SURFACES, SUBMANIFOLDS, AND ALIGNED FOX REIMBEDDING IN NON-HAKEH 3-MANIFOLDS

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Abstract. Understanding non-Haken 3-manifolds is central to many current endeavors in 3-manifold topology. We describe some results for closed orientable surfaces in non-Haken manifolds, and extend Fox’s theorem for submanifolds of the 3-sphere to submanifolds of general non-Haken manifolds. In the case where the submanifold has connected boundary, we show also that the ∂-connected sum decomposition of the submanifold can be aligned with such a structure on the submanifold’s complement.

1. Introduction

A closed orientable irreducible 3-manifold \( N \) is called Haken if it contains a closed orientable incompressible surface; otherwise \( N \) is non-Haken. In Section 2 we describe some results for surfaces in non-Haken manifolds. Generalizing a theorem of Fox ([F]), we show in Section 3 that a 3-dimensional submanifold of a non-Haken manifold \( N \) is homeomorphic either to a handlebody complement in \( N \) or the complement of a handlebody in \( S^3 \). Sections 2 and 3 are independent, but both represent progress towards understanding submanifolds of non-Haken manifolds. In Section 4 we combine the techniques from Section 2 with the results from Section 3 to show that if the submanifold \( M \subset N \) is \( \partial \)-reducible and has connected boundary, then the embedding can be chosen to align a full collection of separating \( \partial \)-reducing disks in \( M \) with similar disks in the complement of \( M \).

2. Handlebodies in non-Haken manifolds

Let \( N \) be a closed orientable 3-manifold, \( F \) a closed orientable surface of non-trivial genus imbedded in \( N \). Recall that \( F \) is compressible if there exists an essential simple closed curve on \( F \) that bounds an imbedded disk \( D \) in \( N \) with interior disjoint from \( F \). \( D \) is a compressing disk for \( F \).

Definition 1. Suppose \( F \) is a separating closed surface in an orientable irreducible closed 3-manifold \( N \). \( F \) is reducible if there exists an essential simple closed curve on \( F \) that bounds compressing disks on both sides of \( F \). The union of the two compressing disks is a reducing sphere for \( F \).
Suppose $S$ is a collection of disjoint reducing spheres for $F$. A reducing sphere $S \in S$ is redundant if a component of $F - S$ that is adjacent to $S \cap F$ is planar. $S$ is complete if, for any disjoint reducing sphere $S'$, $S'$ is redundant in $S \cup S'$.

Let $\sigma(S)$ denote the number of components of $F - S$ that are not planar surfaces.

Since $N$ is irreducible, any sphere in $N$ is necessarily separating. Suppose a reducing sphere $S'$ is added to a collection $S$ of disjoint reducing spheres. If $S'$ is redundant, the number of non-planar complementary components in $F$ is unchanged, since $S'$ necessarily separates the component of $F - S$ that it intersects and the union of two planar surfaces along a single boundary component is still planar. If $S'$ is not redundant, then the number of non-planar complementary components in $F$ increases by one. Thus we have:

**Lemma 2.** Suppose $S \subset S'$ are two collections of disjoint reducing spheres for $F$ in $N$. Then $\sigma(S) \leq \sigma(S')$. Equality holds if and only if each sphere $S'$ in $S' - S$ is redundant in $S' \cup S$. In particular, $S$ is complete if and only if for every collection $S'$ such that $S \subset S'$, $\sigma(S) = \sigma(S')$.

Let $H$ be a handlebody imbedded in $N$. $H$ has an unknotted core if there exists a pair of transverse simple closed curves $c, d \subset \partial H$ such that $c \cap d$ is a single point, $d$ bounds an imbedded disk in $H$ and $c$ (the core) bounds an imbedded disk in $N$ transverse to $\partial H$. Note that the interior of the latter imbedded disk may intersect $H$.

**Lemma 3.** Let $F$ be a connected, closed, separating, orientable surface in a closed orientable irreducible 3-manifold $N$. Suppose that $F$ has compressing disks to both sides. Then at least one of the following must hold:

1. $F$ is a Heegaard surface for $N$.
2. $N$ is Haken.
3. There exist disjoint compressing disks for $F$ on opposite sides of $F$.

**Proof.** The proof is an application of the generalized Heegaard decomposition described in [ST]. Since $F$ is compressible to both sides, we can construct a handle decomposition of $N$ starting at $F$ so that $F$ appears as a “thick” surface in the decomposition. If $F$ is not a Heegaard surface, then this decomposition contains a “thin” surface $G$ adjacent to $F$. If $G$ is incompressible in $N$, then $N$ is Haken. If $G$ is compressible we apply [CG] to obtain the required disjoint compressing disks for $F$.

**Theorem 4.** Let $H$ be a handlebody of genus $g$ imbedded in a closed orientable irreducible non-Haken 3-manifold $N$. Let $G$ be the complement of $H$ in $N$. Let $F = \partial H = \partial G$. Suppose $F$ is compressible in $G$. Then at least one of the following must hold:

1. The Heegaard genus of $N$ is less than or equal to $g$.
2. $F$ is reducible.
3. $H$ has an unknotted core.

**Proof.** The proof is by induction on the genus of $H$. If $g = 1$, then the result of compressing $F$ into $G$ is a 2-sphere, necessarily bounding a ball in $N$. If a ball it bounds lies in $G$, then the Heegaard genus of $N$ is $\leq 1$. If a ball it bounds contains $H$, then $H$ is an unknotted solid torus in $N$, and so it has an unknotted core.
Suppose then that $\text{genus}(H) = g > 1$ and assume inductively that the theorem is true for handlebodies of genus $g - 1$. Suppose that $G$, the complement of $H$, has compressible boundary. If $G$ is a handlebody, then $G \cup F$ is a Heegaard splitting of genus $g$ and we are done. So suppose $G$ is not a handlebody. Then by Lemma 3 there are disjoint compressing disks on opposite sides of $F$, say $D$ in $H$ and $E$ in $G$. Without loss of generality, we can assume that $D$ is non-separating. Compress $H$ along $D$ to obtain a new handlebody $H_1$ with boundary $F_1$; let $G_1$ be the complement of $H_1$.

If $\partial E$ is inessential in $F_1$, then it bounds a disk in $H_1 \subset H$ as well, so $F$ is reducible.

If $\partial E$ is essential in $F_1$, then $E$ is a compressing disk in $G_1$ and so we can apply the inductive hypothesis to $H_1$. If 1 or 3 holds, then it holds for $H$, and we are done. Suppose instead $F_1$ is reducible. Let $S$ be a collection of disjoint reducing spheres for $F_1$ chosen to maximize $\sigma$ among all possible such collections and then, subject to that condition, further choose $S$ to minimize $|E \cap S|$. Clearly $E \cap S$ contains no closed curves, else replacing a subdisk lying in the disk collection $S \cap G_1$ with an innermost disk of $E - S$ would reduce $|E \cap S|$. Similarly, we have

**Claim 1.** Suppose $\epsilon$ is an arc component of $\partial E - S$ and $F_0$ is the component of $F_1 - S$ in which $\epsilon$ lies. If $\epsilon$ separates $F_0$ (so the ends of $\epsilon$ necessarily lie on the same component of $\partial F_0$), then neither component of $F_0 - \epsilon$ is planar.

**Proof of Claim 1.** Let $c_0$ be the closed curve component of $\partial F_0 \subset S \cap F_1$ on which the ends of $\epsilon$ lie and, of the two arcs into which the ends of $\epsilon$ divide $c_0$, let $\alpha$ be adjacent to a planar component of $F_0 - \epsilon$. Then the curve $\epsilon \cup \alpha$ clearly bounds a disk in both $G_1$ and $H_1$, and then so does the curve $\epsilon' = \epsilon \cup (c_0 - \alpha)$. Let $S'$ be a sphere in $N$ intersecting $F_1$ in $\epsilon'$ and $S_0$ be the reducing sphere in $S$ containing $c_0$. Replacing $S_0$ with $S'$ (or just deleting $S_0$ if $\epsilon'$ is inessential in $F_1$) gives a new collection $S'$ of disjoint reducing spheres, intersecting $\partial E$ in at least two fewer points. Moreover $\sigma(S') = \sigma(S)$ since the only change in the complementary components in $F_1$ is to add to one component and delete from another a planar surface along an arc in the boundary. Then the collection $S'$ contradicts our initial choice for $S$, a contradiction that proves the claim.

Let $H'$ be the closed complement of $S$ in $H_1$, so $H'$ is itself a collection of handlebodies.

**Claim 2.** Either $F$ is reducible or $\partial H'$ is compressible in $N - H'$.

**Proof of Claim 2.** If $\partial E$ is disjoint from $S$ and is inessential in $\partial H'$, then $\partial E$ bounds a disk in $H'$, hence in $H$, so $F$ is reducible. If $\partial E$ is disjoint from $S$ and is essential in $\partial H'$, then $E$ compresses $\partial H'$ in $N - H'$, verifying the claim. Finally, if $E$ intersects $S$, consider an outermost disk $A$ cut off from $E$ by $S$. According to Claim 1, this disk, together with a subdisk of $S$, constitute a disk $E'$ that compresses $\partial H'$ in $N - H'$, proving the claim.

Following Claim 2, either $F$ is reducible or the inductive hypothesis applies to a component $H_0$ of $H'$. If 2 holds for $H_0$, then consider a reducing sphere $S$ for $H_0$, isotoped so that the curve $c = S \cap \partial H_0$ is disjoint from the disks $S \cap H_0$. The disk $S - H_0$ may intersect $H_1$; by general position with respect to the dual 1-handles, each component of intersection is a disk parallel to a component of $S \cap H_1$. But each such disk can be replaced by the corresponding disk in $S - H_1$ so that in the
end c also bounds a disk in \( N - H_1 \). After this change, \( S \) is a reducing sphere for \( F_1 \) in \( N \) and, since \( c \) is essential in \( H_0 \), \( \sigma(S \cup S) > \sigma(S) \), contradicting our initial choice for \( S \). Thus in fact 1 or 3 holds for \( H_0 \), hence also for \( H \).

In the specific case \( N = S^3 \), we apply precisely the same argument, combined with Waldhausen’s theorem \([W]\) on Heegaard splittings of \( S^3 \), to obtain:

\textbf{Corollary 5.} Let \( H \) be a handlebody imbedded in \( S^3 \), and suppose \( G \), the complement of \( H \), has compressible boundary. Then either \( H \) has an unknotted core or the boundary of \( H \) is reducible.

This corollary is similar to \((\text{MT}, \text{Theorem 1.1})\), but no reimbedding of \( S^3 - H \) is required.

## 3. COMPLEMENTS OF HANDLEBODIES IN NON-HAKEN MANIFOLDS

In \([F]\) (see also \([\text{MT}]\) for a brief version) Fox showed that any compact connected 3-dimensional submanifold \( M \) of \( S^3 \) is homeomorphic to the complement of a union of handlebodies in \( S^3 \). We generalize this result to non-Haken manifolds, showing that a submanifold \( M \) of a non-Haken manifold \( N \) has an almost equally simple description, that is, \( M \) is homeomorphic to the complement of handlebodies either in \( S^3 \) or in \( N \).

\textbf{Definition 6.} Let \( N \) be a compact irreducible 3-manifold, and let \( M \) be a compact 3-submanifold of \( N \). We will say the complement of \( M \) in \( N \) is \( \text{standard} \) if it is homeomorphic to a collection of handlebodies or to \( N \# (\text{collection of handlebodies}) \). (We regard \( B^3 \) as a handlebody of genus 0.)

Note that in the latter case \( M \) is actually homeomorphic to the complement of a collection of handlebodies in \( S^3 \).

\textbf{Theorem 7.} Let \( N \) be a closed orientable irreducible non-Haken 3-manifold, and let \( M \) be a connected compact 3-submanifold of \( N \) with non-empty boundary. Then \( M \) is homeomorphic to a submanifold of \( N \) whose complement is standard.

\textbf{Proof.} The proof will be by induction on \( n + g \) where \( n \) is the number of components of \( \partial M \) and \( g \) is the genus of \( \partial M \), that is, the sum of the genera of its components. If \( n + g = 1 \), then \( \partial M \) is a single sphere. Since \( N \) is irreducible, the sphere bounds a 3-ball in \( N \). So either \( M \) or its complement is a 3-ball and in either case the proof is immediate.

To verify the inductive step, suppose first that \( \partial M \) has multiple components \( T_1, \ldots, T_n, n \geq 2 \). Each component \( T_i \) must bound a distinct component \( J_i \) of \( N - M \) since each must be separating in the non-Haken manifold \( N \). Let \( M' = M \cup J_n \); by inductive assumption \( M' \) can be reimbedded so that its complement is standard. After the reimbedding, remove \( J_n \) from \( M' \) to recover a homeomorph of \( M \) and adjoin \( J_1 \) (now homeomorphic either to a handlebody or to \( N \# (\text{handlebody}) \)) instead. Reimbed the resulting manifold so that its complement is standard and remove \( J_1 \) to recover \( M \), now with standard complement.

Henceforth we can therefore assume that \( \partial M \) is connected and not a sphere. Since \( N \) is non-Haken there exists a compressing disk \( D \) for \( \partial M \) in \( N - \partial M \); the compressing disk lies either in \( M \) or in its closed complement \( \overline{M} \).
Case 1. \( \partial D \) is non-separating on \( \partial M \).

If \( D \) lies inside \( M \), compress \( M \) along \( D \) to obtain \( M' \) and use the induction hypothesis to find an imbedding of \( M' \) with standard complement. Reconstruct \( M \) by attaching a trivial 1-handle to \( M' \), thus simultaneously attaching a trivial 1-handle to the complement.

If \( D \) lies outside \( M \), attach a 2-handle to \( M \) corresponding to \( D \) to obtain \( M' \), whose connected boundary has lower genus. Invoking the inductive hypothesis, imbed \( M' \) in \( N \) with standard complement. Reconstruct \( M \) from \( M' \) by removing a co-core of the attached 2-handle, thus adding a 1-handle to the complement of \( M' \).

Case 2. \( \partial D \) is separating on \( \partial M \).

Suppose \( D \) lies outside \( M \). Then \( D \) also separates the closed complement \( J \) of \( M \) into two components, \( J_1 \) and \( J_2 \), since \( H_2(N) = 0 \). Denote the components of \( \partial M - \partial D \) by \( \partial_1 \subset J_1 \) and \( \partial_2 \subset J_2 \), both of positive genus. Let \( M' = M \cup J_2 \). Reimbed \( M' \) so that its complement is standard. The boundary of \( M' \) consists of \( \partial_1 \) together with a disk. Since the complement of \( M' \) is standard, there is a non-separating compressing disk \( D' \) for \( \partial M' \) contained in the complement of \( M' \). \( D' \) is also a non-separating compressing disk for the reimbedded \( \partial M \) (which is contained in \( M' \)). Apply case 1 to this new imbedding of \( M \).

We can now suppose that the only compressing disks for \( \partial M \) are separating compressing disks lying inside \( M \). Choose a family \( D \) of such \( \partial \)-reducing disks for \( M \) that is maximal in the sense that no component of \( M' = M - D \) is itself \( \partial \)-compressible. Since each compressing disk is separating, \( \text{genus}(\partial M') > 0 \), so \( \partial M' \) is compressible in \( N \). Such a compressing disk \( E \) cannot lie inside \( M' \), by construction, so it lies in the connected manifold \( N - M' \); let \( M_1 \) be the component of \( M' \) on whose boundary \( \partial E \) lies. Since each disk in \( D \) was separating, \( M \) has the simple topological description that it is the boundary-connect sum of the components of \( M' \). So \( M \) can easily be reconstructed from \( M' \) in \( N - M' \) by doing boundary connect sum along arcs connecting each component of \( M' - M_1 \) to \( M_1 \) in \( N - (M' \cup E) \). After this reimbedding of \( M \), \( E \) is a compressing disk for \( \partial M \) that lies outside \( M \), so we can conclude the proof via one of the previous cases.

4. Aligned Fox reimbedding

Now we combine results from the previous two sections and consider this question: If \( M \) is a connected 3-submanifold of a non-Haken manifold \( N \) and \( M \) is \( \partial \)-reducible, to what extent can a reimbedding of \( M \), so that its complement is standard, have its \( \partial \)-reducing disks aligned with meridian disks of its complement. Obviously non-separating disks in \( M \) cannot have boundaries matched with meridian disks of \( N - M \), since \( N \) contains no non-separating surfaces. But at least in the case when \( \partial M \) is connected, this is the only restriction.

Definition 8. For \( M \) a compact irreducible orientable 3-manifold, define a disjoint collection of separating \( \partial \)-reducing disks \( D \subset M \) to be full if each component of \( M - D \) is either a solid torus or is \( \partial \)-irreducible.

For \( M \) reducible, \( D \subset M \) is full if there is a prime decomposition of \( M \) so that for each summand \( M' \) of \( M \) containing the boundary, \( D \cap M' \) is full in \( M' \).

\( M \subset N \) a 3-submanifold is aligned to a standard complement if the complement of \( M \) is standard and there is a (complete) collection of reducing spheres \( S \) for \( \partial M \) so that \( S \cap M \) is a full collection of \( \partial \)-reducing disks for \( M \).
There is a uniqueness theorem, presumably well known, for full collections of disks, which is most easily expressed for irreducible manifolds:

**Lemma 9.** Suppose $M$ is an irreducible orientable 3-manifold with boundary and $M$ is expressed as a boundary connect sum in two ways: $M = M_1 \natural M_2 \natural \ldots \natural M_n = M_1^* \natural M_2^* \natural \ldots \natural M_n^*$, where each $M_i, M_i^*$ is either a solid torus or $\partial$-irreducible. Then, after rearrangement, $n^* = n$ and $M_i \cong M_i^*$.

**Proof.** One can easily prove the theorem from first principles, along the lines of, e.g. [H Theorem 3.21], the standard proof of the corresponding theorem for a connected sum. But a cheap start is to just double $M$ along its boundary to get a manifold $DM$. The decompositions above double to give connected sum decompositions of $DM$ in which each factor consists of either $S^1 \times S^2$ or the double of an irreducible, $\partial$-irreducible manifold, which is then necessarily irreducible. Then [H Theorem 3.21] implies that $n = n^*$ and that the two original decompositions of $M$ also each contain the same number of solid tori. After removing these, we are reduced to the case in which the only $\partial$-reducing disks in $M$ are separating and $n^* = n$.

Following the outline suggested by the proof of [H Theorem 3.21], choose a disk $D$ that separates $M$ into the component $M_n$ and the component $M_1 \natural M_2 \natural \ldots \natural M_{n-1}$. Choose disks $E_1, \ldots, E_{n-1}$ that separate $M$ into the components $M_1^* \natural M_2^* \natural \ldots \natural M_{n}^*$. Choose the disks to minimize the number of intersection components in $D \cap \bigcup \{E_i\}$. Since each manifold is irreducible and $\partial$-irreducible, a standard innermost disk, outermost arc argument, the genus of $\partial$-reducing disks in a standard innermost disk, outermost arc argument (in $D$) shows that $D$ is then disjoint from $\{E_i\}$, so $D \subset M_n^*$ (say). Since $M_n^*$ is $\partial$-irreducible, $D$ is $\partial$-parallel in $M_n^*$. So in fact (with no loss of generality) $M_n \cong M_n^*$ and $M_1 \natural M_2 \natural \ldots \natural M_{n-1} \cong M_1^* \natural M_2^* \natural \ldots \natural M_{n-1}^*$. The result follows by induction.

**Theorem 10.** Let $N$ be a closed orientable irreducible non-Haken 3-manifold, and $M$ be a connected compact 3-submanifold of $N$ with connected boundary. Then $M$ can be reimbedded in $N$ with standard complement so that $M$ is aligned to the standard complement.

**Proof.** The proof is by induction on the genus of $\partial M$. Unless $M$ has a separating $\partial$-reducing disk, there is nothing beyond the result of Theorem 8 to prove. So we assume that $M$ does have a separating $\partial$-reducing disk; in particular, the genus of $\partial M$ is $g \geq 2$. We inductively assume that the theorem has been proven whenever the genus of $\partial M$ is less than $g$.

The first observation is that it suffices to find an embedding of $M$ in $N$ so that there is some reducing sphere $S$ for $\partial M$ in $N$, for such a reducing sphere divides $J = N - M$ into two components $J_1$ and $J_2$. Apply the inductive hypothesis to $M \cup J_1$ to reimbed it with an aligned complement $J_1'$. Notice that by a standard innermost disk argument, the reducing spheres can be taken to be disjoint from $S$. After this reimbedding, apply the inductive hypothesis to $M \cup J_2'$ to reimbed it so that its complement $J_2'$ is aligned. After this reimbedding, $M$ has aligned complement $J_1' \cup_{S-M} J_2'$.

Our goal then is to find a reimbedding of $M$ so that afterwards $\partial M$ has a reducing sphere. First use Theorem 4 to reimbed $M$ in $N$ so that its complement $J$ is standard, i.e. either a handlebody or $N \#$ (handlebody). Since $M$ is $\partial$-reducible, Lemma 3 applies: either $M$ is itself a handlebody (in which case the required reimbedding of $M$ is easy) or there are disjoint compressing disks $D$ in $J$ and $E$ in $M$. Since $J$ is standard, $D$ can be chosen to be non-separating in $J$. Then $\partial E$ is
not homologous to \( \partial D \) in \( \partial M \), so \( \partial E \) is either separating in \( \partial M \) or non-separating in \( \partial M - \partial D \). In the latter case, two copies of \( E \) can be banded together along an arc in \( \partial M - \partial D \) to create a separating essential disk in \( M \) that is disjoint from \( D \). The upshot is that we may as well assume that \( D \subset J \) is non-separating and \( E \subset M \) is separating.

Add a 2-handle to \( M \) along \( D \) to get \( M' \), still with standard complement \( J' \). Dually, \( M \) can be viewed as the complement of the neighborhood of an arc \( \alpha \subset M' \). If \( \partial E \) is inessential in \( \partial M' \), it bounds a disk \( D' \) in \( J' \subset J \). Then the sphere \( D' \cup E \) is a reducing sphere for \( M \) as required. So we may as well assume that \( \partial E \) is essential in \( \partial M' \) and of course still separates \( M' \). By the inductive assumption, \( M' \) can be embedded in \( N \) so that its complement is aligned, but note that this does not immediately mean that \( \partial E \) itself bounds a disk in \( N - M' \). Let \( S \) be a complete collection of reducing spheres for \( \partial M' \) intersecting \( M' \) in a full collection of disks.

\( E \) divides \( M' \) into two components, \( U \) and \( V \) with, say, \( \alpha \subset U \). If \( M' \) is reducible (i.e. contains a punctured copy of \( N \)) an innermost (in \( E \)) disk argument ensures that the reducing sphere is disjoint from \( E \). By possibly tubing \( E \) to that reducing sphere, we can ensure that the \( N \)-summand, if it lies in \( M' \), lies in \( U \subset M' \). That is, we can arrange that \( V \) is irreducible. \( E \) extends to a full collection of disks in \( M' \), with the new disks dividing \( U \) and \( V \) into \( \partial \)-connected sums: \( U = U_1 \natural \cdots \natural U_m, V = V_1 \natural \cdots \natural V_n, m, n \geq 1 \), with each \( U_i, V_j \) either \( \partial \)-irreducible or a solid torus (with one of the \( U_i \) possibly containing \( N \) as a connect summand). By Lemma 10, some component \( V' \) of \( M' - S \) is homeomorphic to \( V_n \). Tube together all components of \( S \) incident to \( V' \) along arcs in \( \partial V' \) to get a reducing sphere \( S' \) dividing \( M' \) into two components, one homeomorphic to \( V_n \) and the other homeomorphic to \( U_1 \natural V_2 \natural \cdots \natural V_{n-1} \). The latter homeomorphism carries \( \alpha \subset U \) to an arc \( \alpha' \) that is disjoint from the reducing sphere \( S' \). Then \( M' - \eta(\alpha') \) is homeomorphic to \( M \) and admits the reducing sphere \( S' \). In other words, the reembedding of \( M \) that replaces \( M' - \eta(\alpha) \) with \( M' - \eta(\alpha') \) makes \( \partial M \) reducible in \( N \), completing the argument.

**Corollary 11.** Given \( M \subset N \) as in Theorem 10, suppose \( D \) is a full set of disks in \( M \). Then, with at most one exception, each component of \( M - D \) embeds in \( S^3 \).

**Proof.** Following Theorem 10 reembed \( M \) in \( N \) with the standard complement so that \( M \) is aligned to the standard complement. Then there is a collection \( S \) of disjoint spheres in \( N \) so that, via Lemma 10, \( M - S \) and \( M - D \) are homeomorphic. Since \( N \) is irreducible, each component but at most one of \( N - S \) is a punctured 3-ball. Finally, each component of \( N - S \) contains at most one component of \( M - S \) since each component of \( S \) is separating.

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