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# POISSON INTEGRALS ASSOCIATED TO DUNKL OPERATORS FOR DIHEDRAL GROUPS

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Abstract. In this paper we study the boundary behavior of Poisson integrals associated to Dunkl differential-difference operators for dihedral groups and the boundary integral representations for functions on the unit disc of $\mathbb{C}$ annihilated by the Laplace operator corresponding to these differential-difference operators.

## 1. Notation and statement of the main result

For every integer $k$ such that $k \geq 1$, let $D_k$ be the dihedral group of order $2k$; that is, $D_k$ consists of the rotations $z \mapsto ze^{\frac{2\pi il}{k}}$ and the reflections $z \mapsto \overline{z}\, e^{\frac{2\pi il}{k}}$, $0 \leq l \leq k-1$, $z \in \mathbb{C}$.

If $k \geq 1$ is a fixed integer and $\alpha$ is a fixed positive real number, we associate to the group $D_k$ the weight function $h$ defined by

$$h(z) = \left| \frac{z^k - \overline{z}^k}{2i} \right|^{\alpha},$$

which is a product of powers of the linear functions on $\mathbb{R}^2 \cong \mathbb{C}$ whose zero-sets are the mirrors of the reflections in $D_k$.

The complex Dunkl operators are defined for a complex-valued function $f$ of class $C^1$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ by

$$T_h f(z) = \frac{\partial f(z)}{\partial z} + \alpha \sum_{l=0}^{k-1} \frac{f(z) - f(\overline{z}\omega^{2l})}{z - \overline{z}\omega^{2l}}$$

and

$$\overline{T}_h f(z) = \frac{\partial f(z)}{\partial \overline{z}} - \alpha \sum_{l=0}^{k-1} \frac{f(z) - f(\overline{z}\omega^{2l})}{z - \overline{z}\omega^{2l}} \omega^{2l}$$

where $\omega = e^{\frac{\pi i}{k}}$. As in [D1], let $\Delta_h$ denote the $h$-Laplacian operator, $\Delta_h = 4\, T_h \overline{T}_h$, and say that a complex-valued function $f$ on $\mathbb{D}$ is $h$-harmonic if it is of class $C^2$ and $\Delta_h f = 0$ on $\mathbb{D}$.

An orthogonal basis for $L^2(h(e^{i\theta})^2 d\theta)$ on the unit circle was constructed in [D2], and this led to the definition of a Poisson kernel $P$ which reproduces $h$-harmonic

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polynomials in $\mathbb{D}$ from their boundary values, and which is given by

$$P(z,w)=\frac{1-|z|^2|w|^2}{B(\alpha,\alpha+1)\,|1-z\overline{w}|^2}\int_0^1\frac{u^{\alpha-1}(1-u)^{\alpha}du}{[(1-u)|1-z^kw^k|^2+u|1-z^k\overline{w}^k|^2]^{\alpha}} \tag{1.1}$$

for $z,w\in\mathbb{C}$ such that $|zw|<1$ (see [D2, Theorems 1.3 and 2.1]).

Define the Poisson integral $P[f]$ of a function $f\in L^1(h(e^{i\theta})^2d\theta)$ by

$$P[f](z)=c_\alpha\int_{-\pi}^{\pi}f(e^{i\theta})P(z,e^{i\theta})h(e^{i\theta})^2d\theta$$

for $z\in\mathbb{D}$, where $c_\alpha=\left(\int_{-\pi}^{\pi}h(e^{i\theta})^2d\theta\right)^{-1}$.

Let $S^1$ denote the unit circle $\{z\in\mathbb{C}\colon |z|=1\}$ and $C(S^1)$ be the space of all complex-valued continuous functions on $S^1$. The main purpose of this paper is to prove the following result:

**Theorem 1.1.** *If $f\in C(S^1)$, then the function $F$ defined by*

$$F(z)=\begin{cases}P[f](z), & \text{if } z\in\mathbb{D},\\ f(z), & \text{if } z\in S^1,\end{cases} \tag{1.2}$$

*is continuous on $\overline{\mathbb{D}}$.*

Trivially, we have the following corollary.

**Corollary 1.2.** *If $f\in C(S^1)$, then*

$$\sup_{w\in S^1}|P[f](rw)-f(w)|\to 0\quad (r\to 1).$$

It should be noted that the boundary behavior of Poisson integrals associated to Dunkl operators acting on functions on $\mathbb{R}^d$, $d\geq 2$, and corresponding to the abelian group $\mathbb{Z}_2^d$, was considered in Theorem 5.5.7 of [DX].

Theorem 1.1 will be used in Section 3 to study the boundary behavior of Poisson integrals and give conditions under which an $h$-harmonic function in the disc is the Poisson integral of some type of function or measure on the unit circle.

*Remark* 1.3. If $f\in L^1(h(e^{i\theta})^2d\theta)$, then $P[f]$ is $h$-harmonic on $\mathbb{D}$. Indeed, as stated in the proof of Theorem 1.3 of [D2], if $\{\psi_n(z),\overline{z}\overline{\psi_n}(z)\colon n\geq 0\}$ denotes the orthonormal basis of $L^2(c_\alpha h_1(e^{i\theta})^2d\theta)$ associated to $h_1(x+iy):=|y|^\alpha$, the Cauchy kernel associated to $h$ is given by

$$C(z,w)=\sum_{n=0}^{\infty}\sum_{l=0}^{k-1}z^l\psi_n(z^k)\,\overline{w}^l\,\overline{\psi_n}(w^k)\qquad (|zw|<1), \tag{1.3}$$

with $\overline{T}_h(z^l\psi_n(z^k))=0$ for $0\leq l\leq k-1$ and $n\geq 0$. Using (1.3) and the expression of the $\psi_n$ in terms of Heisenberg polynomials (see [D1, Proposition 3.11] and [D2]), we easily get that for fixed $w\in S^1$, $\overline{T}_hC(z,w)=0$ for all $z\in\mathbb{D}$, so that $z\mapsto P(z,w)$ is $h$-harmonic on $\mathbb{D}$, because $P(z,w)=C(z,w)+\overline{z}w\,C(w,z)$ and $\Delta_h(z\,C(z,w))=z\Delta_hC(z,w)+4\,\overline{T}_hC(z,w)$ (see [D1, Proposition 2.2]). From this we easily deduce that $P[f]$ is $h$-harmonic on $\mathbb{D}$.

Thus Theorem 1.1 provides the solution of the analogue of the Dirichlet problem for the unit disc of $\mathbb{C}$ associated to the $h$-Laplacian operator. The uniqueness theorem that corresponds to this existence theorem is proved in Section 3.

## 2. Proof of the main result

For $z \in \mathbb{D}$ and $\theta \in (-\pi, \pi]$, set

$$a(z,\theta) = |1 - z^k e^{ik\theta}|^2$$

and

$$b(z,\theta) = |1 - z^k e^{-ik\theta}|^2.$$

We consider the integral

$$\int_{-\pi}^{\pi}\int_0^1 \frac{u^{\alpha-1}(1-u)^\alpha du}{[(1-u)\,a(z,\theta) + u\,b(z,\theta)]^\alpha}(\sin^2 k\theta)^\alpha d\theta \qquad (z \in \mathbb{D})$$

in two pieces: the first is taken over $A_1(z) = \{\theta \in (-\pi,\pi] \colon b(z,\theta) \leq 2\,a(z,\theta)\}$, and the second over the complement of $A_1(z)$, denoted by $A_2(z)$; for $j \in \{1,2\}$ and $z \in \mathbb{D}$, set

$$I_j(z) = \int_{A_j(z)}\int_0^1 \frac{u^{\alpha-1}(1-u)^\alpha du}{[(1-u)\,a(z,\theta) + u\,b(z,\theta)]^\alpha}(\sin^2 k\theta)^\alpha d\theta. \tag{2.1}$$

Then we have the following

**Lemma 2.1.** *Set $K = 2\pi\left(\frac{1+\sqrt{2}}{2}\right)^{2\alpha}$. For any $z \in \mathbb{D}$,*

$$I_1(z) \leq \frac{K}{\alpha},$$

*and if $z \in \mathbb{D}$ satisfies $|z| \geq \frac{1}{2}$, then*

$$I_2(z) \leq K\left(\int_0^1 \frac{t^{\alpha-1}}{(1+t)^\alpha}dt + 2^{k+1}\int_1^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\frac{1}{2}}}dt\right).$$

*Proof.* Let $z \in \mathbb{D}$. We have $|\sin k\theta| \leq \frac{1}{2}(\sqrt{a(z,\theta)} + \sqrt{b(z,\theta)})$, so that for any $\theta \in A_1(z)$, $(\sin^2 k\theta)^\alpha \leq \left(\frac{1+\sqrt{2}}{2}\right)^{2\alpha} a(z,\theta)^\alpha$ and

$$I_1(z) \leq K\int_0^1 u^{\alpha-1}du = \frac{K}{\alpha}.$$

If $\theta \in A_2(z)$ and $u \in [0,1]$, then $(\sin^2 k\theta)^\alpha \leq K' b(z,\theta)^\alpha$ where $K' = \left(\frac{1+\frac{1}{\sqrt{2}}}{2}\right)^{2\alpha}$ and $(1-u)\,a(z,\theta) + u\,b(z,\theta) \geq a(z,\theta) + u\frac{b(z,\theta)}{2}$, so that

$$\begin{aligned} I_2(z) &\leq K'\int_{A_2(z)} \frac{b(z,\theta)^\alpha}{a(z,\theta)^\alpha}\int_0^1 \frac{u^{\alpha-1}}{\left(1 + u\frac{b(z,\theta)}{2a(z,\theta)}\right)^\alpha}du\,d\theta \\ &= \frac{K}{2\pi}\int_{A_2(z)}\int_0^{c(z,\theta)} \frac{t^{\alpha-1}}{(1+t)^\alpha}dt\,d\theta, \end{aligned}$$

where $c(z,\theta) = \frac{b(z,\theta)}{2\,a(z,\theta)} > 1$. Then

$$I_2(z) \leq K\int_0^1 \frac{t^{\alpha-1}}{(1+t)^\alpha}dt + \frac{K}{2\pi}I_3(z)$$

with

$$\begin{aligned}I_3(z) &= \int_{A_2(z)} \int_1^{c(z,\theta)} \frac{t^{\alpha-1}}{(1+t)^\alpha} dt\, d\theta \\ &= \int_1^\infty \frac{t^{\alpha-1}}{(1+t)^\alpha} \lambda(A_2(z) \cap \{\theta \in (-\pi,\pi]\colon c(z,\theta) \geq t\}) dt,\end{aligned}$$

where $\lambda$ denotes the Lebesgue measure on $(-\pi,\pi]$.

For any $t \geq 1$, we have

$$\begin{aligned}\lambda(\{\theta \in (-\pi,\pi]\colon c(z,\theta) \geq t\}) &\leq \lambda(\{\theta \in (-\pi,\pi]\colon b(z,\theta) \geq a(z,\theta)(1+t)\}) \\ &\leq \lambda(\{\theta \in (-\pi,\pi]\colon a(z,\theta) \leq \frac{4}{1+t}\})\end{aligned}$$

because $b(z,\theta) \leq 4$. Set $z = \rho e^{i\phi}$, $M = 4^{k+1}$, and if $A$ is a Borel set of $\mathbb{R}$, let $\chi_A$ be the characteristic function of $A$; if $\rho \geq \frac{1}{2}$, then

$$\begin{aligned}&\lambda\left(\left\{\theta \in (-\pi,\pi]\colon a(z,\theta) \leq \frac{4}{1+t}\right\}\right) \\ &\quad \leq \lambda\left(\left\{\theta \in (-\pi,\pi]\colon \left|\frac{1}{\rho^k} - e^{ik(\theta+\phi)}\right|^2 \leq \frac{M}{1+t}\right\}\right) \\ &\quad = \lambda\left(\left\{u \in (-\pi,\pi]\colon \left|\frac{1}{\rho^k} - e^{iku}\right|^2 \leq \frac{M}{1+t}\right\}\right) \\ &\quad = \frac{1}{k}\int_{-k\pi}^{k\pi} \chi_{\left\{\theta\in(-k\pi,k\pi]\colon\left|\frac{1}{\rho^k}-e^{i\theta}\right|^2\leq\frac{M}{1+t}\right\}} d\theta \\ &\quad = \int_{-\pi}^{\pi} \chi_{\left\{\theta\in(-\pi,\pi]\colon\left|\frac{1}{\rho^k}-e^{i\theta}\right|^2\leq\frac{M}{1+t}\right\}} d\theta \\ &\quad \leq \lambda\left(\left\{\theta \in (-\pi,\pi]\colon |1-e^{i\theta}| \leq 2\sqrt{\frac{M}{1+t}}\right\}\right) \\ &\quad \leq 2\pi\sqrt{\frac{M}{1+t}},\end{aligned}$$

so that

$$I_3(z) \leq 2^{k+2}\pi \int_1^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\frac{1}{2}}} dt,$$

which concludes the proof of the lemma. □

*Proof of Theorem* 1.1. By formula (1.1), $P[f]$ is continuous on $\mathbb{D}$. Now let $z_0 \in S^1$. Since

$$c_\alpha \int_{-\pi}^{\pi} P(z,e^{i\theta}) h(e^{i\theta})^2 d\theta = 1 \qquad (z \in \mathbb{D}), \tag{2.2}$$

we have, for any $z \in \mathbb{D}$,

$$P[f](z) - f(z_0) = c_\alpha \int_{-\pi}^{\pi} [f(e^{i\theta}) - f(z_0)] P(z,e^{i\theta}) h(e^{i\theta})^2 d\theta.$$

Let $\varepsilon > 0$. There exists $\delta \in (0,1]$ such that for any $w \in S^1$ satisfying $|w - z_0| \leq \delta$, $|f(w) - f(z_0)| \leq \frac{\varepsilon}{2}$.

Using formula (1.1) and the fact that for any $z \in \mathbb{D}$ satisfying $|z - z_0| \leq \frac{\delta}{2}$, then if $|e^{i\theta} - z_0| \geq \delta$, we have $|1 - ze^{-i\theta}| \geq \frac{\delta}{2}$, we obtain, if $|z - z_0| \leq \frac{\delta}{2}$:

$$|P[f](z) - f(z_0)| \leq \frac{\varepsilon}{2} + 2\, c_\alpha ||f||_\infty \frac{1 - |z|^2}{B(\alpha, \alpha+1)(\frac{\delta}{2})^2} I(z) \tag{2.3}$$

with $||f||_\infty = \sup_{w \in S^1} |f(w)|$ and

$$\begin{aligned} I(z) &= \int_{|e^{i\theta} - z_0| \geq \delta} \int_0^1 \frac{u^{\alpha-1}(1-u)^\alpha du}{[(1-u)|1 - z^k e^{ik\theta}|^2 + u|1 - z^k e^{-ik\theta}|^2]^\alpha} (\sin^2 k\theta)^\alpha d\theta \\ &\leq I_1(z) + I_2(z), \end{aligned}$$

where $I_1(z)$ and $I_2(z)$ are given by formula (2.1). Then it follows from Lemma 2.1 that $I(z)$ is bounded on $\{z \in \mathbb{D}\colon |z - z_0| \leq \frac{\delta}{2}\}$ by a constant depending only on $\alpha$ and $k$. Consequently, (2.3) implies that there is $\eta \in (0, \frac{\delta}{2}]$ such that for any $z \in \mathbb{D}$ satisfying $|z - z_0| \leq \eta$, we have $|P[f](z) - f(z_0)| \leq \varepsilon$, which completes the proof of the theorem. $\square$

## 3. Applications

In this section, we study the boundary behavior of Poisson integrals and establish that under certain conditions an $h$-harmonic function in the disc is the Poisson integral of some type of function or measure on the unit circle.

Let $1 \leq p \leq \infty$ and $f \in L^p(h(e^{i\theta})^2 d\theta)$. For fixed $r \in [0, 1)$, for all $z, w \in S^1$, we have

$$0 \leq P(rz, w) \leq \frac{1 - r^2}{(1-r)^2} \frac{1}{(1 - r^k)^{2\alpha}}, \tag{3.1}$$

so that the function $P[f]_r$ defined on $S^1$ by $P[f]_r(z) = P[f](rz)$ is bounded on $S^1$ and hence is in $L^p(h(e^{i\theta})^2 d\theta)$.

If $1 < p < \infty$, then, using Hölder's inequality and (2.2), we get

$$|P[f]_r(w)|^p \leq c_\alpha \int_{-\pi}^{\pi} |f(e^{it})|^p P(rw, e^{it}) h(e^{it})^2 dt \quad (w \in S^1),$$

which is also true if $p = 1$. It now follows from Fubini's theorem that

$$||P[f]_r||_{L^p(h(e^{i\theta})^2 d\theta)} \leq ||f||_{L^p(h(e^{i\theta})^2 d\theta)}, \tag{3.2}$$

when $1 \leq p < \infty$, since

$$P(rw, w') = P(rw', w), \qquad w, w' \in S^1. \tag{3.3}$$

Note that (3.2) holds when $p = \infty$.

**Theorem 3.1.** (a) *Let $1 \leq p < \infty$ and $f \in L^p(h(e^{i\theta})^2 d\theta)$. Then*

$$||P[f]_r - f||_p \to 0 \quad (r \to 1),$$

*where the $L^p$-norm is with respect to $h(e^{i\theta})^2 d\theta$.*

(b) *If $f \in L^\infty(h(e^{i\theta})^2 d\theta)$, then, as $r \to 1$, the functions $P[f]_r$ converge to $f$ in the weak-star topology on $L^\infty(h(e^{i\theta})^2 d\theta)$.*

(c) *Let $\mu$ be a complex Borel measure on $S^1$, and $P[\mu]$ be the Poisson integral of the measure $\mu$ on $S^1$; that is,*

$$P[\mu](z) = c_\alpha \int_{S^1} P(z, w)\, d\mu(w) \qquad (z \in \mathbb{D}).$$

*Then, as* $r \to 1$, *the measures* $P[\mu](re^{i\theta})h(e^{i\theta})^2 d\theta$ *converge to* $\mu$ *in the weak-star topology of the dual space of* $C(S^1)$.

(d) *If* $P[\mu](z) = 0$ *for all* $z \in \mathbb{D}$, *then* $\mu = 0$.

*Proof.* To prove (a), we may assume $f \in C(S^1)$, because of (3.2) and the density of continuous functions in $L^p(h(e^{i\theta})^2 d\theta)$. In this case, the result follows easily from Corollary 1.2.

If $f \in L^\infty(h(e^{i\theta})^2 d\theta)$, $g \in L^1(h(e^{i\theta})^2 d\theta)$ and $\mu$ is a complex Borel measure on $S^1$, then by (3.3) and Fubini's theorem,

$$\int_{-\pi}^{\pi} g(e^{i\theta}) P[f]_r(e^{i\theta}) h(e^{i\theta})^2 d\theta = \int_{-\pi}^{\pi} f(e^{i\theta}) P[g]_r(e^{i\theta}) h(e^{i\theta})^2 d\theta$$

and

$$\int_{-\pi}^{\pi} g(e^{i\theta}) P[\mu](re^{i\theta}) h(e^{i\theta})^2 d\theta = \int_{S^1} P[g]_r(w)\, d\mu(w),$$

so that part (b) follows from part (a) applied to $g$, and if in addition $g$ is continuous on $S^1$, Corollary 1.2 applied to $g$ completes the proof of (c).

Part (d) follows from part (c) and the uniqueness assertion of the Riesz representation theorem. ☐

The following proposition is the uniqueness theorem corresponding to Theorem 1.1.

**Proposition 3.2.** *Let* $f$ *be a complex-valued continuous function on* $\overline{\mathbb{D}}$. *Assume* $f$ *is h-harmonic on* $\mathbb{D}$. *Then*

$$f(z) = c_\alpha \int_{-\pi}^{\pi} f(e^{i\theta}) P(z, e^{i\theta})\, h(e^{i\theta})^2 d\theta, \qquad z \in \mathbb{D}.$$

*Proof.* Without loss of generality, we may assume that $f$ is real. Put $u = f - F$ where $F$ is the function defined by formula (1.2). By Theorem 1.1 and Remark 1.3, $u$ is continuous on $\overline{\mathbb{D}}$ and $h$-harmonic on $\mathbb{D}$. Thus, by Theorem 4.2 in [R], we obtain

$$\max_{\overline{\mathbb{D}}}(u) = \max_{S^1}(u) = 0,$$

so that $u \leq 0$ on $\mathbb{D}$. The same argument shows that $-u \leq 0$. Hence $f = F$ on $\mathbb{D}$, and the proof is complete. ☐

**Theorem 3.3.** *Assume* $u$ *is a complex-valued h-harmonic function on* $\mathbb{D}$. *Write* $u_r(w) = u(rw)$ *for* $r \in [0, 1)$ *and* $w \in S^1$.

(a) *If* $1 < p \leq \infty$, *then* $u$ *is the Poisson integral of a function* $f \in L^p(h(e^{i\theta})^2 d\theta)$ *if and only if*

$$\sup_{0 \leq r < 1} ||u_r||_{L^p(h(e^{i\theta})^2 d\theta)} < \infty. \tag{3.4}$$

(b) *u is the Poisson integral of a complex Borel measure on* $S^1$ *if and only if*

$$\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |u(re^{i\theta})|\, h(e^{i\theta})^2 d\theta < \infty. \tag{3.5}$$

(c) *u is positive if and only if u is the Poisson integral of a bounded positive Borel measure on* $S^1$.

(d) *u is the Poisson integral of a function* $f \in L^1(h(e^{i\theta})^2 d\theta)$ *if and only if the* $u_r$ *converge in* $L^1(h(e^{i\theta})^2 d\theta)$.

(e) *$u$ is the Poisson integral of a function $f \in C(S^1)$ if and only if the $u_r$ converge uniformly.*

*Proof.* Inequality (3.2) shows that (3.4) is a necessary condition. If $u = P[\mu]$, where $\mu$ is a complex Borel measure on $S^1$, then, using Fubini's theorem, (3.3) and (2.2), we obtain

$$\sup_{0\leq r<1} \int_{-\pi}^{\pi} |u(re^{i\theta})|\, h(e^{i\theta})^2 d\theta \leq ||\mu||,$$

where $||\mu||$ denotes the total variation of $\mu$, so that (3.5) holds.

Since for fixed $r \in [0,1)$, $\Delta_h(u(rz)) = r^2(\Delta_h u)(rz) = 0$ on $\mathbb{D}$, we have by Proposition 3.2,

$$u(rz) = c_\alpha \int_{-\pi}^{\pi} u(re^{i\theta}) P(z, e^{i\theta})\, h(e^{i\theta})^2 d\theta, \qquad z \in \mathbb{D}. \tag{3.6}$$

Assume $u$ satisfies (3.4) or (3.5). Let $\{r_n\}$ be a sequence in $[0,1)$ that increases to 1. If $1 < p \leq \infty$, let $f_n = u_{r_n}$; by (3.4), the sequence $\{f_n\}$ is bounded in $L^p(h(e^{i\theta})^2 d\theta)$, so that it follows from the Banach-Alaoglu theorem that there is a subsequence $\{f_{n_j}\}$, $n_1 < n_2 < \ldots$, that converges weak-star to an element $f$ of $L^p(h(e^{i\theta})^2 d\theta)$. Since for any $z \in \mathbb{D}$, $P(z,.) \in L^q(h(e^{i\theta})^2 d\theta)$, $q = p/(p-1)$, we obtain, using (3.6),

$$u(z) = \lim_{j\to\infty} u(r_{n_j} z) = \lim_{j\to\infty} c_\alpha \int_{-\pi}^{\pi} f_{n_j}(e^{i\theta}) P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta = P[f](z)$$

for all $z \in \mathbb{D}$.

The proof of (b) is the same except that now the measures $u(r_n e^{i\theta}) h(e^{i\theta})^2 d\theta$ have bounded norms, so that there is a subsequence that converges weak-star to a complex Borel measure on $S^1$.

If $u \geq 0$, the measures $u(r_n e^{i\theta}) h(e^{i\theta})^2 d\theta$ are positive and have bounded norms, because (3.6) yields

$$c_\alpha \int_{-\pi}^{\pi} u(r_n e^{i\theta}) h(e^{i\theta})^2 d\theta = u(0).$$

Then, as in the proof of (b), there is a subsequence that converges weak-star to a finite Borel measure $\mu$ on $S^1$, and the measure $\mu$ is now positive.

We have already proved that if $u$ is the Poisson integral of a function $f \in L^1(h(e^{i\theta})^2 d\theta)$ (respectively $f \in C(S^1)$), then the $u_r$ converge to $f$ in $L^1(h(e^{i\theta})^2 d\theta)$ (respectively uniformly on $S^1$). Conversely, if the $u_r$ converge to a function $f$ in $L^1(h(e^{i\theta})^2 d\theta)$, then, using (3.6) and (3.1), we get

$$u(z) = \lim_{r\to 1} u(rz) = \lim_{r\to 1} c_\alpha \int_{-\pi}^{\pi} u(re^{i\theta}) P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta = P[f](z)$$

for all $z \in \mathbb{D}$.

If the $u_r$ converge uniformly, they converge to a continuous $g$, and then $u = P[g]$ by a similar argument. □

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