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BOUNDEDNESS OF ADMISSIBLE AREA FUNCTION ON NONISOTROPIC LIPSCHITZ SPACE

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ABSTRACT. Let B be the unit ball in C^n, let S be the unit sphere, and let $S_\beta(f)$ be the admissible area function. In this paper, we show that if $f \in Lip_\alpha(S)$, then $S_\beta(f) \in Lip_\alpha(S)$ and there exists a constant C such that $\|S_\beta(f)\|_{Lip_\alpha} \leq C\|f\|_{Lip_\alpha}$.

1. INTRODUCTION

Let B be the unit ball in C^n, $d\nu$ the normalized Lebesgue measure on B, $d\sigma$ the normalized surface measure on the boundary S of B.

For $\xi \in S$ and $0 < \delta \leq 2$, let $Q_\delta(\xi) = \{\eta \in S : d(\xi, \eta) = |1 - \langle \xi, \eta \rangle| < \delta\}$ be a nonisotropic ball of S.

We denote by $f \in Lip_\alpha(S)$ the nonisotropic Lipschitz space of order $0 < \alpha < 1$ if

$$\sup_{\xi,\eta \in S} \frac{|f(\xi) - f(\eta)|}{d(\xi, \eta)^\alpha} := \|f\|_{Lip_\alpha} < \infty.$$

We shall follow the convention of identifying the function f on the unit sphere with invariant harmonic extensions into the unit ball defined via the Poisson formula:

$$f(z) = \int_S P(z, \xi) f(\xi) d\sigma(\xi),$$

where $P(z, \xi)$ is the Poisson-Szegö kernel

$$P(z, \xi) = \frac{(1 - |z|^2)^n}{|1 - \langle z, \xi \rangle|^{2n}}.$$

For $F \in C^1(B)$, let $DF = (\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial y_1}, \cdots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial y_n})$ $(k = 1, 2, \cdots, n)$ be the real gradient of F and $\nabla F = (\frac{\partial F}{\partial z_1}, \frac{\partial F}{\partial z_2}, \cdots, \frac{\partial F}{\partial z_n})$ the complex gradient of F, $z_k = x_k + iy_k$.

Let $\widetilde{\nabla}$ denote the invariant gradient on B, that is,

$$\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0),$$

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where φ_z is the involution automorphism of B satisfying $\varphi_z(0) = z, \varphi_z(z) = 0$. It has been shown in [1] that for $a \in B$,

$$\varphi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{\frac{1}{2}} Q_a z}{1 - \langle z, a\rangle}$$

where $P_a z = \frac{\langle z,a\rangle}{|a|^2} a, P_0 z = 0, Q_a = I - P_a$ and

$$\varphi_a'(0) = -(1 - |a|^2)P_a - \sqrt{1 - |a|^2} Q_a.$$

Now, by a simple computation we get that

$$\begin{aligned} |\widetilde{\nabla} f(z)|^2 &= |\nabla(f \circ \varphi_z)(0)|^2 = |\nabla f(z)\varphi_z'(0)|^2 = |\varphi_{\bar z}'(0)\nabla f(z)|^2 \\ &= (1 - |z|^2)^2 |P_{\bar z}\nabla f(z)|^2 + (1 - |z|^2)|Q_{\bar z}\nabla f(z)|^2 \\ &= (1 - |z|^2)(|\nabla f(z)|^2 - |\langle \nabla f(z), \bar z\rangle|^2) \\ &= (1 - |z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2) \end{aligned} \tag{1.1}$$

where Rf is the radial derivative of f.

If $u \in C^1(B)$, the admissible area function is defined on S by

$$S_\beta u(\xi) = \{\int_{D_\beta(\xi)} |\widetilde{\nabla} u|^2 (1 - |z|^2)^{-(n+1)} d\nu(z)\}^{\frac{1}{2}}$$

where $D_\beta(\xi)$ denotes the admissible approach region with $\beta > 1$:

$$D_\beta(\xi) = \{z \in B : |1 - \langle z, \xi\rangle| < \frac{\beta}{2}(1 - |z|^2)\}.$$

In [2] S. Y. Chang proved the boundedness of S_β on L^p $(1 < p < +\infty)$; P. Ahern and J. Bruna [3] studied the characterization of Hardy-Sobolev spaces by $S_\beta(f)$.

The aim of this paper is to study the behavior of $S_\beta(f)$ where $f \in Lip_\alpha(S)\,(0 < \alpha < 1)$.

Theorem 1.1. *If $f \in Lip_\alpha(S)\,(0 < \alpha < 1)$ and $S_\beta(f) < +\infty$ a.e.on S, then $S_\beta f \in Lip_\alpha(S)$ and there exists a constant C such that*

$$\|S_\beta(f)\|_{Lip_\alpha} \le C\|f\|_{Lip_\alpha}.$$

Throughout this paper we shall use the letter C to denote constants, and it may change from line to line.

2. Preliminaries

Lemma 2.1. *Let $z \in B$, $\xi \in S$. We have*

$$|\widetilde{\nabla} P(z,\xi)| \le n\left(\frac{(1 - |z|^2)^n}{|1 - \langle z, \xi\rangle|^{2n}} + \frac{(1 - |z|^2)^{n+1}}{|1 - \langle z, \xi\rangle|^{2n+1}} + \frac{(1 - |z|^2)^{n+\frac{1}{2}}}{|1 - \langle z, \xi\rangle|^{2n+\frac{1}{2}}}\right).$$

Proof. Notice firstly that $P(z,\xi) = \bar P(z,\xi)$ and

$$|\widetilde{D} P(z,\xi)| = |\widetilde{\nabla} P(z,\xi)|.$$

To calculate $|\widetilde{\nabla} P(z,\xi)|$ we will use formula (1.1). We have

$$\nabla P(z,\xi) = -n\left(\frac{(1 - |z|^2)^{n-1}}{|1 - \langle z, \xi\rangle|^{2n}}\bar z - \frac{(1 - |z|^2)^n (1 - \overline{\langle z, \xi\rangle})}{|1 - \langle z, \xi\rangle|^{2n+2}}\bar\xi\right)$$

and

$$\begin{aligned} P_{\bar z}\nabla P(z,\xi) &= -n\left(\frac{(1-|z|^2)^{n-1}}{|1-\langle z,\xi\rangle|^{2n}} - \frac{(1-|z|^2)^n(1-\overline{\langle z,\xi\rangle})\langle\bar\xi,\bar z\rangle}{|z|^2|1-\langle z,\xi\rangle|^{2n+2}}\right)\bar z,\\ Q_{\bar z}\nabla P(z,\xi) &= -n\frac{(1-|z|^2)^n(1-\overline{\langle z,\xi\rangle})(|z|^2\bar\xi-\langle\bar\xi,\bar z\rangle\bar z)}{|z|^2|1-\langle z,\xi\rangle|^{2n+2}}. \end{aligned}$$

Hence

$$\begin{aligned} |\widetilde\nabla P(z,\xi)| &\le (1-|z|^2)|P_{\bar z}\nabla P(z,\xi)| + (1-|z|^2)^{\frac12}|Q_{\bar z}\nabla P(z,\xi)|\\ &\le n\left(\frac{(1-|z|^2)^n}{|1-\langle z,\xi\rangle|^{2n}} + \frac{(1-|z|^2)^{n+1}}{|1-\langle z,\xi\rangle|^{2n+1}} + \frac{(1-|z|^2)^{n+\frac12}}{|1-\langle z,\xi\rangle|^{2n+\frac12}}\right). \end{aligned}$$

□

The following lemma is well known; see [4].

Lemma 2.2. *Let $f\in L^1(S)$ and $0<\alpha<1$. Then the norm $\|f\|_{Lip_\alpha}$ is equivalent to*

$$\sup_Q \frac{1}{|Q|^{1+\frac{\alpha}{n}}}\int_Q |f-f_Q|d\sigma$$

where $f_Q=\frac{1}{|Q|}\int_Q f(\xi)d\sigma(\xi)$.

Lemma 2.3. *Let $f\in Lip_\alpha(S),\gamma\ge 0$. Let Q_0 be a nonisotropic ball of radius $\delta>0$ and center η^*. If $|1-\langle\eta,\eta^*\rangle|<\frac{\delta}{16}$, then there exists a constant C depending only on n,γ so that for any $z\in B$ that satisfies $|1-\langle z,\eta\rangle|<\frac{\delta}{16}$, we have*

$$\int_{S\setminus Q_0}\frac{|f(\xi)-f_{Q_0}|}{|1-\langle z,\xi\rangle|^{2n+\gamma}}d\sigma(\xi)\le C\delta^{-(n+\gamma-\alpha)}\|f\|_{Lip_\alpha}.$$

Proof. For $\xi\in S\setminus Q_0$ and $z\in B$ that satisfies $|1-\langle z,\eta\rangle|<\frac{\delta}{16},|1-\langle\eta,\eta^*\rangle|<\frac{\delta}{16}$, we have

$$\begin{aligned} |f(\xi)-f_{Q_0}| &\le \frac{1}{|Q_0|}\int_{Q_0}|f(\xi)-f(\eta')|d\sigma(\eta')\\ &\le \frac{1}{|Q_0|}\int_{Q_0}|1-\langle\xi,\eta'\rangle|^\alpha d\sigma(\eta')\|f\|_{Lip_\alpha}\\ &\le C|1-\langle z,\xi\rangle|^\alpha\|f\|_{Lip_\alpha}. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{S\setminus Q_0}\frac{|f(\xi)-f_{Q_0}|}{|1-\langle z,\xi\rangle|^{2n+\gamma}}d\sigma(\xi)\\ &\le C\|f\|_{Lip_\alpha}\int_{S\setminus Q_0}\frac{1}{|1-\langle z,\xi\rangle|^{2n+\gamma-\alpha}}d\sigma(\xi)\\ &\le C\delta^{-(n+\gamma-\alpha)}\|f\|_{Lip_\alpha}. \end{aligned}$$

□

To be convenient, we denote $r(z)=(1-|z|^2)$ $(z\in B)$.

Lemma 2.4. *Let Q_0 be a nonisotropic ball of radius $\delta > 0$ and center η^*, let R be a nonisotropic ball of radius $r(z)$ and center at η^*, and let $f \in Lip_\alpha(S)$. Then*

$$|f_R - f_{Q_0}| \le C(1 + |\ln[\frac{r(z)}{\delta}]|)r(z)^\alpha \|f\|_{Lip_\alpha} \qquad (r(z) > \delta), \tag{2.1}$$

$$|f_R - f_{Q_0}| \le C(1 + |\ln[\frac{r(z)}{\delta}]|)\delta^\alpha \|f\|_{Lip_\alpha} \qquad (r(z) < \delta). \tag{2.2}$$

Proof. We only prove (2.1), the proof of (2.2) being similar.

Since $r(z) > \delta$, we choose k such that $2^k\delta < r(z) \le 2^{k+1}\delta$. Then $\frac{2^{k+1}\delta}{r(z)} \le 2$ and $k \le \log_2 \frac{r(z)}{\delta}$, and

$$\begin{aligned} |f_R - f_{Q_0}| &= |\frac{1}{|R|}\int_R (f(\xi) - f_{Q_0})d\sigma(\xi)| \\ &\le C(r(z))^{-n}\int_R |f(\xi) - f_{Q_0}|d\sigma(\xi) \\ &\le C(r(z))^{-n}\int_{Q_{k+1}} |f(\xi) - f_{Q_0}|d\sigma(\xi). \end{aligned}$$

As in the proof of Lemma 2.3 we have

$$\begin{aligned} \int_{Q_{k+1}} |f(\xi) - f_{Q_0}|d\sigma(\xi) &\le C(k+2)|Q_{k+1}|^{1+\frac{\alpha}{n}}\|f\|_{Lip_\alpha} \\ &\le C(k+2)|2^{k+1}\delta|^{n+\alpha}\|f\|_{Lip_\alpha}. \end{aligned}$$

Thus

$$\begin{aligned} |f_R - f_{Q_0}| &\le C(k+2)|2^{k+1}\delta|^{n+\alpha}(r(z))^{-n}\|f\|_{Lip_\alpha} \\ &\le C2^{n+\alpha}(2 + |\log_2[\frac{r(z)}{\delta}]|)(r(z))^\alpha\|f\|_{Lip_\alpha}. \end{aligned}$$

Since $1 + |\ln[\frac{r(z)}{\delta}]|$ is equivalent to $2 + |\log_2[\frac{r(z)}{\delta}]|$, (2.1) is proved, which completes the proof of the lemma. □

Lemma 2.5. *Let R' be a nonisotropic ball of radius $4\alpha r(z)$ and center at η^*, and let $f \in Lip_\alpha(S)$. Then*

$$\int_S \frac{|f(\xi) - f_{R'}|}{|1 - \langle z, \xi\rangle|^{2n+\gamma}}d\sigma(\xi) \le C(r(z))^{-(n+\gamma-\alpha)}\|f\|_{Lip_\alpha}.$$

Proof. Argue as in the proof of Lemma 2.3. □

Lemma 2.6. *Let Q_0 be a nonisotropic ball of radius δ and center at η^*. If $\frac{\beta}{2}r(z) \ge \frac{\delta}{16}, |1 - \langle \eta, \eta^*\rangle| < \frac{\delta}{16}, |1 - \langle z, \eta\rangle| < \frac{\beta}{2}r(z)$, we have*

$$\int_S \frac{|f(\xi) - f_{Q_0}|}{|1 - \langle z, \xi\rangle|^{2n+\gamma}}d\sigma(\xi) \le C(r(z))^{-(n+\gamma-\alpha)}(1 + |\ln[\frac{r(z)}{\delta}]|)\|f\|_{Lip_\alpha}.$$

Proof.

$$\int_S \frac{|f(\xi) - f_{Q_0}|}{|1 - \langle z, \xi\rangle|^{2n+\gamma}}d\sigma(\xi) \le \int_S \frac{|f(\xi) - f_{R'}|}{|1 - \langle z, \xi\rangle|^{2n+\gamma}}d\sigma(\xi) + |f_{R'} - f_{Q_0}|\int_S \frac{d\sigma(\xi)}{|1 - \langle z, \xi\rangle|^{2n+\gamma}}.$$

By Lemma 2.4 and Lemma 2.5, we have

$$\int_S \frac{|f(\xi)-f_{R'}|}{|1-\langle z,\xi\rangle|^{2n+\gamma}}d\sigma(\xi)\le C(r(z))^{-(n+\gamma-\alpha)}\|f\|_{Lip_\alpha},$$
$$|f_{R'}-f_{Q_0}|\le C(r(z))^{\alpha}(1+|\ln[\frac{r(z)}{\delta}]|)\|f\|_{Lip_\alpha}.$$

We also have (see [1, P17])

$$\int_S \frac{d\sigma(\xi)}{|1-\langle z,\xi\rangle|^{2n+\gamma}}\approx r(z)^{-(n+\gamma)}.$$

Hence

$$\begin{aligned}&\int_S \frac{|f(\xi)-f_{Q_0}|}{|1-\langle z,\xi\rangle|^{2n+\gamma}}d\sigma(\xi)\\ &\le C(r(z))^{-(n+\gamma-\alpha)}\|f\|_{Lip_\alpha}+C(r(z))^{-(n+\gamma-\alpha)}(1+|\ln[\frac{r(z)}{\delta}]|)\|f\|_{Lip_\alpha}\\ &\le C(r(z))^{-(n+\gamma-\alpha)}(1+|\ln[\frac{r(z)}{\delta}]|)\|f\|_{Lip_\alpha}.\end{aligned}$$

□

3. The proof of Theorem 1.1

Let $h_Q(\xi)=(f(\xi)-f_Q)\chi_{S\setminus Q}(\xi)$. To complete the proof of Theorem 1.1, we need the following theorem.

Theorem 3.1. *Let $f\in Lip_\alpha(\xi)$, let Q be a nonisotropic ball of radius δ and center at η^*, and let $\frac{1}{16}Q=\{\xi\in S:|1-\langle\xi,\eta^*\rangle|<\frac{\delta}{16}\}$. Suppose that there is an $\eta'\in\frac{1}{16}Q$ so that $S_\beta(h_Q)(\eta')<\infty$. Then there exists a constant C, depending only on n and β, such that $S_\beta(h_Q)(\eta)<\infty$ and $|S_\beta(h_Q)(\eta)-S_\beta(h_Q)(\eta')|\le C\delta^\alpha\|f\|_{Lip_\alpha}$ for all $\eta\in\frac{1}{16}Q$.*

Proof. Let $D_\beta^-(\eta)=D_\beta(\eta)\cap\{\frac{\beta r(z)}{2}\le\frac{\delta}{16}\}$ and $D_\beta^+(\eta)=D_\beta(\eta)\cap\{\frac{\beta r(z)}{2}>\frac{\delta}{16}\}$.

We have

$$\begin{aligned}S_\beta(h_Q)(\eta)&=\{\int_{D_\beta(\eta)}|\widetilde{\nabla}\int_S P(z,\xi)h_Q(\xi)d\sigma(\xi)|^2r(z)^{-(n+1)}d\nu(z)\}^{\frac12}\\ &=\{\int_{D_\beta^-(\eta)}|\widetilde{\nabla}\int_S P(z,\xi)h_Q(\xi)d\sigma(\xi)|^2r(z)^{-(n+1)}d\nu(z)\}^{\frac12}\\ &\quad+\{\int_{D_\beta^+(\eta)}|\widetilde{\nabla}\int_S P(z,\xi)h_Q(\xi)d\sigma(\xi)|^2r(z)^{-(n+1)}d\nu(z)\}^{\frac12}\\ &=\{\int_{D_\beta^-(\eta)}(\int_S|\widetilde{\nabla}P(z,\xi)||h_Q(\xi)|d\sigma(\xi))^2r(z)^{-(n+1)}d\nu(z)\}^{\frac12}\\ &\quad+\{\int_{D_\beta^+(\eta)}(\int_S|\widetilde{\nabla}P(z,\xi)||h_Q(\xi)|d\sigma(\xi))^2r(z)^{-(n+1)}d\nu(z)\}^{\frac12}\\ &:=I_1+I_2.\end{aligned}\tag{3.1}$$

By Lemma 2.1, we have

$$I_1 \leq n\Big\{\int_{D_\beta^-(\eta)}\Big(\int_{S\setminus Q}\Big(\frac{(1-|z|^2)^n}{|1-\langle z,\xi\rangle|^{2n}}+\frac{(1-|z|^2)^{n+1}}{|1-\langle z,\xi\rangle|^{2n+1}}+\frac{(1-|z|^2)^{n+\frac12}}{|1-\langle z,\xi\rangle|^{2n+\frac12}}\Big)$$
$$\times|f(\xi)-f_Q|d\sigma(\xi)\Big)^2 r(z)^{-(n+1)}d\nu(z)\Big\}^{\frac12}.$$

If $z\in D_\beta^-(\eta)$, by Lemma 2.3, we get

$$\left(\int_{S\setminus Q}\frac{(1-|z|^2)^n}{|1-\langle z,\xi\rangle|^{2n}}|f(\xi)-f_Q|d\sigma(\xi)\right)^2 \leq C(r(z))^{2n}\delta^{-2(n-\alpha)}\|f\|^2_{Lip_\alpha},$$
$$\left(\int_{S\setminus Q}\frac{(1-|z|^2)^{n+1}}{|1-\langle z,\xi\rangle|^{2n+1}}|f(\xi)-f_Q|d\sigma(\xi)\right)^2 \leq C(r(z))^{2n+2}\delta^{-2(n+1-\alpha)}\|f\|^2_{Lip_\alpha},$$
$$\left(\int_{S\setminus Q}\frac{(1-|z|^2)^{n+\frac12}}{|1-\langle z,\xi\rangle|^{2n+\frac12}}|f(\xi)-f_Q|d\sigma(\xi)\right)^2 \leq C(r(z))^{2n+1}\delta^{-2(n+\frac12-\alpha)}\|f\|^2_{Lip_\alpha}.$$

It follows that

$$\begin{aligned} I_1 &\leq C\Big\{\int_{D_\beta^-(\eta)}[(r(z))^{2n}\delta^{-2(n-\alpha)}+(r(z))^{2n+2}\delta^{-2(n+1-\alpha)} \\ &\qquad +(r(z))^{2n+1}\delta^{-2(n+\frac12-\alpha)}]r(z)^{-(n+1)}\|f\|^2_{Lip_\alpha}d\nu(z)\Big\}^{\frac12} \\ &= C\Big(\delta^{-2(n-\alpha)}\int_{D_\beta^-(\eta)}(r(z))^{n-1}d\nu(z)+\delta^{-2(n+1-\alpha)}\int_{D_\beta^-(\eta)}(r(z))^{n+1}d\nu(z) \\ &\qquad +\delta^{-2(n+\frac12-\alpha)}\int_{D_\beta^-(\eta)}(r(z))^{n}d\nu(z)\Big)^{\frac12}\|f\|_{Lip_\alpha} \\ &\leq C\Big(\delta^{-2(n+1-2\alpha)}\int_{D_\beta^-(\eta)}d\nu(z)\Big)^{\frac12}\|f\|_{Lip_\alpha}. \end{aligned}$$

Arguing as in [2, P116], we get

$$\int_{D_\beta^-(\eta)}d\nu(z)\leq C\delta^{n+1}.$$

Hence

$$I_1\leq C\delta^{\alpha}\|f\|_{Lip_\alpha}. \tag{3.2}$$

Next, we estimate I_2:

$$\begin{aligned} I_2 &= \Big\{\int_{D_\beta^+(\eta)}\Big(\int_S|\widetilde{\nabla}P(z,\xi)||h_Q(\xi)|d\sigma(\xi)\Big)^2 r(z)^{-(n+1)}d\nu(z)\Big\}^{\frac12} \\ &\leq \Big\{\int_{D_\beta^+(\eta)\cap D_\beta(\eta')}\Big(\int_S|\widetilde{\nabla}P(z,\xi)||h_Q(\xi)|d\sigma(\xi)\Big)^2 r(z)^{-(n+1)}d\nu(z)\Big\}^{\frac12} \\ &\quad +\Big\{\int_{D_\beta^+(\eta)\setminus D_\beta(\eta')}\Big(\int_S|\widetilde{\nabla}P(z,\xi)||h_Q(\xi)|d\sigma(\xi)\Big)^2 r(z)^{-(n+1)}d\nu(z)\Big\}^{\frac12}. \end{aligned} \tag{3.3}$$

For the first term above, we have

$$\begin{aligned}
&\Big\{\int_{D_\beta^+(\eta)\cap D_\beta(\eta')}\big(\int_S|\widetilde{\nabla}P(z,\xi)||h_Q(\xi)|d\sigma(\xi)\big)^2 r(z)^{-(n+1)}d\nu(z)\Big\}^{\frac{1}{2}}\\
&\le\Big\{\int_{D_\beta(\eta')}\big(\int_S|\widetilde{\nabla}P(z,\xi)||h_Q(\xi)|d\sigma(\xi)\big)^2 r(z)^{-(n+1)}d\nu(z)\Big\}^{\frac{1}{2}}\\
&=S_\beta(h_Q)(\eta').
\end{aligned}\tag{3.4}$$

For the second term above, we have

$$\begin{aligned}
&\Big\{\int_{D_\beta^+(\eta)\setminus D_\beta(\eta')}\big(\int_S|\widetilde{\nabla}P(z,\xi)||h_Q(\xi)|d\sigma(\xi)\big)^2 r(z)^{-(n+1)}d\nu(z)\Big\}^{\frac{1}{2}}\\
&\le n\Big\{\int_{D_\beta^+(\eta)\setminus D_\beta(\eta')}\big(\int_{S\setminus Q}\big(\frac{(1-|z|^2)^n}{|1-\langle z,\xi\rangle|^{2n}}+\frac{(1-|z|^2)^{n+1}}{|1-\langle z,\xi\rangle|^{2n+1}}+\frac{(1-|z|^2)^{n+\frac{1}{2}}}{|1-\langle z,\xi\rangle|^{2n+\frac{1}{2}}}\big)\\
&\qquad\times|f(\xi)-f_Q|d\sigma(\xi)\big)^2 r(z)^{-(n+1)}d\nu(z)\Big\}^{\frac{1}{2}}.
\end{aligned}$$

Using Lemma 2.6, it follows that

$$\begin{aligned}
\big(\int_{S\setminus Q}\frac{(1-|z|^2)^n}{|1-\langle z,\xi\rangle|^{2n}}|f(\xi)-f_Q|d\sigma(\xi)\big)^2 &\le C(r(z))^{2\alpha}(1+|\ln[\frac{r(z)}{\delta}]|)^2\|f\|^2_{Lip_\alpha},\\
\big(\int_{S\setminus Q}\frac{(1-|z|^2)^{n+1}}{|1-\langle z,\xi\rangle|^{2n+1}}|f(\xi)-f_Q|d\sigma(\xi)\big)^2 &\le C(r(z))^{2\alpha}(1+|\ln[\frac{r(z)}{\delta}]|)^2\|f\|^2_{Lip_\alpha},\\
\big(\int_{S\setminus Q}\frac{(1-|z|^2)^{n+\frac{1}{2}}}{|1-\langle z,\xi\rangle|^{2n+\frac{1}{2}}}|f(\xi)-f_Q|d\sigma(\xi)\big)^2 &\le C(r(z))^{2\alpha}(1+|\ln[\frac{r(z)}{\delta}]|)^2\|f\|^2_{Lip_\alpha}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\Big\{\int_{D_\beta^+(\eta)\setminus D_\beta(\eta')}\big(\int_S|\widetilde{\nabla}P(z,\xi)||h_Q(\xi)|d\sigma(\xi)\big)^2 r(z)^{-(n+1)}d\nu(z)\Big\}^{\frac{1}{2}}\\
&\le C\Big\{\int_{D_\beta^+(\eta)\setminus D_\beta(\eta')}(1+|\ln[\frac{r(z)}{\delta}]|)^2 r(z)^{-(n+1-2\alpha)}\|f\|^2_{Lip_\alpha}d\nu(z)\Big\}^{\frac{1}{2}}.
\end{aligned}$$

Notice that $|D_\beta^+(\eta)\setminus D_\beta(\eta')\cap\{r(z)=c'\}|\le C\delta^n$ for all $0<c'<1$ (see [5, P128]) and $\frac{\beta r(z)}{2}>\frac{1}{16}\delta$. So we have $|D_\beta^+(\eta)\setminus D_\beta(\eta')\cap\{r(z)=c'\}|\le C\delta(r(z))^{n-1}$. Thus it follows that

$$\begin{aligned}
&\Big\{\int_{D_\beta^+(\eta)\setminus D_\beta(\eta')}(1+|\ln[\frac{r(z)}{\delta}]|)^2 r(z)^{-(n+1-2\alpha)}\|f\|^2_{Lip_\alpha}d\nu(z)\Big\}^{\frac{1}{2}}\\
&\le C\int_{\frac{\delta}{8\beta}}^1(1+|\ln[\frac{r(z)}{\delta}]|)^2\delta r(z)^{2\alpha-2}dr(z)\|f\|_{Lip_\alpha}\\
&\le C\delta^\alpha\int_{\frac{\delta}{8\beta}}^{+\infty}(1+|\ln t|)t^{-2}dt\|f\|_{Lip_\alpha}\\
&\le C\delta^\alpha\|f\|_{Lip_\alpha}.
\end{aligned}\tag{3.5}$$

Using (3.3)-(3.5), we have

$$I_2\le S_\beta(h_Q)(\eta')+C\delta^\alpha\|f\|_{Lip_\alpha}.\tag{3.6}$$

Combining (3.1) with (3.2), (3.6), it follows that

$$S_\beta(h_Q)(\eta) \leq S_\beta(h_Q)(\eta') + C\delta^\alpha \|f\|_{Lip_\alpha}.$$

Thus $S_\beta(h_Q)(\eta)$ is finite. Reversing the roles of η and η', we obtain

$$|S_\beta(h_Q)(\eta) - S_\beta(h_Q)(\eta')| \leq C\delta^\alpha \|f\|_{Lip_\alpha}.$$

This completes the proof of Theorem 3.1. □

The proof of Theorem 1.1. Write f as

$$\begin{aligned} f(\xi) &= f_Q + (f(\xi) - f_Q)\chi_Q(\xi) + (f(\xi) - f_Q)\chi_{S\setminus Q}(\xi) \\ &= f_Q(\xi) + g_Q(\xi) + h_Q(\xi). \end{aligned} \tag{3.7}$$

Since f_Q is a constant, $S_\beta(f_Q) = 0$. Thus $S_\beta(f_Q)$ is in Lip_α with the Lip_α norm equal to 0. Therefore

$$\begin{aligned} S_\beta(f) &\leq S_\beta(g_Q) + S_\beta(h_Q), \\ S_\beta(h_Q) &\leq S_\beta(g_Q) + S_\beta(f). \end{aligned} \tag{3.8}$$

By the boundedness of $S_\beta(f)$ on L^p $(1 < p < \infty)$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_Q |S_\beta(g_Q)| d\sigma(\xi) &\leq |Q|^{\frac{1}{2}} (\int_Q |S_\beta(g_Q)|^2 d\sigma(\xi))^{\frac{1}{2}} \\ &\leq C|Q|^{1+\frac{\alpha}{n}} \|f\|_{Lip_\alpha}. \end{aligned} \tag{3.9}$$

It also follows from (3.9) that $S_\beta(g_Q)$ is finite almost everywhere. Therefore $S_\beta(h_Q) < +\infty$ at almost every point such that $S_\beta(f) < +\infty$.

Now, we prove $\|S_\beta(f)\|_{Lip_\alpha} \leq C\|f\|_{Lip_\alpha}$.

To show the boundedness of S_β on Lip_α, by Lemma 2.2 and the triangle inequality, it suffices to show that for every $f \in Lip_\alpha(S)$, there is a constant $\lambda = \lambda(Q, f)$ such that

$$\frac{1}{|Q|^{1+\frac{\alpha}{n}}} \int_Q |S_\beta f(\xi) - \lambda| d\sigma(\xi) \leq C\|f\|_{Lip_\alpha}.$$

Let $Q' \subset S$ be any nonisotrophic ball, and $Q = 16Q'$ (that is $Q' = \frac{1}{16}Q$). Since $S_\beta(h_Q)(\eta) < +\infty$ at almost every point, we choose a point $\eta' \in \frac{1}{16}Q$ so that $S_\beta(h_Q)(\eta')$ is finite. Then, by (3.9) and Theorem 3.1,

$$\begin{aligned} &\frac{1}{|Q'|^{1+\frac{\alpha}{n}}} \int_{Q'} |S_\beta f(\xi) - S_\beta(h_Q)(\eta')| d\sigma(\xi) \\ &= \frac{1}{|Q'|^{1+\frac{\alpha}{n}}} \int_{Q'} |S_\beta(g_Q + h_Q)(\xi) - S_\beta(h_Q)(\xi) + S_\beta(h_Q)(\xi) - S_\beta(h_Q)(\eta')| d\sigma(\xi) \\ &= \frac{1}{|Q'|^{1+\frac{\alpha}{n}}} \int_{Q'} |S_\beta(g_Q)(\xi)| d\sigma(\xi) + \frac{1}{|Q'|^{1+\frac{\alpha}{n}}} \int_{Q'} |S_\beta(h_Q)(\xi) - S_\beta(h_Q)(\eta')| d\sigma(\xi) \\ &\leq C\|f\|_{Lip_\alpha}. \end{aligned}$$

Since Q' is arbitrary, the proof is completed. □

Remark 3.2. The referee asked whether the following proposition is true or not:

$$S_\alpha f \in Lip_\alpha(S) \Rightarrow f \in Lip_\alpha(S)??$$

The authors cannot give a positive proof right now.

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