POISSON INTEGRALS ASSOCIATED TO DUNKL OPERATORS
FOR DIHEDRAL GROUPS

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Abstract. In this paper we study the boundary behavior of Poisson integrals
associated to Dunkl differential-difference operators for dihedral groups and
the boundary integral representations for functions on the unit disc of C anni-
hilated by the Laplace operator corresponding to these differential-difference
operators.

1. Notation and statement of the main result

For every integer $k$ such that $k \geq 1$, let $D_k$ be the dihedral group of order $2k$;
that is, $D_k$ consists of the rotations $z \mapsto ze^{\frac{2\pi i}{k}}$ and the reflections $z \mapsto ze^{\frac{2\pi i}{k}l}$,
$0 \leq l \leq k - 1$, $z \in \mathbb{C}$.

If $k \geq 1$ is a fixed integer and $\alpha$ is a fixed positive real number, we associate to
the group $D_k$ the weight function $h$ defined by
$$h(z) = \left| \frac{z^k - z^k}{2i} \right|^\alpha,$$
which is a product of powers of the linear functions on $\mathbb{R}^2 \cong \mathbb{C}$ whose zero-sets are
the mirrors of the reflections in $D_k$.

The complex Dunkl operators are defined for a complex-valued function $f$
of class $C^1$ on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ by
$$T_h f(z) = \frac{\partial f(z)}{\partial z} + \alpha \sum_{l=0}^{k-1} \frac{f(z) - f(z \omega^l)}{z - \omega^l}$$
and
$$\overline{T}_h f(z) = \frac{\partial f(z)}{\partial \overline{z}} - \alpha \sum_{l=0}^{k-1} \frac{f(z) - f(z \omega^l)}{z - \overline{\omega}^l} \omega^l$$
where $\omega = e^{\frac{2\pi i}{k}}$. As in [D1], let $\Delta_h$ denote the $h$-Laplacian operator, $\Delta_h = 4 T_h \overline{T}_h$,
and say that a complex-valued function $f$ on $D$ is $h$-harmonic if it is of class $C^2$
and $\Delta_h f = 0$ on $D$.

An orthogonal basis for $L^2(h(e^{i\theta})^2 d\theta)$ on the unit circle was constructed in [D2],
and this led to the definition of a Poisson kernel $P$ which reproduces $h$-harmonic

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polynomials in $\mathbb{D}$ from their boundary values, and which is given by

\begin{equation}
P(z, w) = \frac{1 - |zw|^2}{B(a, a + 1)|1 - zw|^2} \int_0^1 u^{a-1}(1 - u)^a du
\end{equation}

for $z, w \in \mathbb{C}$ such that $|zw| < 1$ (see [D2, Theorems 1.3 and 2.1]).

Define the Poisson integral $P[f]$ of a function $f \in L^1(h(e^{i\theta})^2 d\theta)$ by

\[ P[f](z) = c_0 \int_{-\pi}^{\pi} f(e^{i\theta})P(z, e^{i\theta})h(e^{i\theta})^2 d\theta \]

for $z \in \mathbb{D}$, where $c_0 = \left( \int_{-\pi}^{\pi} h(e^{i\theta})^2 d\theta \right)^{-1}$.

Let $S^1$ denote the unit circle $\{ z \in \mathbb{C} : |z| = 1 \}$ and $C(S^1)$ be the space of all complex-valued continuous functions on $S^1$. The main purpose of this paper is to prove the following result:

**Theorem 1.1.** If $f \in C(S^1)$, then the function $F$ defined by

\begin{equation}
F(z) = \begin{cases} 
P[f](z), & \text{if } z \in \mathbb{D}, \\
f(z), & \text{if } z \in S^1, 
\end{cases}
\end{equation}

is continuous on $\overline{\mathbb{D}}$.

Trivially, we have the following corollary.

**Corollary 1.2.** If $f \in C(S^1)$, then

\[ \sup_{w \in S^1} |P[f](rw) - f(w)| \to 0 \quad (r \to 1). \]

It should be noted that the boundary behavior of Poisson integrals associated to Dunkl operators acting on functions on $\mathbb{R}^d$, $d \geq 2$, and corresponding to the abelian group $\mathbb{Z}_d^4$, was considered in Theorem 5.5.7 of [DX].

Theorem 1.1 will be used in Section 3 to study the boundary behavior of Poisson integrals and give conditions under which an $h$-harmonic function in the disc is the Poisson integral of some type of function or measure on the unit circle.

**Remark 1.3.** If $f \in L^1(h(e^{i\theta})^2 d\theta)$, then $P[f]$ is $h$-harmonic on $\mathbb{D}$. Indeed, as stated in the proof of Theorem 1.3 of [D2], if $\{ \psi_n(z), \overline{\psi_n(z)} : n \geq 0 \}$ denotes the orthonormal basis of $L^2(c_\alpha h_1(e^{i\theta})^2 d\theta)$ associated to $h_1(x + iy) := |y|^\alpha$, the Cauchy kernel associated to $h$ is given by

\begin{equation}
C(z, w) = \sum_{n=0}^{\infty} \sum_{l=0}^{k-1} z^l \psi_n(z^k) \overline{w^l \psi_n(w^k)} \quad (|zw| < 1),
\end{equation}

with $\overline{T_h(z^l \psi_n(z^k))} = 0$ for $0 \leq l \leq k - 1$ and $n \geq 0$. Using (1.3) and the expression of the $\psi_n$ in terms of Heisenberg polynomials (see [D1] Proposition 3.11 and [D2]), we easily get that for fixed $w \in S^1$, $\overline{T_h C(z, w) = 0}$ for all $z \in \mathbb{D}$, so that $z \mapsto P(z, w)$ is $h$-harmonic on $\mathbb{D}$, because $P(z, w) = C(z, w) + \overline{C(w, z)}$ and $\Delta_h(z C(z, w)) = z \Delta_h C(z, w) + 4 T_h C(z, w)$ (see [D1] Proposition 2.2). From this we easily deduce that $P[f]$ is $h$-harmonic on $\mathbb{D}$.

Thus Theorem 1.1 provides the solution of the analogue of the Dirichlet problem for the unit disc of $\mathbb{C}$ associated to the $h$-Laplacian operator. The uniqueness theorem that corresponds to this existence theorem is proved in Section 3.
2. Proof of the Main Result

For \( z \in \mathbb{D} \) and \( \theta \in (-\pi, \pi] \), set

\[
a(z, \theta) = |1 - z^k e^{ik\theta}|^2
\]

and

\[
b(z, \theta) = |1 - z^k e^{-ik\theta}|^2.
\]

We consider the integral

\[
\int_{-\pi}^{\pi} \int_0^1 \frac{u^{\alpha-1}(1-u)^{2\alpha} du}{[(1-u)a(z,\theta) + ub(z,\theta)]^\alpha} (\sin^2 k\theta)^\alpha d\theta \quad (z \in \mathbb{D})
\]

in two pieces: the first is taken over \( A_1(z) = \{ \theta \in (-\pi, \pi]: b(z, \theta) \leq 2a(z, \theta) \} \), and the second over the complement of \( A_1(z) \), denoted by \( A_2(z) \); for \( j \in \{1, 2\} \) and \( z \in \mathbb{D} \), set

\[
I_j(z) = \int_{A_j(z)} \int_0^1 \frac{u^{\alpha-1}(1-u)^{2\alpha} du}{[(1-u)a(z,\theta) + ub(z,\theta)]^\alpha} (\sin^2 k\theta)^\alpha d\theta.
\]

Then we have the following

**Lemma 2.1.** Set \( K = 2\pi\left(\frac{1+\sqrt{2}}{2}\right)^{2\alpha} \). For any \( z \in \mathbb{D} \),

\[
I_1(z) \leq \frac{K}{\alpha},
\]

and if \( z \in \mathbb{D} \) satisfies \( |z| \geq \frac{1}{2} \), then

\[
I_2(z) \leq K \left( \int_0^1 \frac{t^{\alpha-1}}{(1+t)^{\alpha+\frac{1}{2}}} dt + 2^k \int_1^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\frac{1}{2}}} dt \right).
\]

**Proof.** Let \( z \in \mathbb{D} \). We have \( |\sin k\theta| \leq \frac{1}{2} (\sqrt{a(z,\theta)} + \sqrt{b(z,\theta)}) \), so that for any \( \theta \in A_1(z) \), \((\sin^2 k\theta)^\alpha \leq \left(\frac{1+\sqrt{2}}{2}\right)^{2\alpha} a(z,\theta)\alpha \) and

\[
I_1(z) \leq K \int_0^1 u^{\alpha-1} du = \frac{K}{\alpha}.
\]

If \( \theta \in A_2(z) \) and \( u \in [0, 1] \), then \((\sin^2 k\theta)^\alpha \leq K' b(z,\theta)\alpha \) where \( K' = \left(\frac{1+\sqrt{2}}{2}\right)^{2\alpha} \)

and \( (1-u)a(z,\theta) + ub(z,\theta) \geq a(z,\theta) + \frac{b(z,\theta)}{2a(z,\theta)} \), so that

\[
I_2(z) \leq K' \int_{A_2(z)} \frac{b(z,\theta)^\alpha}{a(z,\theta)\alpha} \int_0^1 \frac{u^{\alpha-1}}{\left(1 + u \frac{b(z,\theta)}{2a(z,\theta)}\right)^\alpha} du d\theta
\]

\[
= \frac{K}{2\pi} \int_{A_2(z)} \int_0^{c(z,\theta)} t^{\alpha-1} \frac{du}{(1+t)^{\alpha}} dt d\theta,
\]

where \( c(z, \theta) = \frac{b(z,\theta)}{2a(z,\theta)} > 1 \). Then

\[
I_2(z) \leq K \int_0^1 \frac{t^{\alpha-1}}{(1+t)^{\alpha}} dt + \frac{K}{2\pi} I_3(z)
\]
with

\[ I_3(z) = \int_{A_2(z)} \int_1^{c(z, \theta)} \frac{t^{\alpha-1}}{(1 + t)^\alpha} dt \ d\theta \]

\[ = \int_1^\infty \frac{t^{\alpha-1}}{(1 + t)^\alpha} \lambda(A_2(z) \cap \{ \theta \in (-\pi, \pi] : c(z, \theta) \geq t \}) dt, \]

where \( \lambda \) denotes the Lebesgue measure on \((-\pi, \pi]\).

For any \( t \geq 1 \), we have

\[ \lambda(\{ \theta \in (-\pi, \pi] : c(z, \theta) \geq t \}) \leq \lambda(\{ \theta \in (-\pi, \pi] : b(z, \theta) \geq \alpha(z, \theta)(1 + t) \}) \]

\[ \leq \lambda(\{ \theta \in (-\pi, \pi] : a(z, \theta) \leq \frac{4}{1 + t} \}) \]

because \( b(z, \theta) \leq 4 \). Set \( z = \rho e^{i\phi} \), \( M = 4^{k+1} \), and if \( A \) is a Borel set of \( \mathbb{R} \), let \( \chi_A \) be the characteristic function of \( A \); if \( \rho \geq \frac{1}{2} \), then

\[ \lambda \left( \left\{ \theta \in (-\pi, \pi] : a(z, \theta) \leq \frac{4}{1 + t} \right\} \right) \]

\[ \leq \lambda \left( \left\{ \theta \in (-\pi, \pi] : \left| \frac{1}{\rho^k} - e^{ik(\theta + \phi)} \right|^2 \leq \frac{M}{1 + t} \right\} \right) \]

\[ = \lambda \left( \left\{ u \in (-\pi, \pi] : \left| \frac{1}{\rho^k} - e^{iku} \right|^2 \leq \frac{M}{1 + t} \right\} \right) \]

\[ = \frac{1}{k} \int_{-k\pi}^{k\pi} \chi \left( \left\{ \theta \in (-k\pi, k\pi] : \left| \frac{1}{\rho^k} - e^{i\theta} \right|^2 \leq \frac{M}{1 + t} \right\} \right) d\theta \]

\[ = \int_{-\pi}^{\pi} \chi \left( \left\{ \theta \in (-\pi, \pi] : \left| \frac{1}{\rho^k} - e^{i\theta} \right|^2 \leq \frac{M}{1 + t} \right\} \right) d\theta \]

\[ \leq \lambda \left( \left\{ \theta \in (-\pi, \pi] : |1 - e^{i\theta}| \leq 2\sqrt{\frac{M}{1 + t}} \right\} \right) \]

\[ \leq 2\pi \sqrt{\frac{M}{1 + t}}, \]

so that

\[ I_3(z) \leq 2^{k+2} \pi \int_1^\infty \frac{t^{\alpha-1}}{(1 + t)^\alpha} dt, \]

which concludes the proof of the lemma.

**Proof of Theorem 1.1.** By formula (1.1), \( P[f] \) is continuous on \( \mathbb{D} \). Now let \( z_0 \in S^1 \).

Since

\[ c_\alpha \int_{-\pi}^{\pi} P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta = 1 \quad (z \in \mathbb{D}), \]

we have, for any \( z \in \mathbb{D} \),

\[ P[f](z) - f(z_0) = c_\alpha \int_{-\pi}^{\pi} [f(e^{i\theta}) - f(z_0)] P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta. \]

Let \( \varepsilon > 0 \). There exists \( \delta \in (0, 1] \) such that for any \( w \in S^1 \) satisfying \( |w - z_0| \leq \delta \), \( |f(w) - f(z_0)| \leq \frac{\varepsilon}{2} \).
Using formula (1.1) and the fact that for any \( z \in \mathbb{D} \) satisfying \( |z - z_0| \leq \frac{\delta}{2} \), then if \( |e^{i\theta} - z_0| \geq \delta \), we have \( |1 - ze^{-i\theta}| \geq \frac{\delta}{2} \), we obtain, if \( |z - z_0| \leq \frac{\delta}{2} \):

\[
|P[f](z) - f(z_0)| \leq \frac{\varepsilon}{2} + 2c_\alpha \|f\|_\infty \frac{1 - |z|^2}{B(\alpha, \alpha + (\frac{\delta}{2})^2)} I(z)
\]
with \( \|f\|_\infty = \sup_{w \in S^1} |f(w)| \) and

\[
I(z) = \int_{|e^{i\theta} - z_0| \geq \delta} \int_0^1 \frac{u^{\alpha-1}(1-u)^{\alpha}du}{[(1-u)|1 - z^k e^{ik\theta}|^2 + u|1 - z^k e^{-ik\theta}|^2]^{\alpha}} (\sin^2 k\theta)^\alpha d\theta
\]

\[
\leq I_1(z) + I_2(z),
\]

where \( I_1(z) \) and \( I_2(z) \) are given by formula (2.1). Then it follows from Lemma 2.1 that \( I(z) \) is bounded on \( \{z \in \mathbb{D} : |z - z_0| \leq \frac{\delta}{2} \} \) by a constant depending only on \( \alpha \) and \( k \). Consequently, (2.3) implies that there is \( \eta \in (0, \frac{\delta}{2}] \) such that for any \( z \in \mathbb{D} \) satisfying \( |z - z_0| \leq \eta \), we have \( |P[f](z) - f(z_0)| \leq \varepsilon \), which completes the proof of the theorem. \( \square \)

3. Applications

In this section, we study the boundary behavior of Poisson integrals and establish that under certain conditions an \( h \)-harmonic function in the disc is the Poisson integral of some type of function or measure on the unit circle.

Let \( 1 \leq p \leq \infty \) and \( f \in L^p(h(e^{i\theta})^2 d\theta) \). For fixed \( r \in [0, 1) \), for all \( z, w \in S^1 \), we have

\[
0 \leq P(rz, w) \leq \frac{1 - r^2}{(1 - r)^2 (1 - r^k)^{2\alpha}},
\]
so that the function \( P[f]_r \), defined on \( S^1 \) by \( P[f]_r(z) = P[f](rz) \) is bounded on \( S^1 \) and hence is in \( L^p(h(e^{i\theta})^2 d\theta) \).

If \( 1 < p < \infty \), then, using Hölder’s inequality and (2.2), we get

\[
|P[f]_r(w)|^p \leq c_\alpha \int_{-\pi}^\pi |f(e^{it})|^p P(rw, e^{it}) h(e^{it})^2 dt \quad (w \in S^1),
\]

which is also true if \( p = 1 \). It now follows from Fubini’s theorem that

\[
\|P[f]_r\|_{L^p(h(e^{i\theta})^2 d\theta)} \leq \|f\|_{L^p(h(e^{i\theta})^2 d\theta)},
\]
when \( 1 \leq p < \infty \), since

\[
P(rw, w') = P(rw', w), \quad w, w' \in S^1.
\]

Note that (3.2) holds when \( p = \infty \).

**Theorem 3.1.** (a) Let \( 1 \leq p < \infty \) and \( f \in L^p(h(e^{i\theta})^2 d\theta) \). Then

\[
\|P[f]_r - f\|_p \to 0 \quad (r \to 1),
\]

where the \( L^p \)-norm is with respect to \( h(e^{i\theta})^2 d\theta \).

(b) If \( f \in L^\infty(h(e^{i\theta})^2 d\theta) \), then, as \( r \to 1 \), the functions \( P[f]_r \) converge to \( f \) in the weak-star topology on \( L^\infty(h(e^{i\theta})^2 d\theta) \).

(c) Let \( \mu \) be a complex Borel measure on \( S^1 \), and \( P[\mu] \) be the Poisson integral of the measure \( \mu \) on \( S^1 \); that is,

\[
P[\mu](z) = c_\alpha \int_{S^1} P(z, w) \, d\mu(w) \quad (z \in \mathbb{D}).
\]
Then, as \( r \to 1 \), the measures \( P_\mu(re^{i\theta})h(e^{i\theta})d\theta \) converge to \( \mu \) in the weak-star topology of the dual space of \( C(S^1) \).

(d) If \( P_\mu(z) = 0 \) for all \( z \in \mathbb{D} \), then \( \mu = 0 \).

Proof. To prove (a), we may assume \( f \in C(S^1) \), because of (3.2) and the density of continuous functions in \( L^p(h(e^{i\theta})^2d\theta) \). In this case, the result follows easily from Corollary 1.2.

If \( f \in L^\infty(h(e^{i\theta})^2d\theta) \), \( g \in L^1(h(e^{i\theta})^2d\theta) \) and \( \mu \) is a complex Borel measure on \( S^1 \), then by (3.3) and Fubini’s theorem,

\[
\int_{-\pi}^{\pi} g(e^{i\theta})P[f](e^{i\theta})h(e^{i\theta})^2d\theta = \int_{-\pi}^{\pi} f(e^{i\theta})P[g](e^{i\theta})h(e^{i\theta})^2d\theta
\]

and

\[
\int_{-\pi}^{\pi} g(e^{i\theta})P[\mu](re^{i\theta})h(e^{i\theta})^2d\theta = \int_{S^1} P[g](w) d\mu(w),
\]

so that part (b) follows from part (a) applied to \( g \), and if in addition \( g \) is continuous on \( S^1 \), Corollary 1.2 applied to \( g \) completes the proof of (c).

Part (d) follows from part (c) and the uniqueness assertion of the Riesz representation theorem.

The following proposition is the uniqueness theorem corresponding to Theorem 1.1.

**Proposition 3.2.** Let \( f \) be a complex-valued continuous function on \( \overline{D} \). Assume \( f \) is \( h \)-harmonic on \( \mathbb{D} \). Then

\[
f(z) = c_\alpha \int_{-\pi}^{\pi} f(e^{i\theta})P(z,e^{i\theta})h(e^{i\theta})^2d\theta, \quad z \in \mathbb{D}.
\]

Proof. Without loss of generality, we may assume that \( f \) is real. Put \( u = f - F \) where \( F \) is the function defined by formula (1.2). By Theorem 1.1 and Remark 1.3, \( u \) is continuous on \( \overline{D} \) and \( h \)-harmonic on \( \mathbb{D} \). Thus, by Theorem 4.2 in [R], we obtain

\[
\max_{S^1} (u) = \max_{\overline{D}} (u) = 0,
\]

so that \( u \leq 0 \) on \( \mathbb{D} \). The same argument shows that \( -u \leq 0 \). Hence \( f = F \) on \( \mathbb{D} \), and the proof is complete.

**Theorem 3.3.** Assume \( u \) is a complex-valued \( h \)-harmonic function on \( \overline{D} \). Write \( u_r(w) = u(rw) \) for \( r \in [0,1) \) and \( w \in S^1 \).

(a) If \( 1 < p \leq \infty \), then \( u \) is the Poisson integral of a function \( f \in L^p(h(e^{i\theta})^2d\theta) \) if and only if

\[
\sup_{0 \leq r < 1} \|u_r\|_{L^p(h(e^{i\theta})^2d\theta)} < \infty.
\]

(b) \( u \) is the Poisson integral of a complex Borel measure on \( S^1 \) if and only if

\[
\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |u(re^{i\theta})| |h(e^{i\theta})^2d\theta < \infty.
\]

(c) \( u \) is positive if and only if \( u \) is the Poisson integral of a bounded positive Borel measure on \( S^1 \).

(d) \( u \) is the Poisson integral of a function \( f \in L^1(h(e^{i\theta})^2d\theta) \) if and only if the \( u_r \) converge in \( L^1(h(e^{i\theta})^2d\theta) \).
(e) *u* is the Poisson integral of a function *f* ∈ *C*(S¹) if and only if the *u_r* converge uniformly.

**Proof.** Inequality (3.2) shows that (3.4) is a necessary condition. If *u* = *P*[μ], where μ is a complex Borel measure on S¹, then, using Fubini's theorem, (3.3) and (2.2), we obtain

\[ \sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |u(re^{i\theta})| h(e^{i\theta})^2 d\theta \leq ||\mu||, \]

where ||μ|| denotes the total variation of μ, so that (3.5) holds.

Since for fixed *r* ∈ [0, 1), \( \Delta_h(u(rz)) = r^2(\Delta_h u)(rz) = 0 \) on D, we have by Proposition 3.2,

\[ u(rz) = c_\alpha \int_{-\pi}^{\pi} u(re^{i\theta}) P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta, \quad z \in \mathbb{D}. \]  

(3.6)

Assume *u* satisfies (3.4) or (3.5). Let \( \{r_n\} \) be a sequence in [0, 1) that increases to 1. If 1 < *p* ≤ ∞, let \( f_n = u_{r_n} \); by (3.4), the sequence \( \{f_n\} \) is bounded in \( L^p(h(e^{i\theta})^2 d\theta) \), so that it follows from the Banach-Alaoglu theorem that there is a subsequence \( \{f_{n_j}\} \), \( n_1 < n_2 < \ldots \), that converges weak-star to an element *f* of \( L^p(h(e^{i\theta})^2 d\theta) \). Since for any *z* ∈ D, \( P(z, \cdot) \in L^q(h(e^{i\theta})^2 d\theta), \) \( q = p/(p - 1) \), we obtain, using (3.6),

\[ u(z) = \lim_{j \to \infty} u(r_{n_j}z) = \lim_{j \to \infty} c_\alpha \int_{-\pi}^{\pi} f_{n_j}(e^{i\theta}) P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta = P[f](z) \]

for all \( z \in \mathbb{D} \).

The proof of (b) is the same except that now the measures \( u(r_n e^{i\theta}) h(e^{i\theta})^2 d\theta \) have bounded norms, so that there is a subsequence that converges weak-star to a complex Borel measure on S¹.

If \( u \geq 0 \), the measures \( u(r_n e^{i\theta}) h(e^{i\theta})^2 d\theta \) are positive and have bounded norms, because (3.6) yields

\[ c_\alpha \int_{-\pi}^{\pi} u(r_n e^{i\theta}) h(e^{i\theta})^2 d\theta = u(0). \]

Then, as in the proof of (b), there is a subsequence that converges weak-star to a finite Borel measure μ on S¹, and the measure μ is now positive.

We have already proved that if *u* is the Poisson integral of a function *f* ∈ \( L^1(h(e^{i\theta})^2 d\theta) \) (respectively *f* ∈ \( C(S^1) \)), then the *u_r* converge to *f* in \( L^1(h(e^{i\theta})^2 d\theta) \) (respectively uniformly on S¹). Conversely, if the *u_r* converge to a function *f* in \( L^1(h(e^{i\theta})^2 d\theta) \), then, using (3.6) and (3.1), we get

\[ u(z) = \lim_{r \to 1} u(rz) = \lim_{r \to 1} c_\alpha \int_{-\pi}^{\pi} u(re^{i\theta}) P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta = P[f](z) \]

for all \( z \in \mathbb{D} \).

If the *u_r* converge uniformly, they converge to a continuous *g*, and then \( u = P[g] \) by a similar argument.

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