

## SURFACES, SUBMANIFOLDS, AND ALIGNED FOX REIMBEDDING IN NON-HAKEN 3-MANIFOLDS

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**ABSTRACT.** Understanding non-Haken 3-manifolds is central to many current endeavors in 3-manifold topology. We describe some results for closed orientable surfaces in non-Haken manifolds, and extend Fox's theorem for submanifolds of the 3-sphere to submanifolds of general non-Haken manifolds. In the case where the submanifold has connected boundary, we show also that the  $\partial$ -connected sum decomposition of the submanifold can be aligned with such a structure on the submanifold's complement.

### 1. INTRODUCTION

A closed orientable irreducible 3-manifold  $N$  is called *Haken* if it contains a closed orientable incompressible surface; otherwise  $N$  is *non-Haken*. In Section 2 we describe some results for surfaces in non-Haken manifolds. Generalizing a theorem of Fox ([F]), we show in Section 3 that a 3-dimensional submanifold of a non-Haken manifold  $N$  is homeomorphic either to a handlebody complement in  $N$  or the complement of a handlebody in  $S^3$ . Sections 2 and 3 are independent, but both represent progress towards understanding submanifolds of non-Haken manifolds. In Section 4 we combine the techniques from Section 2 with the results from Section 3 to show that if the submanifold  $M \subset N$  is  $\partial$ -reducible and has connected boundary, then the embedding can be chosen to align a full collection of separating  $\partial$ -reducing disks in  $M$  with similar disks in the complement of  $M$ .

### 2. HANDLEBODIES IN NON-HAKEN MANIFOLDS

Let  $N$  be a closed orientable 3-manifold,  $F$  a closed orientable surface of non-trivial genus imbedded in  $N$ . Recall that  $F$  is *compressible* if there exists an essential simple closed curve on  $F$  that bounds an imbedded disk  $D$  in  $N$  with interior disjoint from  $F$ .  $D$  is a *compressing disk* for  $F$ .

**Definition 1.** Suppose  $F$  is a separating closed surface in an orientable irreducible closed 3-manifold  $N$ .  $F$  is *reducible* if there exists an essential simple closed curve on  $F$  that bounds compressing disks on both sides of  $F$ . The union of the two compressing disks is a *reducing sphere* for  $F$ .

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Suppose  $\mathbf{S}$  is a collection of disjoint reducing spheres for  $F$ . A reducing sphere  $S \in \mathbf{S}$  is *redundant* if a component of  $F - \mathbf{S}$  that is adjacent to  $S \cap F$  is planar.  $\mathbf{S}$  is *complete* if, for any disjoint reducing sphere  $S'$ ,  $S'$  is redundant in  $\mathbf{S} \cup S'$ .

Let  $\sigma(\mathbf{S})$  denote the number of components of  $F - \mathbf{S}$  that are not planar surfaces.

Since  $N$  is irreducible, any sphere in  $N$  is necessarily separating. Suppose a reducing sphere  $S'$  is added to a collection  $\mathbf{S}$  of disjoint reducing spheres. If  $S'$  is redundant, the number of non-planar complementary components in  $F$  is unchanged, since  $S'$  necessarily separates the component of  $F - \mathbf{S}$  that it intersects and the union of two planar surfaces along a single boundary component is still planar. If  $S'$  is not redundant, then the number of non-planar complementary components in  $F$  increases by one. Thus we have:

**Lemma 2.** *Suppose  $\mathbf{S} \subset \mathbf{S}'$  are two collections of disjoint reducing spheres for  $F$  in  $N$ . Then  $\sigma(\mathbf{S}) \leq \sigma(\mathbf{S}')$ . Equality holds if and only if each sphere  $S'$  in  $\mathbf{S}' - \mathbf{S}$  is redundant in  $\mathbf{S}' \cup \mathbf{S}$ . In particular,  $\mathbf{S}$  is complete if and only if for every collection  $\mathbf{S}'$  such that  $\mathbf{S} \subset \mathbf{S}'$ ,  $\sigma(\mathbf{S}) = \sigma(\mathbf{S}')$ .*

Let  $H$  be a handlebody imbedded in  $N$ .  $H$  has an *unknotted core* if there exists a pair of transverse simple closed curves  $c, d \subset \partial H$  such that  $c \cap d$  is a single point,  $d$  bounds an imbedded disk in  $H$  and  $c$  (the *core*) bounds an imbedded disk in  $N$  transverse to  $\partial H$ . Note that the interior of the latter imbedded disk may intersect  $H$ .

**Lemma 3.** *Let  $F$  be a connected, closed, separating, orientable surface in a closed orientable irreducible 3-manifold  $N$ . Suppose that  $F$  has compressing disks to both sides. Then at least one of the following must hold:*

- (1)  $F$  is a Heegaard surface for  $N$ .
- (2)  $N$  is Haken.
- (3) There exist disjoint compressing disks for  $F$  on opposite sides of  $F$ .

*Proof.* The proof is an application of the generalized Heegaard decomposition described in [ST]. Since  $F$  is compressible to both sides, we can construct a handle decomposition of  $N$  starting at  $F$  so that  $F$  appears as a “thick” surface in the decomposition. If  $F$  is not a Heegaard surface, then this decomposition contains a “thin” surface  $G$  adjacent to  $F$ . If  $G$  is incompressible in  $N$ , then  $N$  is Haken. If  $G$  is compressible we apply [CG] to obtain the required disjoint compressing disks for  $F$ .

**Theorem 4.** *Let  $H$  be a handlebody of genus  $g$  imbedded in a closed orientable irreducible non-Haken 3-manifold  $N$ . Let  $G$  be the complement of  $H$  in  $N$ . Let  $F = \partial H = \partial G$ . Suppose  $F$  is compressible in  $G$ . Then at least one of the following must hold:*

- (1) The Heegaard genus of  $N$  is less than or equal to  $g$ .
- (2)  $F$  is reducible.
- (3)  $H$  has an unknotted core.

*Proof.* The proof is by induction on the genus of  $H$ . If  $g = 1$ , then the result of compressing  $F$  into  $G$  is a 2-sphere, necessarily bounding a ball in  $N$ . If a ball it bounds lies in  $G$ , then the Heegaard genus of  $N$  is  $\leq 1$ . If a ball it bounds contains  $H$ , then  $H$  is an unknotted solid torus in  $N$ , and so it has an unknotted core.

Suppose then that  $\text{genus}(H) = g > 1$  and assume inductively that the theorem is true for handlebodies of genus  $g - 1$ . Suppose that  $G$ , the complement of  $H$ , has compressible boundary. If  $G$  is a handlebody, then  $G \cup_F H$  is a Heegaard splitting of genus  $g$  and we are done. So suppose  $G$  is not a handlebody. Then by Lemma 3 there are disjoint compressing disks on opposite sides of  $F$ , say  $D$  in  $H$  and  $E$  in  $G$ . Without loss of generality, we can assume that  $D$  is non-separating. Compress  $H$  along  $D$  to obtain a new handlebody  $H_1$  with boundary  $F_1$ ; let  $G_1$  be the complement of  $H_1$ .

If  $\partial E$  is inessential in  $F_1$ , then it bounds a disk in  $H_1 \subset H$  as well, so  $F$  is reducible.

If  $\partial E$  is essential in  $F_1$ , then  $E$  is a compressing disk in  $G_1$  and so we can apply the inductive hypothesis to  $H_1$ . If 1 or 3 holds, then it holds for  $H$ , and we are done. Suppose instead  $F_1$  is reducible. Let  $\mathbf{S}$  be a collection of disjoint reducing spheres for  $F_1$  chosen to maximize  $\sigma$  among all possible such collections and then, subject to that condition, further choose  $\mathbf{S}$  to minimize  $|E \cap \mathbf{S}|$ . Clearly  $E \cap \mathbf{S}$  contains no closed curves, else replacing a subdisk lying in the disk collection  $\mathbf{S} \cap G_1$  with an innermost disk of  $E - \mathbf{S}$  would reduce  $|E \cap \mathbf{S}|$ . Similarly, we have

**Claim 1.** *Suppose  $\epsilon$  is an arc component of  $\partial E - \mathbf{S}$  and  $F_0$  is the component of  $F_1 - \mathbf{S}$  in which  $\epsilon$  lies. If  $\epsilon$  separates  $F_0$  (so the ends of  $\epsilon$  necessarily lie on the same component of  $\partial F_0$ ), then neither component of  $F_0 - \epsilon$  is planar.*

*Proof of Claim 1.* Let  $c_0$  be the closed curve component of  $\partial F_0 \subset \mathbf{S} \cap F_1$  on which the ends of  $\epsilon$  lie and, of the two arcs into which the ends of  $\epsilon$  divide  $c_0$ , let  $\alpha$  be adjacent to a planar component of  $F_0 - \epsilon$ . Then the curve  $\epsilon \cup \alpha$  clearly bounds a disk in both  $G_1$  and  $H_1$ , and then so does the curve  $c' = \epsilon \cup (c_0 - \alpha)$ . Let  $S'$  be a sphere in  $N$  intersecting  $F_1$  in  $c'$  and  $S_0$  be the reducing sphere in  $\mathbf{S}$  containing  $c_0$ . Replacing  $S_0$  with  $S'$  (or just deleting  $S_0$  if  $c'$  is inessential in  $F_1$ ) gives a new collection  $\mathbf{S}'$  of disjoint reducing spheres, intersecting  $\partial E$  in at least two fewer points. Moreover  $\sigma(\mathbf{S}') = \sigma(\mathbf{S})$  since the only change in the complementary components in  $F_1$  is to add to one component and delete from another a planar surface along an arc in the boundary. Then the collection  $\mathbf{S}'$  contradicts our initial choice for  $\mathbf{S}$ , a contradiction that proves the claim.

Let  $H'$  be the closed complement of  $\mathbf{S}$  in  $H_1$ , so  $H'$  is itself a collection of handlebodies.

**Claim 2.** *Either  $F$  is reducible or  $\partial H'$  is compressible in  $N - H'$ .*

*Proof of Claim 2.* If  $\partial E$  is disjoint from  $\mathbf{S}$  and is inessential in  $\partial H'$ , then  $\partial E$  bounds a disk in  $H'$ , hence in  $H$ , so  $F$  is reducible. If  $\partial E$  is disjoint from  $\mathbf{S}$  and is essential in  $\partial H'$ , then  $E$  compresses  $\partial H'$  in  $N - H'$ , verifying the claim. Finally, if  $E$  intersects  $\mathbf{S}$ , consider an outermost disk  $A$  cut off from  $E$  by  $\mathbf{S}$ . According to Claim 1, this disk, together with a subdisk of  $\mathbf{S}$ , constitute a disk  $E'$  that compresses  $\partial H'$  in  $N - H'$ , proving the claim.

Following Claim 2, either  $F$  is reducible or the inductive hypothesis applies to a component  $H_0$  of  $H'$ . If 2 holds for  $H_0$ , then consider a reducing sphere  $S$  for  $H_0$ , isotoped so that the curve  $c = S \cap \partial H_0$  is disjoint from the disks  $\mathbf{S} \cap H_0$ . The disk  $S - H_0$  may intersect  $H_1$ ; by general position with respect to the dual 1-handles, each component of intersection is a disk parallel to a component of  $\mathbf{S} \cap H_1$ . But each such disk can be replaced by the corresponding disk in  $\mathbf{S} - H_1$  so that in the

end  $c$  also bounds a disk in  $N - H_1$ . After this change,  $S$  is a reducing sphere for  $F_1$  in  $N$  and, since  $c$  is essential in  $H_0$ ,  $\sigma(\mathbf{S} \cup S) > \sigma(\mathbf{S})$ , contradicting our initial choice for  $\mathbf{S}$ . Thus in fact 1 or 3 holds for  $H_0$ , hence also for  $H$ .

In the specific case  $N = S^3$ , we apply precisely the same argument, combined with Waldhausen's theorem [W] on Heegaard splittings of  $S^3$ , to obtain:

**Corollary 5.** *Let  $H$  be a handlebody imbedded in  $S^3$ , and suppose  $G$ , the complement of  $H$ , has compressible boundary. Then either  $H$  has an unknotted core or the boundary of  $H$  is reducible.*

This corollary is similar to ([MT], Theorem 1.1), but no reembedding of  $S^3 - H$  is required.

### 3. COMPLEMENTS OF HANDLEBODIES IN NON-HAKEN MANIFOLDS

In [F] (see also [MT] for a brief version) Fox showed that any compact connected 3-dimensional submanifold  $M$  of  $S^3$  is homeomorphic to the complement of a union of handlebodies in  $S^3$ . We generalize this result to non-Haken manifolds, showing that a submanifold  $M$  of a non-Haken manifold  $N$  has an almost equally simple description, that is,  $M$  is homeomorphic to the complement of handlebodies either in  $S^3$  or in  $N$ .

**Definition 6.** Let  $N$  be a compact irreducible 3-manifold, and let  $M$  be a compact 3-submanifold of  $N$ . We will say the complement of  $M$  in  $N$  is *standard* if it is homeomorphic to a collection of handlebodies or to  $N \# (\text{collection of handlebodies})$ . (We regard  $B^3$  as a handlebody of genus 0.)

Note that in the latter case  $M$  is actually homeomorphic to the complement of a collection of handlebodies in  $S^3$ .

**Theorem 7.** *Let  $N$  be a closed orientable irreducible non-Haken 3-manifold, and let  $M$  be a connected compact 3-submanifold of  $N$  with non-empty boundary. Then  $M$  is homeomorphic to a submanifold of  $N$  whose complement is standard.*

*Proof.* The proof will be by induction on  $n+g$  where  $n$  is the number of components of  $\partial M$  and  $g$  is the genus of  $\partial M$ , that is, the sum of the genera of its components. If  $n+g = 1$ , then  $\partial M$  is a single sphere. Since  $N$  is irreducible, the sphere bounds a 3-ball in  $N$ . So either  $M$  or its complement is a 3-ball and in either case the proof is immediate.

To verify the inductive step, suppose first that  $\partial M$  has multiple components  $T_1, \dots, T_n, n \geq 2$ . Each component  $T_i$  must bound a distinct component  $J_i$  of  $N - M$  since each must be separating in the non-Haken manifold  $N$ . Let  $M' = M \cup J_n$ ; by inductive assumption  $M'$  can be reembedded so that its complement is standard. After the reembedding, remove  $J_n$  from  $M'$  to recover a homeomorph of  $M$  and adjoin  $J_1$  (now homeomorphic either to a handlebody or to  $N \# (\text{handlebody})$ ) instead. Reembed the resulting manifold so that its complement is standard and remove  $J_1$  to recover  $M$ , now with standard complement.

Henceforth we can therefore assume that  $\partial M$  is connected and not a sphere. Since  $N$  is non-Haken there exists a compressing disk  $D$  for  $\partial M$  in  $N - \partial M$ ; the compressing disk lies either in  $M$  or in its closed complement  $J$ .

**Case 1.**  $\partial D$  is non-separating on  $\partial M$ .

If  $D$  lies inside  $M$ , compress  $M$  along  $D$  to obtain  $M'$  and use the induction hypothesis to find an imbedding of  $M'$  with standard complement. Reconstruct  $M$  by attaching a trivial 1-handle to  $M'$ , thus simultaneously attaching a trivial 1-handle to the complement.

If  $D$  lies outside  $M$ , attach a 2-handle to  $M$  corresponding to  $D$  to obtain  $M'$ , whose connected boundary has lower genus. Invoking the inductive hypothesis, imbed  $M'$  in  $N$  with standard complement. Reconstruct  $M$  from  $M'$  by removing a co-core of the attached 2-handle, thus adding a 1-handle to the complement of  $M'$ .

**Case 2.**  $\partial D$  is separating on  $\partial M$ .

Suppose  $D$  lies outside  $M$ . Then  $D$  also separates the closed complement  $J$  of  $M$  into two components,  $J_1$  and  $J_2$ , since  $H_2(N) = 0$ . Denote the components of  $\partial M - \partial D$  by  $\partial_1 \subset J_1$  and  $\partial_2 \subset J_2$ , both of positive genus. Let  $M' = M \cup J_2$ . Reimbed  $M'$  so that its complement is standard. The boundary of  $M'$  consists of  $\partial_1$  together with a disk. Since the complement of  $M'$  is standard, there is a non-separating compressing disk  $D'$  for  $\partial M'$  contained in the complement of  $M'$ .  $D'$  is also a non-separating compressing disk for the reimbedded  $\partial M$  (which is contained in  $M'$ ). Apply case 1 to this new imbedding of  $M$ .

We can now suppose that the only compressing disks for  $\partial M$  are separating compressing disks lying inside  $M$ . Choose a family  $\mathbf{D}$  of such  $\partial$ -reducing disks for  $M$  that is maximal in the sense that no component of  $M' = M - \mathbf{D}$  is itself  $\partial$ -compressible. Since each compressing disk is separating,  $\text{genus}(\partial M') = \text{genus}(\partial M) > 0$ , so  $\partial M'$  is compressible in  $N$ . Such a compressing disk  $E$  cannot lie inside  $M'$ , by construction, so it lies in the connected manifold  $N - M'$ ; let  $M_1$  be the component of  $M'$  on whose boundary  $\partial E$  lies. Since each disk in  $\mathbf{D}$  was separating,  $M$  has the simple topological description that it is the boundary-connect sum of the components of  $M'$ . So  $M$  can easily be reconstructed from  $M'$  in  $N - M'$  by doing boundary connect sum along arcs connecting each component of  $M' - M_1$  to  $M_1$  in  $N - (M' \cup E)$ . After this reimbedding of  $M$ ,  $E$  is a compressing disk for  $\partial M$  that lies outside  $M$ , so we can conclude the proof via one of the previous cases.

#### 4. ALIGNED FOX REIMBEDDING

Now we combine results from the previous two sections and consider this question: If  $M$  is a connected 3-submanifold of a non-Haken manifold  $N$  and  $M$  is  $\partial$ -reducible, to what extent can a reimbedding of  $M$ , so that its complement is standard, have its  $\partial$ -reducing disks aligned with meridian disks of its complement. Obviously non-separating disks in  $M$  cannot have boundaries matched with meridian disks of  $N - M$ , since  $N$  contains no non-separating surfaces. But at least in the case when  $\partial M$  is connected, this is the only restriction.

**Definition 8.** For  $M$  a compact irreducible orientable 3-manifold, define a disjoint collection of separating  $\partial$ -reducing disks  $\mathbf{D} \subset M$  to be *full* if each component of  $M - \mathbf{D}$  is either a solid torus or is  $\partial$ -irreducible.

For  $M$  reducible,  $\mathbf{D} \subset M$  is full if there is a prime decomposition of  $M$  so that for each summand  $M'$  of  $M$  containing the boundary,  $\mathbf{D} \cap M'$  is full in  $M'$ .

$M \subset N$  a 3-submanifold is *aligned to a standard complement* if the complement of  $M$  is standard and there is a (complete) collection of reducing spheres  $\mathbf{S}$  for  $\partial M$  so that  $\mathbf{S} \cap M$  is a full collection of  $\partial$ -reducing disks for  $M$ .

There is a uniqueness theorem, presumably well known, for full collections of disks, which is most easily expressed for irreducible manifolds:

**Lemma 9.** *Suppose  $M$  is an irreducible orientable 3-manifold with boundary and  $M$  is expressed as a boundary connect sum in two ways:  $M = M_1 \natural M_2 \natural \dots \natural M_n = M_1^* \natural M_2^* \natural \dots \natural M_n^*$ , where each  $M_i, M_j^*$  is either a solid torus or  $\partial$ -irreducible. Then, after rearrangement,  $n^* = n$  and  $M_i \cong M_i^*$ .*

*Proof.* One can easily prove the theorem from first principles, along the lines of, e.g. [H, Theorem 3.21], the standard proof of the corresponding theorem for a connected sum. But a cheap start is to just double  $M$  along its boundary to get a manifold  $DM$ . The decompositions above double to give connected sum decompositions of  $DM$  in which each factor consists of either  $S^1 \times S^2$  or the double of an irreducible,  $\partial$ -irreducible manifold, which is then necessarily irreducible. Then [H, Theorem 3.21] implies that  $n = n^*$  and that the two original decompositions of  $M$  also each contain the same number of solid tori. After removing these, we are reduced to the case in which the only  $\partial$ -reducing disks in  $M$  are separating and  $n^* = n$ .

Following the outline suggested by the proof of [H, Theorem 3.21], choose a disk  $D$  that separates  $M$  into the component  $M_n$  and the component  $M_1 \natural M_2 \natural \dots \natural M_{n-1}$ . Choose disks  $E_1, \dots, E_{n-1}$  that separate  $M$  into the components  $M_1^*, M_2^*, \dots, M_n^*$ . Choose the disks to minimize the number of intersection components in  $D \cap (\bigcup \{E_i\})$ . Since each manifold is irreducible and  $\partial$ -irreducible, a standard innermost disk, outermost arc argument (in  $D$ ) shows that in fact  $D$  is then disjoint from  $\{E_i\}$ , so  $D \subset M_n^*$  (say). Since  $M_n^*$  is  $\partial$ -irreducible,  $D$  is  $\partial$ -parallel in  $M_n^*$ . So in fact (with no loss of generality)  $M_n \cong M_n^*$  and  $M_1 \natural M_2 \natural \dots \natural M_{n-1} \cong M_1^* \natural M_2^* \natural \dots \natural M_{n-1}^*$ . The result follows by induction.

**Theorem 10.** *Let  $N$  be a closed orientable irreducible non-Haken 3-manifold, and  $M$  be a connected compact 3-submanifold of  $N$  with connected boundary. Then  $M$  can be reimbedded in  $N$  with standard complement so that  $M$  is aligned to the standard complement.*

*Proof.* The proof is by induction on the genus of  $\partial M$ . Unless  $M$  has a separating  $\partial$ -reducing disk, there is nothing beyond the result of Theorem 7 to prove. So we assume that  $M$  does have a separating  $\partial$ -reducing disk; in particular, the genus of  $\partial M$  is  $g \geq 2$ . We inductively assume that the theorem has been proven whenever the genus of  $\partial M$  is less than  $g$ .

The first observation is that it suffices to find an embedding of  $M$  in  $N$  so that there is some reducing sphere  $S$  for  $\partial M$  in  $N$ , for such a reducing sphere divides  $J = N - M$  into two components  $J_1$  and  $J_2$ . Apply the inductive hypothesis to  $M \cup J_1$  to reimbed it with an aligned complement  $J_2'$ . Notice that by a standard innermost disk argument, the reducing spheres can be taken to be disjoint from  $S$ . After this reimbedding, apply the inductive hypothesis to  $M \cup J_2'$  to reimbed it so that its complement  $J_1'$  is aligned. After this reimbedding,  $M$  has aligned complement  $J_1' \cup_{S-M} J_2'$ .

Our goal then is to find a reimbedding of  $M$  so that afterwards  $\partial M$  has a reducing sphere. First use Theorem 7 to reimbed  $M$  in  $N$  so that its complement  $J$  is standard, i.e. either a handlebody or  $N \#$  (handlebody). Since  $M$  is  $\partial$ -reducible, Lemma 3 applies: either  $M$  is itself a handlebody (in which case the required reimbedding of  $M$  is easy) or there are disjoint compressing disks  $D$  in  $J$  and  $E$  in  $M$ . Since  $J$  is standard,  $D$  can be chosen to be non-separating in  $J$ . Then  $\partial E$  is

not homologous to  $\partial D$  in  $\partial M$ , so  $\partial E$  is either separating in  $\partial M$  or non-separating in  $\partial M - \partial D$ . In the latter case, two copies of  $E$  can be banded together along an arc in  $\partial M - \partial D$  to create a separating essential disk in  $M$  that is disjoint from  $D$ . The upshot is that we may as well assume that  $D \subset J$  is non-separating and  $E \subset M$  is separating.

Add a 2-handle to  $M$  along  $D$  to get  $M'$ , still with standard complement  $J'$ . Dually,  $M$  can be viewed as the complement of the neighborhood of an arc  $\alpha \subset M'$ . If  $\partial E$  is inessential in  $\partial M'$ , it bounds a disk  $D'$  in  $J' \subset J$ . Then the sphere  $D' \cup E$  is a reducing sphere for  $M$  as required. So we may as well assume that  $\partial E$  is essential in  $\partial M'$  and of course still separates  $M'$ . By the inductive assumption,  $M'$  can be embedded in  $N$  so that its complement is aligned, but note that this does not immediately mean that  $\partial E$  itself bounds a disk in  $N - M'$ . Let  $\mathbf{S}$  be a complete collection of reducing spheres for  $\partial M'$  intersecting  $M'$  in a full collection of disks.

$E$  divides  $M'$  into two components,  $U$  and  $V$  with, say,  $\alpha \subset U$ . If  $M'$  is reducible (i.e. contains a punctured copy of  $N$ ) an innermost (in  $E$ ) disk argument ensures that the reducing sphere is disjoint from  $E$ . By possibly tubing  $E$  to that reducing sphere, we can ensure that the  $N$ -summand, if it lies in  $M'$ , lies in  $U \subset M'$ . That is, we can arrange that  $V$  is irreducible.  $E$  extends to a full collection of disks in  $M'$ , with the new disks dividing  $U$  and  $V$  into  $\partial$ -connected sums:  $U = U_1 \natural \dots \natural U_m$ ,  $V = V_1 \natural \dots \natural V_n$ ,  $m, n \geq 1$ , with each  $U_i, V_j$  either  $\partial$ -irreducible or a solid torus (with one of the  $U_i$  possibly containing  $N$  as a connect summand). By Lemma 9, some component  $V'$  of  $M' - \mathbf{S}$  is homeomorphic to  $V_n$ . Tube together all components of  $\mathbf{S}$  incident to  $V'$  along arcs in  $\partial V'$  to get a reducing sphere  $S'$  dividing  $M'$  into two components, one homeomorphic to  $V_n$  and the other homeomorphic to  $U \natural V_1 \natural V_2 \natural \dots \natural V_{n-1}$ . The latter homeomorphism carries  $\alpha \subset U$  to an arc  $\alpha'$  that is disjoint from the reducing sphere  $S'$ . Then  $M' - \eta(\alpha')$  is homeomorphic to  $M$  and admits the reducing sphere  $S'$ . In other words, the reimbedding of  $M$  that replaces  $M' - \eta(\alpha)$  with  $M' - \eta(\alpha')$  makes  $\partial M$  reducible in  $N$ , completing the argument.

**Corollary 11.** *Given  $M \subset N$  as in Theorem 10, suppose  $\mathbf{D}$  is a full set of disks in  $M$ . Then, with at most one exception, each component of  $M - \mathbf{D}$  embeds in  $S^3$ .*

*Proof.* Following Theorem 10 reimbed  $M$  in  $N$  with the standard complement so that  $M$  is aligned to the standard complement. Then there is a collection  $\mathbf{S}$  of disjoint spheres in  $N$  so that, via Lemma 9,  $M - \mathbf{S}$  and  $M - \mathbf{D}$  are homeomorphic. Since  $N$  is irreducible, each component but at most one of  $N - \mathbf{S}$  is a punctured 3-ball. Finally, each component of  $N - \mathbf{S}$  contains at most one component of  $M - \mathbf{S}$  since each component of  $\mathbf{S}$  is separating.

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