

REFLECTION SYMMETRY AND SYMMETRIZABILITY OF HILBERT SPACE OPERATORS

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ABSTRACT. A general factorization theorem for symmetrizable operators relating their spectra to spectra of selfadjoint operators induced by minimal factorizations is established. Its modified version essentially improves and completes a theorem of Jorgensen, which concerns diagonalizing operators with reflection symmetry.

The problem of changing the spectrum of an operator with a view to getting a new spectrum with physical desiderata has been studied by many authors including Segal [8] and Jorgensen (see [3] and references therein). Jorgensen has proposed in [3] an axiomatic approach to solving these kinds of problems based on a notion of reflection symmetry. The aim of this note is to emphasize the relationship between reflection symmetry and symmetrizability, a notion invented at the beginning of the last century (cf. [4], [5], [2], [9] and [6] as well as references therein). Recapitulating some general factorization theorems for symmetrizable operators due to the first named author [6] enables us to improve and complete the main result of [3], Theorem 3.1. In particular, we strengthen part (v) of this theorem by replacing the spectral radius inequality by a more general spectral inclusion (cf. part (v) of our Corollary 2). This means that the new spectrum, not only its spectral radius, can be controlled in a general situation.

We begin by recalling some basic concepts from [6]. Let A be a positive bounded (linear) operator on a (complex) Hilbert space \mathcal{K} with inner product (\cdot, \cdot) . The range $\text{ran } A^{1/2}$ of $A^{1/2}$ becomes a Hilbert space $\mathcal{M}(A^{1/2})$ under the inner product

$$\langle A^{1/2}x, A^{1/2}y \rangle := (u, v),$$

where u and v are unique vectors from $\overline{\text{ran } A}$, the closure of the range of A , such that $A^{1/2}x = A^{1/2}u$ and $A^{1/2}y = A^{1/2}v$. This is a well-known de Branges space (cf. [1]). The unitary operator $V_A : \mathcal{M}(A^{1/2}) \rightarrow \overline{\text{ran } A}$ arises from a densely defined one as follows:

$$(1) \quad V_A(Ax) = A^{1/2}x, \quad x \in \mathcal{K}.$$

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The adjoint operator $V_A^* : \overline{\text{ran } A} \rightarrow \mathcal{M}(A^{1/2})$ satisfies the ensuing equality

$$(2) \quad V_A^*(A^{1/2}x) = Ax, \quad x \in \mathcal{K}.$$

One more important bounded operator $W_A : \mathcal{K} \rightarrow \mathcal{M}(A^{1/2})$ can be defined via

$$(3) \quad W_A(x) = Ax, \quad x \in \mathcal{K}.$$

It is easily seen that the norm of W_A is equal to the square root of the norm of A and that the range of W_A is dense in $\mathcal{M}(A^{1/2})$. Therefore the kernel of the adjoint operator $W_A^* : \mathcal{M}(A^{1/2}) \rightarrow \mathcal{K}$ is trivial. In other words, W_A^* is an imbedding. In fact, the operator W_A^* acts as the identity map, because

$$(4) \quad W_A^*(A^{1/2}x) = A^{1/2}x, \quad x \in \mathcal{K}.$$

Consequently, the operator A can be factorized as follows:

$$(5) \quad W_A^*W_A = A.$$

A pair (\mathcal{L}, W) is said to be a *minimal factorization* of A if \mathcal{L} is a Hilbert space, $W : \mathcal{K} \rightarrow \mathcal{L}$ is a bounded operator with dense range, and $A = W^*W$ (cf. [7]). If (\mathcal{L}', W') is another minimal factorization of A , then there exists a (unique) unitary isomorphism $T : \mathcal{L} \rightarrow \mathcal{L}'$ such that $TW = W'$. Note that the above-defined pair $(\mathcal{M}(A^{1/2}), W_A)$ is a particular minimal factorization of A . It follows from (5) (or from (3)) that the kernels of W_A and A are equal to each other. Hence W_A is injective if and only if A is injective.

The ensuing theorem recapitulates the main results of [6] (see also [7] for extensions and generalizations to the case of unbounded operators).

Theorem. *Let A and B be bounded operators on a Hilbert space \mathcal{K} such that A is positive and AB is selfadjoint.¹ Then there exists a unique bounded selfadjoint operator S on the Hilbert space $\mathcal{M}(A^{1/2})$ such that*

$$(6) \quad S(Ax) = A(Bx), \quad x \in \mathcal{K}.$$

The operators $V := V_A$, $W := W_A$ and S satisfy the following conditions:

- 1° $W^*W = A$,
- 2° $SW = WB$ (equivalently: $W^*SW = AB$),
- 3° $\sigma(S) \subseteq \sigma(B) \cap \mathbb{R}$, where $\sigma(C)$ stands for the spectrum of an operator C ,
- 4° $VSV^*A^{1/2} = A^{1/2}B$.

The system $(\mathcal{M}(A^{1/2}), W, S)$ is unique up to unitary equivalence, i.e. if (\mathcal{L}', W') is a minimal factorization of A and S' is a bounded selfadjoint operator on \mathcal{L}' such that $S'W' = W'B$, then there exists a unitary isomorphism $T : \mathcal{L}' \rightarrow \mathcal{M}(A^{1/2})$ such that

- 5° $TW' = W$,
- 6° $TS' = ST$.

Proof. Notice that $\mathcal{M}(A^{1/2})$ (resp. W) corresponds to \mathcal{H}_A (resp. J^*) in the Introduction of [6], S corresponds to \hat{B} in Theorem 1 of [6], and finally V (resp. VSV^*) corresponds to U (resp. S) in the proof of Theorem 3 of [6]. It is now clear that

¹ Such a B is said to be *symmetrizable* with respect to A ; cf. [4], [5], [2], [9] and [6].

1° is the same as (5), 2° is a direct consequence of (4) and part (i) of Theorem 1 of [6] (it can also be deduced from (3), (6) and (4)), 3° corresponds to part (ii) of Theorem 1 of [6], and 4° coincides in its essence with part (iv) of Theorem 3 of [6] (it can also be inferred from (2), (6) and (1)).

Since (\mathcal{L}', W') and $(\mathcal{M}(A^{1/2}), W)$ are minimal factorizations of A , there exists a unitary isomorphism $T : \mathcal{L}' \rightarrow \mathcal{M}(A^{1/2})$ that satisfies 5°. It is now a matter of routine to show that 2°, 5° and $S'W' = W'B$ imply 6° (check 6° on $\text{ran } W'$). \square

Regarding the Theorem, we see that if $A = W^*W$, $SW = WB$ and S is self-adjoint (A, B, W and S are bounded operators), then A is positive and AB is selfadjoint.

Corollary 1. *Let J, P, U be bounded operators on a Hilbert space \mathcal{H} . Assume that P is an orthogonal projection such that PJP is positive and*

$$(7) \quad (PJP)UP = (UP)^*PJP.$$

Then the compression operators $A := PJ|_{\mathcal{K}}$ and $B := PU|_{\mathcal{K}}$ acting on $\mathcal{K} := \text{ran } P$ fulfill all the assumptions of the Theorem.

The next corollary² improves and extends Theorem 3.1 of [3]. The assumptions of Corollary 2 are essentially weaker than those of Theorem 3.1 of [3], as shown in the Remark ensuing Corollary 2. For the convenience of the reader, we follow the notation and the way of numbering which appear in Theorem 3.1 of [3]. Let us point out that the assertion (ix) of Corollary 2 does not appear in Theorem 3.1 of [3]; in turn, the condition (viii) of this theorem concerns the question of the existence of an injective W .

Corollary 2. *Let \mathcal{K} be a closed linear subspace of a Hilbert space \mathcal{H} , and let P be the orthogonal projection of \mathcal{H} onto \mathcal{K} . Assume U is a bounded operator on \mathcal{H} leaving \mathcal{K} invariant and J is a bounded operator on \mathcal{H} such that*

- (i) $JU|_{\mathcal{K}} = U^*J|_{\mathcal{K}}$ (equivalently: $JUP = U^*JP$),
- (ii) PJP is positive.

Then the following statements are valid:

- a) *There exist a unitary operator $V : \mathcal{M}((PJ|_{\mathcal{K}})^{1/2}) \rightarrow \overline{\text{ran}(PJ|_{\mathcal{K}})}$, a bounded operator $W : \mathcal{K} \rightarrow \mathcal{M}((PJ|_{\mathcal{K}})^{1/2})$ with dense range, and a bounded selfadjoint operator S on the Hilbert space $\mathcal{M}((PJ|_{\mathcal{K}})^{1/2})$ such that*
 - (iii) $SW = WU|_{\mathcal{K}}$,
 - (iv) $W^*W = PJ|_{\mathcal{K}}$,
 - (v) $\sigma(S) \subseteq \sigma(U|_{\mathcal{K}}) \cap \mathbb{R}$ and $\|S\| \leq \text{sp}(U)$, where $\text{sp}(U)$ stands for the spectral radius of U ,
 - (viii) $\ker W = \ker(PJ|_{\mathcal{K}})$, where “ker” is the abbreviation for “kernel”,
 - (ix) $VSV^*(PJ|_{\mathcal{K}})^{1/2} = (PJ|_{\mathcal{K}})^{1/2}(U|_{\mathcal{K}})$.
- b) *The system $(\mathcal{M}((PJ|_{\mathcal{K}})^{1/2}), W, S)$ is unique up to unitary equivalence, i.e. if \mathcal{L}' is a Hilbert space, $W' : \mathcal{K} \rightarrow \mathcal{L}'$ is a bounded operator with dense range,*

² Applying Corollary 2 (or Corollary 1) to $\mathcal{K} = \mathcal{H}$, we get back to the Theorem. In turn, if we apply Theorem 3.1 of [3] to $\mathcal{K} = \mathcal{H}_0$ (our space \mathcal{H} is denoted in [3] by \mathcal{H}_0), we arrive at a very particular situation: $J = W =$ the identity operator and $S = U = U^*$.

and S' is a bounded selfadjoint operator on \mathcal{L}' such that $W'^*W' = PJ|_{\mathcal{K}}$ and $S'W' = W'U|_{\mathcal{K}}$, then there exists a unitary isomorphism $T : \mathcal{L}' \rightarrow \mathcal{M}((PJ|_{\mathcal{K}})^{1/2})$ such that

- (vi) $TW' = W$,
- (vii) $TS' = ST$.

Proof. Since $PUP = UP$, we deduce from (i) that

$$(PJP)UP = PJUP = PU^*JP = (UP)^*JP = (PUP)^*JP = (UP)^*PJP,$$

which means that (7) holds. This enables us to apply Corollary 1. In particular, we get the inclusion $\sigma(S) \subseteq \sigma(U|_{\mathcal{K}})$, which leads to

$$\|S\| = \text{sp}(S) \leq \text{sp}(U|_{\mathcal{K}}) = \lim_{n \rightarrow \infty} \|U^n|_{\mathcal{K}}\|^{1/n} \leq \lim_{n \rightarrow \infty} \|U^n\|^{1/n} = \text{sp}(U).$$

This completes the proof. \square

Remark. Let us notice that condition (i) of Corollary 2 results from the following two conditions (at this stage we need not assume that $U(\mathcal{K}) \subseteq \mathcal{K}$):

- (i-a) $JUX = U^*$,
- (i-b) $XJ|_{\mathcal{K}}$ is the identity operator on \mathcal{K} (equivalently: $XJP = P$),

where X is a bounded operator on \mathcal{H} . Indeed, (i-b) and (i-a) imply

$$JUX = (JUX)Jx = U^*Jx, \quad x \in \mathcal{K}.$$

On the other hand, neither (i-a) nor (i-b) follows from (i) because the operator $J := 0$ satisfies (i) and (ii), but it fails to satisfy (i-a) and (i-b) (provided $U \neq 0$ and $\mathcal{K} \neq \{0\}$).

Let us look at particular choices for X . If J is a bijection, then the condition (i-b) is satisfied with $X = J^{-1}$ (in fact, it is sufficient to assume that J has a left inverse X); for such an X the condition (i-a) means that the operators U and U^* are similar. If J is a unitary operator (or if J is an isometry), then the condition (i-b) is satisfied with $X = J^*$; now (i-a) can be interpreted to mean that the operators U and U^* are unitarily equivalent. Finally, the case $J^{-1} = J^* = J$ considered in Theorem 3.1 of [3] follows from any of the two mentioned above (in the terminology of [3], such a J is called a period-2 unitary operator); therefore this theorem, except for its part (c), is a consequence of Corollary 2 (the contractiveness of W results from that of J via the condition (iv) of Corollary 2). What is more, the assumption of the unitarity of J made in Theorem 3.1 of [3] is superfluous (however, now W need not be a contraction).

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