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A NEW PROOF OF THE J^2-CONDITION FOR REAL RANK ONE SIMPLE LIE ALGEBRAS AND THEIR CLASSIFICATION

PAOLO CIATTI

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Abstract. In this paper a new purely algebraic proof of the J^2-condition for the nilpotent Iwasawa algebras in real rank one simple Lie algebras is presented, yielding the classification of real rank one simple Lie algebras.

1. Introduction

In [2] and [3] it is shown how H-type groups can be used to obtain a classification of rank one noncompact symmetric spaces, and they are given a uniform description that avoids the explicit use of the Cayley algebra to treat the only exceptional case. These papers derive from the observation of A. Korányi [6], who in 1985 noticed that if a semi-simple symmetric space G/K has rank one, the nilpotent subgroup N obtained from the Iwasawa decomposition $G = ANK$ belongs to the class, introduced in 1980 by A. Kaplan [5], of the H-type groups. The main result in [2] is the proof that, among the H-type groups, the Iwasawa groups are exactly those that satisfy the so-called J^2-property. This allows the authors to classify the rank one symmetric spaces classifying the H-type groups with the J^2-property.

Here we present a new and shorter proof of the J^2-property which is based on the study of the action on N of the centralizer M of A in K. Our proof is purely algebraic and avoids the use of the complexification of $\mathfrak{g}$, which seems quite unnatural in this context.

This paper consists of this introduction and three more sections. In Section 2 we provide some generalities on real semi-simple Lie algebras, mainly concerning the decomposition in restricted root spaces, and the definitions of H-type Lie algebras and of the J^2-property. In Section 3 we prove two general formulae yielding the action of the Lie algebra of M on the restricted root spaces, from which the J^2-property easily follows. Finally, in the last section we present a new way to classify the H-type algebras satisfying the J^2-property, which rests on the theory of Clifford modules avoiding the use of division algebras as done in [2].

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2. Generalities

Since the problem of classifying simply connected rank one symmetric spaces is essentially equivalent to that of classifying real rank one simple Lie algebras [4], in this paper we deal with the latter ones. Let $\mathfrak{g}$ be a real rank one simple Lie algebra with Killing form B and Cartan involution θ. Denote by Δ a set of restricted roots. The set Δ is either equal to $\{\pm\alpha\}$, or $\{\pm\alpha, \pm 2\alpha\}$. In the first case Δ is called A_1 and in the second case BC_1. Correspondingly set $\mathfrak{n} = \mathfrak{g}_\alpha$ and $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. The subspace $\mathfrak{n}$ is the nilpotent subalgebra of $\mathfrak{g}$ that appears in the Bruhat decomposition

$$\mathfrak{g} = \theta\mathfrak{n} \oplus (\mathfrak{a} \oplus \mathfrak{m}) \oplus \mathfrak{n}. \tag{2.1}$$

We are especially interested in discussing the structure of $\mathfrak{n}$. For this purpose we shall use the following notion introduced by Kaplan in [5].

Definition 2.1. A nilpotent Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ endowed with an inner product $\langle \cdot, \cdot \rangle$, with centre $\mathfrak{z}$ and $\mathfrak{z}^\perp = \mathfrak{v}$, is an H-type algebra if the linear map J_Z defined for Z in $\mathfrak{z}$ by

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle \quad \text{for all } X, Y \text{ in } \mathfrak{v} \tag{2.2}$$

satisfies

$$J_Z^2 = J_Z \circ J_Z = -\|Z\|^2\, I \quad \text{for all } Z \text{ in } \mathfrak{z}. \tag{2.3}$$

From (2.3) it follows by polarization that for Z, Z' in $\mathfrak{z}$,

$$J_Z J_{Z'} + J_{Z'} J_Z = -2 \langle Z, Z' \rangle\, I. \tag{2.4}$$

Thus, $J : \mathfrak{z} \to \mathrm{End}\,(\mathfrak{v})$ extends to a representation of the Clifford algebra $\mathcal{C}\,(d_{2\alpha})$. From (2.2) it follows easily that (see for instance [2])

$$[X, J_Z X] = \|X\|^2 Z \quad \text{for } X \text{ in } \mathfrak{v} \text{ and } Z \text{ in } \mathfrak{z}. \tag{2.5}$$

The first application of the H-type algebras in the study of real simple Lie algebras is due to Korányi, who in [6] noticed that if $\mathfrak{g}$ is endowed with the inner product defined by

$$\langle X, Y \rangle = -c\, B\,(X, \theta Y)\,,$$

with c chosen in such a way that $(\alpha \mid \alpha) = 1/2$ (where $(\cdot \mid \cdot)$ is the inner product induced on the dual of $\mathfrak{a}$ by $\langle \cdot \mid \cdot \rangle$), setting

$$J_Z X = [Z, \theta X] \qquad \text{for } Z \text{ in } \mathfrak{g}_{2\alpha} \text{ and } X \text{ in } \mathfrak{g}_\alpha, \tag{2.6}$$

J_Z satisfies (2.3), and therefore $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is an H-type algebra. After that, he discovered in collaboration with M. Cowling, A. Dooley, and F. Ricci the following property that characterizes the H-type algebras that derive from the decomposition (2.1) of a real rank one simple Lie algebra (see [2], [3]).

Definition 2.2. Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be an H-type algebra with centre $\mathfrak{z}$. One says that $\mathfrak{n}$ satisfies the J^2-condition if for all X in $\mathfrak{v}$ and all orthogonal pairs (Z, Z') in $\mathfrak{z}$, there exists Z'' in $\mathfrak{z}$ (possibly depending on X, Z, and Z') satisfying

$$J_Z J_{Z'} X = J_{Z''} X.$$

This condition is trivially satisfied if $\mathfrak{n}$ is degenerate, i.e. $\mathfrak{z} = \{0\}$, or if $\dim \mathfrak{z} = 1$. It is equivalent to requiring that J_Z preserves the subspace

$$\mathbb{R}X \oplus J_{\mathfrak{z}} X = \{aX + J_{Z'} X\ a \in \mathbb{R},\ Z' \in \mathfrak{z}\}$$

for any X in $\mathfrak{v}$ and any Z in $\mathfrak{z}$.

3. The action of $\mathfrak{m}$

In this section we analyze the subalgebra $\mathfrak{m}$ and its action on $\mathfrak{n}$. It is well known that the elements of $\mathfrak{m}$ are fixed by θ and that

$$\mathfrak{m} = \operatorname{span}\{[X, \theta Y] : \ X, Y \in \mathfrak{g}_\beta \text{ with } \beta \in \{\alpha, 2\alpha\} \text{ and } \langle X, Y\rangle = 0\}.$$

The next result holds in a wider context and was proved in [1] as Corollary 5.2. We present the simple proof here in an attempt to make the paper self-contained.

Proposition 3.1. *Pick a unit vector X in $\mathfrak{g}_\alpha$ and two vectors Z_1, Z_2 in $\mathfrak{g}_{2\alpha}$. Then*

$$[[Z_1, \theta Z_2], X] = J_1 J_2 X, \tag{3.1}$$

where $J_1 X$ and $J_2 X$ stand for $J_{Z_1} X$ and $J_{Z_2} X$, respectively. Moreover,

$$[Z_1, \theta Z_2] = [\theta X, J_1 J_2 X] + [\theta J_1 X, J_2 X]. \tag{3.2}$$

Proof. Formula (3.1) follows from the Jacobi identity and (2.6). Formula (3.2) is also obtained by Jacobi plugging $Z_2 = [X, J_2 X]$ in the left-hand side and then using (2.6) and (3.1). □

Lemma 3.2. *Let Y, W be in $\mathfrak{g}_\alpha$. Then*

$$[W, \theta Y] + [\theta W, Y] = 2\langle W, Y\rangle H, \tag{3.3}$$

where H lies in $\mathfrak{a}$ and satisfies $\|H\|^2 = 1$.

Proof. Let $L = [\theta Y, W] + [\theta W, Y]$. Then L lies in $\mathfrak{g}_0$, and $\theta L = -L$. Therefore, $L = cH$, with the real constant c given by

$$c = \langle H, L\rangle = \langle [H, Y], W\rangle + \langle [H, W], Y\rangle = 2\langle W, Y\rangle.$$

□

The next result whose proof uses essentially the Jacobi identity and the fact that the elements of $\mathfrak{m}$ are fixed by θ yields a formula describing the action of $\mathfrak{m}$ on $\mathfrak{g}_\alpha$.

Proposition 3.3. *For X, Y, W in $\mathfrak{g}_\alpha$, the following formula holds:*

$$\begin{aligned}[][[X, \theta Y], W] &= \tfrac{1}{2}\langle W, X\rangle Y - \tfrac{1}{2}\langle W, Y\rangle X - \tfrac{1}{2}\langle X, Y\rangle W \\ &\quad + \tfrac{1}{2} J_{[X,Y]} W + \tfrac{1}{2} J_{[X,W]} Y + \tfrac{1}{2} J_{[W,Y]} X.\end{aligned} \tag{3.4}$$

Proof. If one among X, Y, and W is zero, then (3.4) is trivial. Assume that X, Y, and W are not zero. We prove the following identity:

$$[[X, \theta Y], W] = -\langle W, Y\rangle X + [[X, W], \theta Y] - [[Y, \theta W], X]. \tag{3.5}$$

By the Jacobi identity one obtains

$$[[X, \theta Y], W] = [[W, \theta Y], X] + [[X, W], \theta Y].$$

By (3.3), $[W, \theta Y]$ decomposes into the sum of a term in $\mathfrak{a}$ and a term in $\mathfrak{m}$. Since $\mathfrak{m}$ is fixed by θ, one then obtains (3.5).

Applying (3.5) with (Y, W, X) in place of (X, Y, W) to the last term in (3.5) yields

$$[[X, \theta Y], W] = \langle W, X\rangle Y - \langle W, Y\rangle X + [[X, Y], \theta W] + [[X, W], \theta Y] + [[W, \theta X], Y].$$

In the same way from this relation one obtains by (3.5),

$$\begin{aligned} 2[[X, \theta Y], W] &= \langle W, X\rangle Y - \langle W, Y\rangle X - \langle X, Y\rangle W \\ &\quad + [[X, Y], \theta W] + [[X, W], \theta Y] + [[W, Y], \theta X],\end{aligned}$$

from which (3.4) follows by (2.6). □

When 2α is not a root the last three terms in (3.4) vanish. In particular, for X, Y, and W in $\mathfrak{g}_{2\alpha}$, (3.4) with the last three terms omitted yields the action of $\mathfrak{m}$ on $\mathfrak{g}_{2\alpha}$.

Lemma 3.4. *Suppose* $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ *is an H-type algebra with the* J^2*-property. If* Z_1, Z_2 *are orthogonal vectors in* $\mathfrak{z}$ *and* X *is a unit vector in* $\mathfrak{v}$*, then*

$$J_{[X,J_1J_2X]}X = J_{[J_1X,J_2X]}X = J_1J_2X. \tag{3.6}$$

Proof. If one among Z_1 and Z_2 is zero, then (3.6) is trivial. Assume $\|Z_1\| = \|Z_2\| = 1$, and let Z_3 in $\mathfrak{z}$ satisfy $J_1J_2X = J_3X$. Then,

$$J_{[X,J_1J_2X]}X = J_{[X,J_3X]}X = J_3X = J_1J_2X$$

by (2.5), proving the first relation. The same argument, writing $J_2X = J_3J_1X$, proves also the second equality. □

The next result was stated and proved the first time in [2]. We present a simple proof here, which is based through Lemma 3.5 on (3.4) and (3.2).

Theorem 3.5. *The nilpotent subalgebras appearing in the Bruhat decomposition of a simple real rank one Lie algebra satisfy the* J^2*-condition.*

Lemma 3.6. *Suppose* $\alpha, 2\alpha$ *in* Δ*. Take a unit vector* X *in* $\mathfrak{g}_\alpha$ *and two orthogonal vectors* Z_1*,* Z_2 *in* $\mathfrak{g}_{2\alpha}$*. Then*

$$J_1J_2X = J_{[J_1X,J_2X]}X = J_{[X,J_1J_2X]}X.$$

Proof. We prove the formula

$$J_1J_2X = \frac{1}{3}J_{[J_1X,J_2X]}X + \frac{2}{3}J_{[X,J_1J_2X]}X, \tag{3.7}$$

from which one sees that the J^2-condition holds in $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. The assertion then follows by (3.6). From (3.4) it follows, by (2.5), (2.6), and (2.4), that

$$[[X,\theta J_1X],J_2X] = \frac{1}{2}J_1J_2X + \frac{1}{2}J_2J_1X + \frac{1}{2}J_{[J_2X,J_1X]}X = \frac{1}{2}J_{[J_2X,J_1X]}X. \tag{3.8}$$

On the other hand, the Jacobi identity yields, by (2.5) and (2.6),

$$[[X,\theta J_1X],J_2X] = [[J_2X,\theta J_1X],X] + J_2J_1X.$$

Using (3.2), replace in the last relation $[J_2X,\theta J_1X]$ with $[Z_2,\theta Z_1] - [X,\theta J_2J_1X]$. Then by (3.1) one obtains

$$[[X,\theta J_1X],J_2X] = 2J_2J_1X - [[X,\theta J_2J_1X],X]. \tag{3.9}$$

From (3.4) it follows that

$$[[X,\theta J_2J_1X],X] = \frac{1}{2}J_2J_1X + J_{[X,J_2J_1X]}X,$$

which plugged in (3.9) yields

$$[[X,\theta J_1X],J_2X] = \frac{3}{2}J_2J_1X - J_{[X,J_2J_1X]}X.$$

Now, (3.7) follows by comparing this relation with (3.8). □

4. The classification

In this section we shall obtain the classification of real rank one simple Lie algebras from Theorem 3.5 classifying the H-type algebras that satisfy the J^2-condition. Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be an H-type algebra with inner product $\langle \cdot, \cdot \rangle$ and centre $\mathfrak{z}$, and let $\{Z_1, \ldots, Z_d\}$ be an orthonormal basis of $\mathfrak{z}$.

Proposition 4.1. *If* $\mathfrak{n}$ *satisfies the* J^2*-condition, then* d *belongs to* $\{0, 1, 3, 7\}$.

Proof. From the J^2-condition it follows that if X is a vector in $\mathfrak{v}$ and Z, Z' are orthogonal non-zero vectors in $\mathfrak{z}$ such that

$$\langle J_{Z''} X, J_Z J_{Z'} X \rangle = 0 \quad \text{for all } Z'' \text{ in } \mathfrak{z},$$

then $X = 0$. We prove the proposition inductively showing by the following two lemmas that this property does not hold if d is not in $\{0, 1, 3, 7\}$. □

Lemma 4.2. *Let* $d \in \{2, 4, 5, 11\}$. *Then there is a non-zero vector* X *in* $\mathfrak{v}$ *satisfying*

$$\langle J_1 J_2 X, J_Z X \rangle = 0 \qquad \text{for all } Z \text{ in } \mathfrak{z}.$$

Lemma 4.3. *Let* $d = m+4$, $m \geq 2$. *If* $J_1 J_2 X$ *is orthogonal to* $J_i X$ *for* $i = 1, \ldots, m$, *then*

$$\langle J_1 J_2 X, J_Z X \rangle = 0 \quad \text{for all } Z \text{ in } \mathfrak{z}.$$

Proof. We prove both lemmas at the same time. For $d = 2$, the assertion is trivial. In the other cases it is sufficient to exhibit a commutative set $\{\chi_i\}_{i=1}^n$ of linear, symmetric, involutive endomorphisms of $\mathfrak{v}$ with the property that χ_i commutes with $J_1 J_2$ and anticommutes with J_i, or conversely, for each i. In fact, then $J_1 J_2 X$ is orthogonal to $J_i X$ for any non-zero common eigenvector X of the endomorphisms χ_i. For $d = 4$, one may take $\chi_i = J_1 J_2 J_3 J_4$ for $i = 1, 2, 3, 4$. For $d = 5$, one may take $\chi_i = J_1 J_2 J_3 J_4$ for $i = 1, 2, 3, 4$, and $\chi_5 = J_2 J_4 J_5$. For $d = 11$, one may take $\chi_i = J_1 J_3 J_5 J_7$ for $i = 2, 4, 6, 8, 9, 10, 11$ and $\chi_i = J_2 J_4 J_6 J_8$ for $i = 1, 3, 5, 7$. Finally, Lemma 4.3 follows by induction on m taking for $d = m + 4$, $\chi_i = J_{m+1} J_{m+2} J_{m+3} J_{m+4}$, $i = m+1, m+2, m+3, m+4$. □

From now on assume that $\mathfrak{n}$ derives from the Bruhat decomposition of a simple real rank one Lie algebra. In particular, $\mathfrak{n}$ satisfies the J^2-property.

Lemma 4.4. *Suppose* $d_{2\alpha} \in \{3, 7\}$ *and set* $\lambda = J_1 J_2 J_3$. *If* X *is a non-zero eigenvector of* λ, *then any* Y *in the orthogonal complement in* $\mathfrak{v}$ *of* $\mathbb{R}X \oplus J_{\mathfrak{z}} X$ *is an eigenvector of* λ *belonging to the same eigenvalue as* X.

Proof. By the J^2-property, $\langle J_Z X, Y \rangle = \langle J_Z J_{Z'} X, Y \rangle = 0$ for all Z, Z' in $\mathfrak{z}$. From this, by the Jacobi identity, it follows that $\operatorname{ad}[X, \theta Y]$ commutes with J_Z for any Z. Hence, $\operatorname{ad}[X, \theta Y]$ commutes with λ, yielding the statement since $Y = 2\,[[X, \theta Y], X]$. □

From this lemma one immediately deduces the following result.

Proposition 4.5. *If* $d_{2\alpha} = 3$, *then the irreducible* $\mathcal{C}(3)$*-modules in which* $\mathfrak{g}_\alpha$ *splits are isotypic, i.e.* λ *is either equal to plus, or minus the identity.*

Proposition 4.6. *If* $d_{2\alpha} = 7$, *then* $d_\alpha = 8$.

Proof. Without loss of generality, we can assume that λ has a non-zero eigenvector X corresponding to the eigenvalue 1. We prove that any vector orthogonal to $\mathbb{R}X \oplus J_3X$ is necessarily zero. This will prove the assertion since dim $(\mathbb{R}X \oplus J_3X) = 8$.

Let Y be orthogonal to $\mathbb{R}X \oplus J_3X$. This implies, by the J^2-property, that also J_4Y is orthogonal to $\mathbb{R}X \oplus J_3X$. Therefore, from Lemma 4.4 it follows that $\lambda J_4 Y = J_4 Y$. On the other hand, since $\lambda J_4 = -J_4\lambda$ and $\lambda Y = Y$, $\lambda J_4 Y = -J_4 Y$. Hence, $Y = 0$. ☐

We summarize in the next theorem the results of this section.

Theorem 4.7. *For each real rank one simple Lie algebra there is an integer* n *such that one of the following holds:*

1. $d_{2\alpha} = 0$ *and* $d_\alpha = n$.
2. $d_{2\alpha} = 1$ *and* $d_\alpha = 2n$.
3. $d_{2\alpha} = 3$, $d_\alpha = 4n$, *and* $\lambda = \pm I$.
4. $d_{2\alpha} = 7$ *and* $d_\alpha = 8$.

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DIPARTIMENTO METODI E MODELLI MATEMATICI, UNIVERSITÀ DI PADOVA, VIA BELZONI 7, PADOVA, ITALY
E-mail address: `ciatti@dmsa.unipd.it`