REFLECTION SYMMETRY AND SYMMETRIZABILITY OF HILBERT SPACE OPERATORS

ZOLTÁN SEBESTYÉN AND JAN STOCHEL

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Abstract. A general factorization theorem for symmetrizable operators relating their spectra to spectra of selfadjoint operators induced by minimal factorizations is established. Its modified version essentially improves and completes a theorem of Jorgensen, which concerns diagonalizing operators with reflection symmetry.

The problem of changing the spectrum of an operator with a view to getting a new spectrum with physical desiderata has been studied by many authors including Segal [8] and Jorgensen (see [3] and references therein). Jorgensen has proposed in [3] an axiomatic approach to solving these kinds of problems based on a notion of reflection symmetry. The aim of this note is to emphasize the relationship between reflection symmetry and symmetrizability, a notion invented at the beginning of the last century (cf. [4], [5], [2], [9] and [6] as well as references therein). Recapitulating some general factorization theorems for symmetrizable operators due to the first named author [6] enables us to improve and complete the main result of [3], Theorem 3.1. In particular, we strengthen part (v) of this theorem by replacing the spectral radius inequality by a more general spectral inclusion (cf. part (v) of our Corollary 2). This means that the new spectrum, not only its spectral radius, can be controlled in a general situation.

We begin by recalling some basic concepts from [6]. Let $A$ be a positive bounded (linear) operator on a (complex) Hilbert space $\mathcal{K}$ with inner product $(\cdot,\cdot)$. The range $\text{ran} A^{1/2}$ of $A^{1/2}$ becomes a Hilbert space $\mathcal{M}(A^{1/2})$ under the inner product

$$
(A^{1/2}x, A^{1/2}y) := (u, v),
$$

where $u$ and $v$ are unique vectors from $\overline{\text{ran} A}$, the closure of the range of $A$, such that $A^{1/2}x = A^{1/2}u$ and $A^{1/2}y = A^{1/2}v$. This is a well-known de Branges space (cf. [1]). The unitary operator $V_A : \mathcal{M}(A^{1/2}) \to \overline{\text{ran} A}$ arises from a densely defined one as follows:

$$
(1) \quad V_A(Ax) = A^{1/2}x, \quad x \in \mathcal{K}.
$$

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1727
The adjoint operator $V_A^* : \text{ran} A \to \mathcal{M}(A^{1/2})$ satisfies the ensuing equality
\begin{equation}
V_A^*(A^{1/2}x) = Ax, \quad x \in \mathcal{K}.
\end{equation}

One more important bounded operator $W_A : \mathcal{K} \to \mathcal{M}(A^{1/2})$ can be defined via
\begin{equation}
W_A(x) = Ax, \quad x \in \mathcal{K}.
\end{equation}

It is easily seen that the norm of $W_A$ is equal to the square root of the norm of $A$ and that the range of $W_A$ is dense in $\mathcal{M}(A^{1/2})$. Therefore the kernel of the adjoint operator $W_A^* : \mathcal{M}(A^{1/2}) \to \mathcal{K}$ is trivial. In other words, $W_A^*$ is an imbedding. In fact, the operator $W_A^*$ acts as the identity map, because
\begin{equation}
W_A^*(A^{1/2}x) = A^{1/2}x, \quad x \in \mathcal{K}.
\end{equation}

Consequently, the operator $A$ can be factorized as follows:
\begin{equation}
W_A^*W_A = A.
\end{equation}

A pair $(\mathcal{L}, W)$ is said to be a *minimal factorization* of $A$ if $\mathcal{L}$ is a Hilbert space, $W : \mathcal{K} \to \mathcal{L}$ is a bounded operator with dense range, and $A = W^*W$ (cf. [7]). If $(\mathcal{L}', W')$ is another minimal factorization of $A$, then there exists a (unique) unitary isomorphism $T : \mathcal{L} \to \mathcal{L}'$ such that $TW = W'$. Note that the above-defined pair $(\mathcal{M}(A^{1/2}), W_A)$ is a particular minimal factorization of $A$. It follows from [3] (or from [3]) that the kernels of $W_A$ and $A$ are equal to each other. Hence $W_A$ is injective if and only if $A$ is injective.

The ensuing theorem recapitulates the main results of [8] (see also [7] for extensions and generalizations to the case of unbounded operators).

**Theorem.** Let $A$ and $B$ be bounded operators on a Hilbert space $\mathcal{K}$ such that $A$ is positive and $AB$ is selfadjoint$^1$ Then there exists a unique bounded selfadjoint operator $S$ on the Hilbert space $\mathcal{M}(A^{1/2})$ such that
\begin{equation}
S(Ax) = A(Bx), \quad x \in \mathcal{K}.
\end{equation}

The operators $V := V_A$, $W := W_A$ and $S$ satisfy the following conditions:
1. $W^*W = A$,
2. $SW = WB$ (equivalently: $W^*SW = AB$),
3. $\sigma(S) \subseteq \sigma(B) \cap \mathbb{R}$, where $\sigma(C)$ stands for the spectrum of an operator $C$,
4. $VSV^*A^{1/2} = A^{1/2}B$.

The system $(\mathcal{M}(A^{1/2}), W, S)$ is unique up to unitary equivalence, i.e. if $(\mathcal{L}', W')$ is a minimal factorization of $A$ and $S'$ is a bounded selfadjoint operator on $\mathcal{L}'$ such that $S'W' = W'B$, then there exists a unitary isomorphism $T : \mathcal{L}' \to \mathcal{M}(A^{1/2})$ such that
5. $TW' = W$,
6. $TS' = ST$.

**Proof.** Notice that $\mathcal{M}(A^{1/2})$ (resp. $W$) corresponds to $\mathcal{H}_A$ (resp. $J^*$) in the Introduction of [5], $S$ corresponds to $B$ in Theorem 1 of [5], and finally $V$ (resp. $VSV^*$) corresponds to $U$ (resp. $S$) in the proof of Theorem 3 of [5]. It is now clear that

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$^1$ Such a $B$ is said to be *symmetrizable* with respect to $A$; cf. [1], [5], [2], [9] and [6].
1° is the same as [5], 2° is a direct consequence of [6] and part (i) of Theorem 1 of [6] (it can also be deduced from [5], (ii) and (iii)), 3° corresponds to part (ii) of Theorem 1 of [6], and 4° coincides in its essence with part (iv) of Theorem 3 of [6] (it can also be inferred from (2), (3) and (1)).

Since \((L', W')\) and \((M(A^{1/2}), W)\) are minimal factorizations of \(A\), there exists a unitary isomorphism \(T : L' \rightarrow M(A^{1/2})\) that satisfies 5°. It is now a matter of routine to show that 2°, 5° and \(SW' = W'B\) imply 6° (check 6° on ran \(W'\)).

Regarding the Theorem, we see that if \(A = W^*W\), \(SW = WB\) and \(S\) is self-adjoint (\(A, B, W\) and \(S\) are bounded operators), then \(A\) is positive and \(AB\) is selfadjoint.

**Corollary 1.** Let \(J, P, U\) be bounded operators on a Hilbert space \(H\). Assume that \(P\) is an orthogonal projection such that \(PJP\) is positive and

\[
(PJP)^*PJP = (UP)^*PJP.
\]

Then the compression operators \(A := PJ|\mathcal{K}\) and \(B := PU|\mathcal{K}\) acting on \(\mathcal{K} := \text{ran } P\) fulfill all the assumptions of the Theorem.

The next corollary improves and extends Theorem 3.1 of [3]. The assumptions of Corollary 2 are essentially weaker than those of Theorem 3.1 of [3], as shown in the Remark ensuing Corollary 2. For the convenience of the reader, we follow the notation and the way of numbering which appear in Theorem 3.1 of [3]. Let us point out that the assertion (ix) of Corollary 2 does not appear in Theorem 3.1 of [3]; in turn, the condition (viii) of this theorem concerns the question of the existence of an injective \(W\).

**Corollary 2.** Let \(\mathcal{K}\) be a closed linear subspace of a Hilbert space \(\mathcal{H}\), and let \(P\) be the orthogonal projection of \(\mathcal{H}\) onto \(\mathcal{K}\). Assume \(U\) is a bounded operator on \(\mathcal{H}\) leaving \(\mathcal{K}\) invariant and \(J\) is a bounded operator on \(\mathcal{H}\) such that

\[
(JU)|\mathcal{K} = U^*J|\mathcal{K} \quad \text{(equivalently: } JUP = U^*JP)\]

(i) \(PJ|\mathcal{K}\) is positive.

Then the following statements are valid:

- \(a\) There exist a unitary operator \(V : M((PJ|\mathcal{K})^{1/2}) \rightarrow \text{ran}(PJ|\mathcal{K})\), a bounded operator \(W : \mathcal{K} \rightarrow M((PJ|\mathcal{K})^{1/2})\) with dense range, and a bounded selfadjoint operator \(S\) on the Hilbert space \(M((PJ|\mathcal{K})^{1/2})\) such that
- \(b\) The system \((M((PJ|\mathcal{K})^{1/2}), W, S)\) is unique up to unitary equivalence, i.e. if \(L'\) is a Hilbert space, \(W' : \mathcal{K} \rightarrow L'\) is a bounded operator with dense range,

\[\text{spectral radius of } \sigma(U|\mathcal{K}) \leq \text{sp}(U), \text{ where sp}(U) \text{ stands for the spectral radius of } U,\]

\(\text{ker } W = \ker(PJ|\mathcal{K}), \text{ where } "\ker\" \text{ is the abbreviation for } "\text{kernel},\]

\(VSV^*(PJ|\mathcal{K})^{1/2} = (PJ|\mathcal{K})^{1/2}(U|\mathcal{K}).\]

Applying Corollary 2 (or Corollary 12) to \(\mathcal{K} = \mathcal{H}\), we get back to the Theorem. In turn, if we apply Theorem 3.1 of [3] to \(\mathcal{K} = \mathcal{H}_0\) (our space \(\mathcal{H}\) is denoted in [3] by \(\mathcal{H}_0\), we arrive at a very particular situation: \(J = W = \text{the identity operator and } S = U = U^*\).
and $S'$ is a bounded selfadjoint operator on $L'$ such that $W^*W' = PJ|_K$ and $S'W' = W'U|_K$, then there exists a unitary isomorphism $T : L' \to \mathcal{M}(PJ|_K^{1/2})$ such that

(vi) $TW' = W'$,

(vii) $TS' = ST$.

Proof. Since $PUP = UP$, we deduce from (i) that

$$(PJP)UP = PJUP = P(U^*)JP = (PU^*)JP = (UP^*)JP = (UP)^*JP,$$

which means that (7) holds. This enables us to apply Corollary 1. In particular, we get the inclusion $\sigma(S) \subseteq \sigma(U|_K)$, which leads to

$$\|S\| = \text{sp}(S) \leq \text{sp}(U|_K) = \lim_{n \to \infty} \|U^n|_K\|^{1/n} \leq \lim_{n \to \infty} \|U^n\|^{1/n} = \text{sp}(U).$$

This completes the proof. \qed

Remark. Let us notice that condition (i) of Corollary 2 results from the following two conditions (at this stage we need not assume that $U(K) \subseteq K$):

(i-a) $JUX = U^*$,

(i-b) $XJ|_K$ is the identity operator on $K$ (equivalently: $XJP = P$),

where $X$ is a bounded operator on $H$. Indeed, (i-b) and (i-a) imply

$$JUX = (JUX)Jx = U^*Jx, \quad x \in K.$$ 

On the other hand, neither (i-a) nor (i-b) follows from (i) because the operator $J := 0$ satisfies (i) and (ii), but it fails to satisfy (i-a) and (i-b) (provided $U \neq 0$ and $K \neq \{0\}$).

Let us look at particular choices for $X$. If $J$ is a bijection, then the condition (i-b) is satisfied with $X = J^{-1}$ (in fact, it is sufficient to assume that $J$ has a left inverse $X$); for such an $X$ the condition (i-a) means that the operators $U$ and $U^*$ are similar. If $J$ is a unitary operator (or if $J$ is an isometry), then the condition (i-b) is satisfied with $X = J^*J$; now (i-a) can be interpreted to mean that the operators $U$ and $U^*$ are unitarily equivalent. Finally, the case $J^{-1} = J^* = J$ considered in Theorem 3.1 of [3], such a $J$ is called a period-2 unitary operator); therefore this theorem, except for its part (c), is a consequence of Corollary 2 (the contractiveness of $W$ results from that of $J$ via the condition (iv) of Corollary 2). What is more, the assumption of the unitarity of $J$ made in Theorem 3.1 of [3] is superfluous (however, now $W$ need not be a contraction).

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References


Department of Applied Analysis, Eötvös University, H-1117 Budapest, Pázmány Péter sétány 1/c, Hungary
E-mail address: sebesty@cs.elte.hu

Instytut Matematyki, Uniwersytet Jagielloński, Reymonta 4, 30-059 Kraków, Poland
E-mail address: stochel@im.uj.edu.pl