ∞ as a fixed point. The following statements are equivalent:*

(i) *C_{ϕ_s} is hyponormal for some $s \geq 0$.*

(ii) *$C_{\mathbf{\Phi}}$ is a strongly continuous semigroup of subnormal composition operators.*

Moreover, if (i) *or* (ii) *hold, then there exists a sequence $\{\lambda_i\}_{i=1}^n$ of positive real numbers such that*

$$C_{\mathbf{\Phi}} \cong \{{C_\psi}^{\lambda_i}\} \quad \text{where} \quad \psi(z) := \frac{1}{z+2}.$$

The purpose of these notes is to exhibit a model, as a semigroup of multiplication operators, of $\{C_{\phi_a}^\lambda\}$, as well as to describe certain cyclicity properties of it.

2. A model for subnormal semigroups of operators

The following is a well-known result due to T. Itô [12].

Theorem 2.1. *Every strongly continuous semigroup $(\{A_t\}_{t\geq 0}, \mathcal{H})$ of subnormal operators can be extended to a strongly continuous semigroup $\{N_t\}_{t\geq 0}$ consisting of normal operators acting on a Hilbert space $\mathcal{K}$ that contains $\mathcal{H}$.*

Definition 2.2. A normal extension $(\{N_t\}_{t\geq 0}, \mathcal{K})$ of $(\{A_t\}_{t\geq 0}, \mathcal{H})$ is said to be minimal if $\mathcal{K}$ has no proper subspace that contains $\mathcal{H}$ and reduces each N_t. Equivalently, $(\{N_t\}_{t\geq 0}, \mathcal{K})$ is the m.n.e. of $(\{A_t\}_{t\geq 0}, \mathcal{H})$ if

$$\mathcal{K} = \bigvee \{N_t^* \xi : t \in [0,\infty), \xi \in \mathcal{H}\}.$$

Theorem 2.3 (T. Itô)**.** *Any two minimal normal extensions (m.n.e.) of $(\{A_t\}_{t\geq 0}, \mathcal{H})$ are unitarily equivalent. If $(\{N_t\}_{t\geq 0}, \mathcal{K})$ is a minimal normal extension of $(\{A_t\}_{t\geq 0}, \mathcal{H})$, then $||N_t||_{\mathcal{K}} = ||A_t||_{\mathcal{H}}$, $\forall\, t \in [0,\infty)$.*

Remark 2.4. Let $\{N_t\}_{t\geq 0}$ be a strongly continuous semigroup of normal operators. Since the Fuglede-Putnam Theorem implies that N_t commutes with N_s^* for all $s,t \in [0,\infty)$, the selfadjoint operators $U_t := N_t^* N_t$ also form a one-parameter semigroup.

Proposition 2.5. *Let $\{N_t\}_{t\geq 0}$ be a strongly continuous semigroup of normal operators. Then the operators $N_t,\ t>0,$ have one and the same null subspace.*

Proof. It suffices to show that if $N_s f = 0$, then $N_u f = 0$ for $u > s$, and $N_{\frac{s}{2}} f = 0$.

The first of these assertions follows from the identity $N_{u-s}N_s = N_u$. The second is a consequence of the relations

$$||N_s f||^2 = \langle N_s^* N_s f, f\rangle = \langle U_s f, f\rangle$$

and

$$\langle U_s f, f\rangle = \langle U_{\frac{s}{2}}^2 f, f\rangle = \langle U_{\frac{s}{2}} f, U_{\frac{s}{2}} f\rangle = ||U_{\frac{s}{2}} f||^2.$$

□

Definition 2.6. A semigroup of operators $\mathcal{T} = \{T_s\}_{s>0}$ is said to be non-degenerate if the common null space $N(\mathcal{T})$ of the operators T_s is $\{0\}$.

Proposition 2.7. *If $\{N_t\}_{t\geq 0}$ is the minimal normal extension of a non-degenerate subnormal semigroup $\{S_t\}_{t\geq 0}$, then $\{N_t\}_{t\geq 0}$ itself is non-degenerate.*

Proof. This follows from the definition of m.n.e., Theorem 2.3 and the fact that the kernel of a normal operator reduces it. □

The following result also involves semigroups of normal operators (cf. [17]). A proof can be found in [14].

Theorem 2.8. *Let $\{N_t\}_{t>0}$ be a non-degenerate, normalized (i.e., $\|N_t\| = 1,\ \forall t > 0$), strongly continuous semigroup of normal operators. Then $\{N_t\}_{t>0}$ has a unique spectral representation of the form:*

$$N_t = \int_{\Pi^+} e^{-tz}\, dE(z)$$

where $\Pi^+ := \{z \,|\, \mathrm{Re}(z) \geq 0\}$ and $E(z)$ is the spectral measure on Π^+ of the infinitesimal generator of the semigroup $\{N_t\}_{t>0}$.

Definition 2.9. A strongly continuous semigroup $\mathcal{T} = \{T(s)\}_{s\geq 0} \subset \mathcal{L}(\mathcal{H})$ is said to be cyclic if there is a vector $x_0 \in \mathcal{H}$ such that $\bigvee\{T(s)x_0 : s > 0\} = \mathcal{H}$. The vector x_0 is said to be cyclic for $\mathcal{T}$. If there is family of vectors $\{x_s\}_{s>0} \subseteq \mathcal{H}$ such that $T(s)x_t = x_{s+t},\ \forall s,t > 0$, and $\bigvee\{x_s : s > 0\} = \mathcal{H}$, then $\mathcal{T}$ is said to be quasicyclic, and the family $\{x_s\}_{s>0}$ is called a quasicyclic family for $\mathcal{T}$.

Using Theorem 2.8 we can slightly modify the proof of a result by R. Frankfurt ([9]) to establish our next result. We will need first the following definition.

Definition 2.10. Let $\Pi^+ := \{z \,|\, \mathrm{Re}(z) \geq 0\}$. Define a map τ from Π^+ to the half-line $\{x \geq 0\}$ by $\tau(z) := \mathrm{Re}(z)$. If μ is a given measure on Π^+ define a measure ν on $\{x \geq 0\}$ by $d\nu(x) = d\mu(\tau^{-1}(x))$. The measure μ is said to have minimal exponential type if the Laplace-Stieltjes integral $\int_0^\infty e^{-sx}\, d\nu(x)$ converges for all $s > 0$. In this case we have that $e^{-sz} \in L^2(\mu)\ \ \forall s > 0$. Denote by $H^2(\mu)$ the $L^2(d\mu)$-closed linear span of these functions.

Theorem 2.11. *Let $\{T(s)\}_{s\geq 0}$, be a quasicyclic subnormal semigroup of contractions on a Hilbert space $\mathcal{H}$, with the property $||T(s)|| = e^{ks}$ for all $s > 0$, where $k = \ln \|T(1)\|$. Then there is a measure μ defined on the right halfplane Π^+ and having minimal exponential type such that the semigroup $\{T(s)\}$ is unitarily equivalent to the semigroup of multiplication by $e^{s(k-z)}$ on $H^2(d\mu)$.*

Proof. Let $\{N(s)\}_{s>0}$ be the minimal normal semigroup extension of the (normalized) semigroup $\{e^{-sk}T(s)\}_{s>0}$, acting on some Hilbert space $\mathcal{K} \supset \mathcal{H}$. Let $\{x_s\}_{s>0}$ be a quasicyclic family for $T(s)_{s>0}$. Then the minimality of $\{N(s)\}_{s>0}$ implies that it is non-degenerate, normalized (Theorem 2.3), and

$$\mathcal{K} = \bigvee\{N(t)^* x_s : s, t > 0\}.$$

By Theorem 2.8, $N(s)$ has a spectral representation of the form

$$N(s) = \int_{\Pi^+} e^{-sz}\, dE(z) \tag{2.1}$$

where $E(z)$ is a spectral measure on Π^+. For each $s, t > 0$, define a complex measure $\mu_{s,t}$ on Π^+ by

$$d\mu_{s,t}(z) = \langle dE(z)x_s, x_t\rangle.$$

If $s = t$, we will write $\mu_{t,t} = \mu_t$. Then we have, for any $s, t, u, v > 0$,

$$d\mu_{s+u,t+v} = e^{(uz+v\bar{z})} d\mu_{s,t}(z). \tag{2.2}$$

Indeed, let f be any continuous function with compact support in Π^+. Then

$$\begin{aligned}
\int_{\Pi^+} f(z)\, d\mu_{s+u,t+v}(z) &= \int_{\Pi^+} f(z)\langle dE(z)x_{s+u}, x_{t+v}\rangle \\
&= \int_{\Pi^+} f(z)\langle dE(z)N(v)^*N(u)x_s, x_t\rangle \\
&= \langle \int_{\Pi^+} f(z)\, dE(z) \int_{\Pi^+} e^{-(uw+v\bar{w})}\, dE(w)x_s, x_t\rangle \\
&= \langle \int_{\Pi^+} f(z)e^{-(uz+v\bar{z})}\, dE(z)x_s, x_t\rangle \\
&= \int_{\Pi^+} f(z)e^{-(uz+v\bar{z})}\, d\mu_{s,t},
\end{aligned} \tag{2.3}$$

which implies (2.2). In particular, for any $s, t > 0$,

$$d\mu_{s+t}(z) = e^{-2sx} d\mu_t(z)\ (x = \mathrm{Re}(z)).$$

Hence

$$d\mu(z) := e^{2sx} d\mu_s(z)\ \ (s > 0)$$

is a well-defined measure on Π^+. Moreover, since each $d\mu_s$ is finite, it follows that $d\mu$ has *minimal exponential type.* Let $\mathcal{H}_0 = sp\{N(t)^* x_s\}_{s,t>0}$, and define a transformation U from $\mathcal{H}_0$ to $L^2(d\mu)$ by

$$U(N(t)^* x_s) = e^{-(sz+t\overline{z})}$$

for any $s,t>0$, and extend it by linearity to $\mathcal{H}_0$. Thus, if $s,t,u,v>0$,

$$\begin{aligned}\langle N(t)^*x_s, N(v)^*x_u\rangle_{\mathcal{K}} &= \langle N(t)^*N(v)x_s, x_u\rangle_{\mathcal{K}}\\ &= \int_{\Pi^+} e^{-(t\overline{z}+vz)}\langle dE(z)x_s, x_u\rangle\\ &= \int_{\Pi^+} e^{-(t\overline{z}+vz)}e^{-(sz+u\overline{z})}d\mu(z)\\ &= \int_{\Pi^+} e^{-(t\overline{z}+vz)}\overline{e^{-(uz+v\overline{z})}}d\mu(z)\\ &= \langle e^{-(sz+t\overline{z})}, e^{-(uz+v\overline{z})}\rangle_{L^2(d\mu)},\end{aligned} \tag{2.4}$$

which proves that U is isometric. Hence U extends to an isometric map from $\mathcal{K}$ to $L^2(d\mu)$. In particular, since $U(x_s)=e^{-sz}$, $\forall s>0$, U carries $\mathcal{H}$ isometrically onto $H^2(d\mu)$. It is readily seen that U intertwines the semigroup $\{e^{-k}T(s)\}_{s>0}$, and the semigroup of multiplication by e^{-sz}, $s>0$. The result now follows. □

3. Cyclicity

P. S. Bourdon and J. H. Shapiro have exhaustively studied and described the cyclic behavior of linear fractional composition operators in their book *Cyclic Phenomena for Composition Operators* [3]. Recall that an operator T in a Hilbert space $\mathcal{H}$ is said to be *cyclic* if there is a vector $x\in\mathcal{H}$ whose orbit

$$Orb(T,x) := \{T^n x : n=0,1,\dots\}$$

has dense linear span. T is called *hypercyclic* if there is $y\in\mathcal{H}$ such that $\overline{Orb(T,y)}=\mathcal{H}$.

As usual x (resp. y) is called a *cyclic* (resp. *hypercyclic*) vector for T.

In [3] the reader can find a proof of the following fact: an operator has either no hypercyclic vector or a dense G_δ set of them. As a consequence, Baire's theorem implies: *Every* **countable** *collection of hypercyclic operators has a common hypercyclic vector.* On the other hand, [2, Theorem 2.8] implies that for any $\lambda>0$, $C_{\phi_a}^{\lambda}|_{H_0^2}$ is not cyclic (as a single operator), and that, in fact, the closed linear span of any orbit of $C_{\phi_a}|_{H_0^2}$ has infinite codimension.

Now, cyclicity (resp. hypercyclicity) deals with the fact that the set obtained by applying the integer powers of T to a cyclic (resp. hypercyclic) vector, has dense linear span (resp. is dense); however, in our case we have more powers of $C_{\phi_a}|_{H_0^2}$ in which to rely, and this might turn out to produce a better cyclic behavior in a certain sense.

Definition 3.1. A Hilbert space $\mathcal{H}$ of analytic functions defined on an open region $\Omega\subset\mathbb{C}$ is called a functional Hilbert space if for each $z\in\Omega$ the linear functional $f\mapsto f(z)$ is continuous. In this case, the Riesz representation theorem implies that for each $z\in\Omega$ there is a function $\mathbf{K}_z\in\mathcal{H}$, called a reproducing kernel, such that

$$f(z)=\langle f,\mathbf{K}_z\rangle.$$

One important step in establishing the hyponormality of $C_{\phi_a}: H^2\to H^2$ is to consider the restriction $C_{\phi_a}\big|_{H_0^2}$. It is readily seen, on the other hand, that both H^2 and H_0^2 are functional Hilbert spaces, and since the set $\{z,z^2,\dots,\}$ is an orthonormal basis for H_0^2, it follows (cf. [1]) that the reproducing kernels of H_0^2 are the functions

$$K(z,w) := K_w(z) = \sum_{n=1}^{\infty} z^n \bar{w}^n = \frac{\bar{w}z}{1-\bar{w}z}, \quad z,w \in \Delta.$$

Notice that the kernels K_w are linear fractional self-maps of Δ. Moreover, the symbols $\{\phi_a^\lambda\}$ are, in fact, scalar multiples of these kernels. Indeed,

$$\phi_a(z) = \frac{z}{az+1+a} = -\frac{1}{a}\,\frac{(-\frac{a}{1+a})z}{1+(\frac{a}{1+a})z} = -\frac{1}{a}K_{(-\frac{a}{1+a})}. \tag{3.1}$$

On the other hand, it is well known (cf. [5]) that the adjoint of any composition operator $C_\phi : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a functional Hilbert space contained in $A(\Delta)$, satisfies: $C_\phi^* \mathbf{K}_w = \mathbf{K}_{\phi(w)}$, for all $w \in \Delta$.

Notice that if $u_s := \phi_a^s$, $s \geq 0$, then $C_{\phi_a}^t u_s = u_{t+s}$. The following proposition shows that the set $\{u_s\}_{s>0}$ actually spans H_0^2.

Proposition 3.2. *Let $T(s) := C_{\phi_a}^s|_{H_0^2}$, $s \geq 0$. For all $t \geq 0$, the function $u_t := \phi_a^t$ is a cyclic vector for the semigroup $\{T(s)\}$.*

Proof. Fix $t \in [0,\infty)$. Suppose that $g \in H_0^2$ is orthogonal to $\bigvee\{T(s)u_t : s \geq 0\}$; then

$$\begin{aligned} 0 &= \langle g, T(s)u_t\rangle \\ &= \langle g, u_{s+t}\rangle \\ &= \frac{-1}{(1+a)^{(s+t)}-1}\,\langle g, K_{\{-\frac{(1+a)^{(s+t)}-1}{(1+a)^{(s+t)}}\}}\rangle \\ &= \frac{-1}{(1+a)^{(s+t)}-1}\, g(\frac{1}{(1+a)^{(s+t)}}-1). \end{aligned}$$

Therefore, the analytic function g vanishes along the curve $\{\frac{1}{(1+a)^{(s+t)}}-1,\ s \geq 0\} \subset \Delta$, which means $g \equiv 0$. □

Remark 3.3. In general, for a strongly continuous semigroup $\{T(s)\}_{s\geq 0}$, there are constants $w \in \mathbb{R}$ and $M \geq 1$ such that $\|T(s)\| \leq Me^{ws}$ (cf. [13], Proposition I.5.5). The semigroup $\{C_{\phi_a}^s|_{H_0^2}\}_{s\geq 0}$, however, satisfies

$$\|C_{\phi_a}^s|_{H_0^2}\| = ((1+a)^{-\frac{1}{2}})^s = e^{s\ln\|C_{\phi_a}|_{H_0^2}\|};$$

cf. [10, Theorem 2.14].

As a consequence of the preceding discussion we have

Theorem 3.4. *The semigroup $\{T(s)\}_{s\geq 0}$ is unitarily equivalent to the semigroup of multiplication by $e^{s(\ln\|T(1)\|-z)}$ on $H^2(\mu)$, where μ is a measure defined on Π^+, having minimal exponential type.*

Now we establish a general result on quasicyclicity of the adjoint of a semigroup of composition operators.

Theorem 3.5. *Let $\Phi := \{\phi(s)\}_{s\geq 0}$ be a semigroup of analytic self-maps of Δ, such that $\phi(s)$ has no fixed points in Δ, for $s \neq 0$. Then the adjoint, $\{C_{\phi(s)}^*\}_{s>0} \subseteq \mathcal{L}(H^2)$, of the semigroup of composition operators induced by Φ, is quasicyclic.*

Moreover, for any non (eventually) constant convergent sequence $\{s_k\} \subseteq [0,\infty)$, *there are infinitely many vectors* $f \in H^2$ *such that*

$$\bigvee\{C^*_{\phi(s_k)}f : k > 0\} = H^2. \tag{3.2}$$

Proof. Pick any $w \in \Delta$, and let $x_s := K_{\phi(s)(w)}$ $(= C^*_{\phi(s)}K_w)$. Then

$$\begin{aligned} C^*_{\phi(t)}x_s &= C^*_{\phi(t)}C^*_{\phi(s)}K_{w_0} \\ &= C^*_{\phi(t+s)}K_w \\ &= K_{\phi(t+s)}(w) \\ &= x_{t+s}. \end{aligned} \tag{3.3}$$

The hypothesis on Φ guarantees that the relation

$$s \longrightarrow \phi(s)(w)$$

defines a simple smooth curve contained in Δ; indeed, suppose $t > s$ and that $\phi(t)(w) = \phi(s)(w)$. Then $\phi(s) \circ \phi(t-s)(w) = \phi(s)(w)$, which implies (since $\phi(t)$ is univalent) that $\phi(t-s)(w) = w$ and therefore (since $\phi(t-s)$ has no fixed points in Δ) that $t = s$. Thus if $g \in H^2$, then

$$\begin{aligned} \langle g, x_s\rangle = 0 \quad \forall\, s \quad &\Leftrightarrow \quad \langle g, C^*_{\phi(s)}K_w\rangle = 0 \quad \forall\, s \\ &\Leftrightarrow \quad g(\phi(s)(w)) = 0 \quad \forall\, s \\ &\Leftrightarrow \quad g \equiv 0, \end{aligned} \tag{3.4}$$

by the uniqueness principle for analytic functions. This argument shows that $\{C^*_{\phi(s)}\}_{s>0}$ is quasicyclic. The last assertion can be proved similarly (by putting $f := K_w$). □

Remarks 3.6.

• Actually, Proposition 3.2 can be deduced from Theorem 3.5 due to the fact that $C_{\phi_a}|_{H^2_0} = sM_zC^*_{\sigma_a}M_{\frac{1}{z}}$, where $\sigma_a(z) = sz + s - 1$ with $s = \frac{1}{1+a}$, and M_z and $M_{\frac{1}{z}}$ denote the operators of multiplication by z on H^2 and by $\frac{1}{z}$ on H^2_0, respectively; see [4].

Thus, the semigroup $\{C^\lambda_{\phi_a}|_{H^2_0}\}_{\lambda>0}$ is unitarily equivalent to a semigroup of the form $\{s^tC^*_{\sigma(t)}\}_{t>0}$, where $\{\sigma(t)\}$ is a semigroup of analytic self-maps of Δ such as the one described in Theorem 3.5.

Note also that if $\{S_t\}$ is a quasicyclic subnormal semigroup, k is a positive constant, and we define

$$T_t := k^tS_t \quad (t > 0),$$

then $\{T_t\}_{t>0}$ is also a quasicyclic subnormal semigroup.

• Applying Theorem 3.5 to the semigroup $\{\phi^\lambda_a\}$, equation (3.2) turns into

$$\bigvee\{(C_{\phi_a})^{*s_k}f : k > 0\} = H^2, \tag{3.5}$$

which resembles cyclicity.

In [9] R. Frankfurt points out the fact that if the measure μ, given in Theorem 2.11, is finite, then $U^*(1)$ is a cyclic vector for $\{T(s)\}_{s>0}$. He also gives an example that the converse, in general, is not true. In the present situation, however, we have

Corollary 3.7. *The measure* μ, *corresponding to* $\{e^{-sk}C^s_{\phi_a}|_{H^2_0}\}_{s\geq 0}$ *by Theorem* 2.11, *is a probability measure.*

Proof. It is readily seen that if $w = 0$ in the proof of Theorem 3.5, then for all $\lambda, \mu > 0$,

$$\langle x_\lambda, x_\mu \rangle_{H_0^2} = \frac{1}{1 - (s^\lambda - 1)(s^\mu - 1)}.$$

Hence

$$\int_{\Pi^+} d\mu = \langle 1, 1 \rangle_{H^2(d\mu)} = \langle x_0, x_0 \rangle_{H_0^2} = 1.$$

□

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