

GRADIENT RANGES OF BUMPS ON THE PLANE

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ABSTRACT. For a \mathcal{C}^1 -smooth bump function $b: \mathbb{R}^2 \rightarrow \mathbb{R}$ we show that the gradient range $\nabla b(\mathbb{R}^2)$ is the closure of its interior, provided that ∇b admits a modulus of continuity $\omega = \omega(t)$ satisfying $\omega(t)/\sqrt{t} \rightarrow 0$ as $t \searrow 0$. The result is a consequence of a more general result about gradient ranges of bump functions $b: \mathbb{R}^n \rightarrow \mathbb{R}$ of the same degree of smoothness. For such bump functions we show that for open sets $G \subset \mathbb{R}^n$, either the intersection $\nabla b(\mathbb{R}^n) \cap G$ is empty or its topological dimension is at least two. The proof relies on a new Morse-Sard type result where the smoothness hypothesis is independent of the dimension n of the space.

1. INTRODUCTION

A real-valued \mathcal{C}^1 function $b: \mathbb{R}^n \rightarrow \mathbb{R}$ is called a bump if its support, $\text{spt}(b)$, defined as the closure of the set $\{x \in \mathbb{R}^n : b(x) \neq 0\}$, is bounded and non-empty. This work is part of an attempt to find characteristic properties, topological and otherwise, of the gradient range $\nabla b(\mathbb{R}^n)$ of a \mathcal{C}^1 -smooth bump b on \mathbb{R}^n . Only the case $n \geq 2$ is interesting since for $n = 1$ it is easy to see that the gradient ranges are precisely those bounded and closed intervals that contain $0 \in \mathbb{R}$ in their interior. Here we address the particular question of whether the gradient range of a \mathcal{C}^1 -smooth bump $b: \mathbb{R}^n \rightarrow \mathbb{R}$ is the closure of its interior (i.e., is regularly closed). Previous works related to this question include [1], [2], [3], [5] and [6]. In particular, we emphasize that for \mathcal{C}^2 -smooth bumps $b: \mathbb{R}^n \rightarrow \mathbb{R}$ (in any finite dimension n) one can use Th. 1, Sect. 3 of [5] in connection with a straightforward argument based on the Inverse Function Theorem to show that $\nabla b(\mathbb{R}^n)$ is regularly closed. It is however not clear how to relax the \mathcal{C}^2 -smoothness condition imposed in [5]. In the infinite-dimensional case the gradient range is far from being regularly closed. Indeed, there exists a \mathcal{C}^1 -smooth bump $b: \ell_2 \rightarrow \mathbb{R}$ such that $\nabla b(\ell_2)$ has empty interior (see [1]). In [6] it is shown that the gradient range of a \mathcal{C}^1 -smooth bump b on the plane is regularly closed and has a connected interior provided that either

- (i) ∇b is of bounded variation, or
- (ii) ∇b admits a modulus of continuity $\omega = \omega(t)$ satisfying $\int_0^1 \frac{dt}{\omega(t)} = \infty$.

The corresponding n -dimensional results are weaker: the gradient range of a \mathcal{C}^1 -smooth bump on \mathbb{R}^n satisfying either (i) or (ii) is at least (topologically) two-dimensional at all of its points (see [6]). We refer to [2] and [6] for additional

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references and background. The main result of this paper establishes the regular closedness of the gradient range for a class of bumps on the plane that is significantly larger than those satisfying (ii) above. We proceed with precise statements of the results.

Theorem. *Let $b: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a \mathcal{C}^1 -smooth bump such that ∇b has a modulus of continuity $\omega = \omega(t)$ satisfying $\omega(t)/\sqrt{t} \rightarrow 0$ as $t \searrow 0$. Then the gradient range $\nabla b(\mathbb{R}^2)$ is regularly closed, i.e., equals the closure of its interior.*

The theorem is easily deduced from the following proposition that concerns bumps on \mathbb{R}^n . In its statement we denote by $\dim M$ the topological dimension of the set M .

Proposition 1. *Let $n \geq 2$ and $b: \mathbb{R}^n \rightarrow \mathbb{R}$ be a \mathcal{C}^1 -smooth bump such that ∇b has a modulus of continuity $\omega = \omega(t)$ satisfying $\omega(t)/\sqrt{t} \rightarrow 0$ as $t \searrow 0$.*

If G is an open set in \mathbb{R}^n such that the intersection $G \cap \nabla b(\mathbb{R}^n)$ is non-empty, then $\dim(\nabla b(\mathbb{R}^n) \cap G) \geq 2$.

The theorem follows from the proposition, because a subset M of \mathbb{R}^2 has topological dimension two precisely when it contains a non-empty open subset (see Th. IV 3 of [4]).

The proof of Proposition 1 relies on the following Morse-Sard type result, in which the smoothness assumption does not depend on the dimension n of the space. A similar result that does not involve Minkowski content and Hölder continuity can be found in [6]. By $\mathcal{U}_\varepsilon(C) = \{y : \text{dist}(y, C) < \varepsilon\}$ we denote the ε -neighbourhood of C . For $s \in [0, n]$, the s -dimensional lower Minkowski content of a set $C \subseteq \mathbb{R}^n$ is defined by

$$\mathcal{M}_*^s(C) = \liminf_{\varepsilon \searrow 0} (2\varepsilon)^{s-n} \mathcal{L}^n(\mathcal{U}_\varepsilon(C))$$

where \mathcal{L}^n denotes the Lebesgue measure on \mathbb{R}^n .

A set C in \mathbb{R}^n is called an irreducible separator if there are two points x and y in \mathbb{R}^n such that C separates x and y , but no proper subset R of C separates them.

Proposition 2. *Assume $s \in (0, 1)$, $f \in \mathcal{C}^1(\mathbb{R}^n)$ and $\nabla f = 0$ on a bounded irreducible separator C . If ∇f is s -Hölder and $\mathcal{M}_*^{n-1+s}(C) = 0$, i.e.,*

$$\liminf_{\varepsilon \searrow 0} \frac{\mathcal{L}^n(\mathcal{U}_\varepsilon(C))}{\varepsilon^{1-s}} = 0,$$

then f is constant on C . The same is true if ∇f admits a modulus of continuity ω with $\lim_{\varepsilon \searrow 0} \omega(\varepsilon)/\varepsilon^s = 0$ and $\mathcal{M}_^{n-1+s}(C) < \infty$.*

The organization of the paper is as follows. In Section 2 we set the notation, and state as lemmata a collection of elementary auxiliary results. The proofs of Propositions 1 and 2 are presented in Section 3.

2. PRELIMINARIES

We use standard notation. In particular, $\|\cdot\|$ always denotes the euclidean norm on \mathbb{R}^n , $B(x, r)$ denotes the open ball with center x and radius r , and $B[x, r]$ the corresponding closed ball.

A non-decreasing, continuous function $\omega: [0, \infty) \rightarrow [0, \infty)$ with $\omega(0) = 0$ is called a modulus of continuity of the vector field $V: \mathbb{R}^n \rightarrow \mathbb{R}^n$ if

$$(2.1) \quad \|V(x) - V(y)\| \leq \omega(\|x - y\|)$$

for all $x, y \in \mathbb{R}^n$. In particular, a vector field V is s -Hölder if it admits a modulus of continuity of the form $\omega(t) = ct^s$ where c is a constant.

We proceed with some auxiliary results of an elementary nature. The Lemmata 1, 2, 4, 5 and 6 are similar to results contained in [6]. We state them here for the convenience of the reader.

Lemma 1. *Let $G \subseteq \mathbb{R}^n$ be bounded and open. Assume $f: \overline{G} \rightarrow \mathbb{R}$ is continuous, differentiable on G , $f = 0$ on ∂G and f is not identically zero on \overline{G} . Then $0 \in \mathbb{R}^n$ is an interior point of $\nabla f(G)$.*

Proof. Let $m = \max f(\overline{G}) > 0$. (In the opposite case f should be replaced by $-f$.) Let $y \in B(0, m/\text{diam } G)$ and put $g(x) = f(x) - y \cdot x$, $x \in \overline{G}$. By continuity and compactness we can find $x_0 \in \overline{G}$, such that $g(x_0) = \max g(\overline{G})$. It is not hard to see from the assumptions that $x_0 \in G$, so $\nabla g(x_0) = 0$, or $\nabla f(x_0) = y$. Hence $B(0, m/\text{diam } G) \subseteq \nabla f(G)$, and $0 \in \mathbb{R}^n$ is an interior point. \square

Lemma 2. *Let $b: \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 -smooth bump. The gradient range $\nabla b(\mathbb{R}^n)$ is a locally connected continuum that contains $0 \in \mathbb{R}^n$ in its interior.*

Proof. The support of b is contained in an open ball B , so $0 \in \text{int } \nabla b(B) = \text{int } \nabla b(\mathbb{R}^n)$ by Lemma 1. The closure \overline{B} of the ball is a locally connected continuum, hence so is $\nabla b(\overline{B}) = \nabla b(\mathbb{R}^n)$ (see §50.II, Th. 5 of [7]). \square

A topological space \mathcal{X} has (topological) dimension 0 at the point $x \in \mathcal{X}$, briefly $\dim_x \mathcal{X} = 0$, if x has arbitrarily small open neighborhoods with empty boundaries. In other words, given any open neighborhood U of x there exists an open neighborhood $V \subset U$, such that $\partial V = \emptyset$.

The space \mathcal{X} is 0-dimensional, briefly $\dim \mathcal{X} = 0$, provided $\dim_x \mathcal{X} = 0$ for all $x \in \mathcal{X}$. It is convenient to define the dimension of the empty set \emptyset to be -1 .

Higher topological dimension is then defined by induction: The space \mathcal{X} has dimension $n \in \mathbb{N}$ at x , briefly $\dim_x \mathcal{X} = n$, if it is false that $\dim_x \mathcal{X} \leq n - 1$, but x has arbitrarily small open neighborhoods with boundaries of dimension at most $n - 1$. The space \mathcal{X} is n -dimensional if $\max_{x \in \mathcal{X}} \dim_x \mathcal{X} = n$.

We shall apply the above definition only to subsets of \mathbb{R}^n (considered with the topology inherited from \mathbb{R}^n).

Our reference on topological dimension is [4] to which we refer for a thorough discussion.

Let \mathcal{X} be a topological space and A, B and C subsets. We say that C separates A and B (in \mathcal{X}) if it is possible to split $\mathcal{X} \setminus C$ into two disjoint relatively open sets containing A and B , respectively. That is, if there exist open sets U and V in \mathcal{X} , such that if we let $U' = U \setminus C$, $V' = V \setminus C$, then $\mathcal{X} \setminus C = U' \cup V'$, $A \subseteq U'$, $B \subseteq V'$ and $U' \cap V' = \emptyset$. (When $A = \{a\}$, $B = \{b\}$ and C separates A and B we also say that C separates a and b , and that C is a separator between a and b .)

We say that A and B are irreducibly separated in \mathcal{X} by C (and that C is an irreducible separator between A and B) if C separates A and B in \mathcal{X} , and any proper subset $R \subset C$ does not. We shall use this concept only when A and B are singletons. In this case it is not hard to show that an irreducible separator is always a closed set (see §46.VII, Th. 3 of [7]).

Lemma 3. *Let \mathcal{X} be a separable metric space, A and B closed and disjoint subsets of \mathcal{X} , and M a subset of \mathcal{X} of dimension at most $n \in \mathbb{N}$. Then there exists a closed set C separating A, B in \mathcal{X} , and with the property that $\dim(M \cap C) \leq n - 1$.*

Proof. This is Prop. B, Ch. III, §5 of [4]. \square

Lemma 4. *Let \mathcal{X} be a locally connected metric space.*

(1) *If A is open and connected and B is a component of $\mathcal{X} \setminus \overline{A}$, then the set ∂B is an irreducible separator between each pair of points $a \in A$ and $b \in B$.*

(2) *Let C be a closed set, A and B two distinct components of $\mathcal{X} \setminus C$, and let $a \in A$ and $b \in B$. The set C is an irreducible separator between a and b if and only if $\partial A = C = \partial B$.*

Proof. (1) is Th. 2 and (2) is Th. 1 in §49.VI of [7]. \square

Lemma 5. *Every irreducible separator between two distinct points of \mathbb{R}^n is closed and connected.*

Proof. This follows from §57.III.1 of [7] that can be applied by §57.I.9(i) of [7]. \square

Lemma 6. *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 function, and Ω an open set in \mathbb{R}^n , such that $\nabla f = 0$ on $\partial\Omega$ and such that $\partial\Omega$ is an irreducible separator between two points $x \in \Omega$ and $y \in \mathbb{R}^n \setminus \overline{\Omega}$. Then f is constant on $\partial\Omega$ if the vector field*

$$V = \begin{cases} \nabla f & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

is curl-free in the distributional sense on \mathbb{R}^n .

Proof. Suppose that V is curl-free: $V_{,i}^j - V_{,j}^i = 0$ holds in the distributional sense on \mathbb{R}^n for each pair of indices $i, j = 1, \dots, n$. Using mollification and a standard result we find a C^1 function $g: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $V = \nabla g$. Now let Λ_1 and Λ_2 be the components of $\mathbb{R}^n \setminus \partial\Omega$ containing x and y , respectively. From Lemma 4 (2) it follows that $\partial\Lambda_1 = \partial\Omega = \partial\Lambda_2$. Obviously, $\Lambda_1 \subseteq \Omega$ and $\Lambda_2 \subseteq \mathbb{R}^n \setminus \overline{\Omega}$. Because Λ_1 is open and connected and $\nabla f = \nabla g$ there, we can assume that $g = f$ on $\overline{\Lambda_1}$. Next, $\nabla g = 0$ on $\mathbb{R}^n \setminus \Omega$, so in particular, $\nabla g = 0$ on Λ_2 . Since Λ_2 is open and connected, g must be constant on $\overline{\Lambda_2} = \Lambda_2 \cup \partial\Omega$. But $f = g$ on $\overline{\Lambda_1} = \Lambda_1 \cup \partial\Omega$, so f is constant on $\partial\Omega$ too. \square

3. THE PROOFS

Proof of Proposition 1. Suppose the assertion of the proposition is false. Then we can clearly find a closed ball $B_0 = B[y_0, R]$ centred at a point y_0 of $K = \nabla b(\mathbb{R}^n)$ such that $\dim M \leq 1$, where $M = K \cap B_0$. In view of Lemma 2 and since $n > 1$ we must have $0 \notin B[y_0, R]$. Let $0 < r < s < R$, and consider the closed and disjoint sets $A = B[y_0, r]$ and $B = \mathbb{R}^n \setminus B(y_0, s)$. By virtue of Lemma 3, we can find a closed set $C = C_{r,s}$ that separates A and B in \mathbb{R}^n and such that $\dim(C \cap M) \leq 0$. Accordingly, there exist open sets $U = U_{r,s}$, $V = V_{r,s}$, such that $\mathbb{R}^n \setminus C = U \cup V$, $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. Since $\partial U \subseteq C$, we have found an open set $U_{r,s}$ in \mathbb{R}^n such that $B[y_0, r] \subseteq U_{r,s} \subseteq B(y_0, s)$ and $\dim(M \cap \partial U_{r,s}) = 0$. The latter implies that $M \cap \partial U_{r,s}$ is a totally disconnected set.

Select $x_0 \in \mathbb{R}^n$ so that $\nabla b(x_0) = y_0$, and define for $0 < r < s \leq R$, the sets Λ_r , $\Lambda_{r,s}$ and Λ_s as the (connected) components of the open sets $(\nabla b)^{-1}(B(y_0, r))$, $(\nabla b)^{-1}(U_{r,s})$ and $(\nabla b)^{-1}(B(y_0, s))$ that contain x_0 , respectively. Then Λ_r , $\Lambda_{r,s}$ and Λ_s are open connected sets, $\Lambda_r \subseteq \Lambda_{r,s} \subseteq \Lambda_s$ and $\nabla b(\partial\Lambda_r) \subseteq \partial B(y_0, r)$, $\nabla b(\partial\Lambda_{r,s}) \subseteq M \cap \partial U_{r,s}$. Since $0 \notin M$ and $\nabla b \in M$ on Λ_s , it follows that $\Lambda_s \subseteq \text{spt}(b)$, the support of b , and hence in particular that the sets are bounded.

We assert that for each $r \in (0, R)$ and each sequence $\{r_j\}$ with $r_j \nearrow r$,

$$(3.1) \quad \Lambda_r = \bigcup_{j=1}^{\infty} \Lambda_{r_j}$$

and

$$(3.2) \quad \Lambda_r = \bigcup_{j=1}^{\infty} \Lambda_{r_j, r}$$

hold. To see this, let $x \in \Lambda_r$ and note that since Λ_r is arcwise connected there is a curve $\gamma \subseteq \Lambda_r$ that connects x and x_0 . Now if we take j so large that $r_j > r - \text{dist}(\nabla b(\gamma), \partial B(y_0, r))/2$, then γ does not intersect $(\nabla b)^{-1}(\partial B(y_0, r_j))$. Since Λ_{r_j} is the component of $(\nabla b)^{-1}(B(y_0, r_j))$ that contains x_0 , it follows that $\gamma \subseteq \Lambda_{r_j}$, and thus in particular, $x \in \Lambda_{r_j}$, establishing (3.1). Then, (3.2) follows from the inclusions $\Lambda_r \subseteq \Lambda_{r,s} \subseteq \Lambda_s$.

Let $0 < r < s < R$. Since Λ_r and $\Lambda_{r,s}$ are bounded, the complements $\mathbb{R}^n \setminus \overline{\Lambda_r}$ and $\mathbb{R}^n \setminus \overline{\Lambda_{r,s}}$ both have precisely one unbounded component. Let A_r and $A_{r,s}$ denote these, respectively. Next, let Ω_r and $\Omega_{r,s}$ denote the components of $\mathbb{R}^n \setminus \overline{A_r}$ and $\mathbb{R}^n \setminus \overline{A_{r,s}}$ that contain x_0 , respectively. It follows from Lemma 4 that $\partial \Omega_r$ and $\partial \Omega_{r,s}$ are irreducible separators (between certain pairs of points). By Lemma 5 they are therefore also connected. Furthermore, we record the following facts: $\Omega_r \subseteq \Omega_{r,s} \subseteq \Omega_s \subseteq \text{cospt}(b)$, the convex hull of the support of b , $\Lambda_r \subseteq \Omega_r$, $\Lambda_{r,s} \subseteq \Omega_{r,s}$ and $\partial \Omega_r \subseteq \partial \Lambda_r$, $\partial \Omega_{r,s} \subseteq \partial \Lambda_{r,s}$. (For example, $\partial \Omega_r \subseteq \partial \Lambda_r$ follows by applying twice the general result stating that for a closed subset B of \mathbb{R}^n the boundary of every component of $\mathbb{R}^n \setminus B$ is contained in ∂B .)

Now recall that $\nabla b \in M \cap \partial U_{r,s}$ on $\partial \Lambda_{r,s}$. Consequently, ∇b maps the connected space $\partial \Omega_{r,s}$ continuously into the totally disconnected space $M \cap \partial U_{r,s}$ and is therefore constant: $\nabla b = y_{r,s}$ on $\partial \Omega_{r,s}$. Next we deduce that ∇b is also constant on each of the boundaries $\partial \Omega_s$, $s \in (0, R)$, by observing that for each $x \in \partial \Omega_s$,

$$(3.3) \quad \lim_{r \nearrow s} \text{dist}(x, \partial \Omega_{r,s}) = 0.$$

The routine verification of (3.3) is left to the interested reader.

Let us summarize what has been achieved so far. To each $r \in (0, R)$ there is assigned an open connected set Ω_r contained in $\text{cospt}(b)$, $\partial \Omega_r$ is an irreducible separator (between certain pairs of points), and $\nabla b = y_r$ on $\partial \Omega_r$, where $y_r \in M \cap \partial B(y_0, r)$. Furthermore, $\Omega_r \subseteq \Omega_s$ when $r < s$. Moreover, since $y_r \neq y_s$, $\partial \Omega_r \cap \partial \Omega_s = \emptyset$ and hence $\overline{\Omega_r} \subset \Omega_s$ when $r < s$.

Assume that for some $r \in (0, R)$ the function $x \mapsto b(x) - y_r \cdot x$ is constant, say equal to c , on the boundary $\partial \Omega_r$. Put $f(x) = b(x) - y_r \cdot x - c$ and $G = \Omega_r$. Since $x_0 \in G$ and $\nabla f(x_0) = y_0 - y_r \neq 0$, it follows that f is not identically zero on G , and therefore, in view of Lemma 1, that $0 \in \mathbb{R}^n$ is an interior point of $\nabla f(G)$. But this implies that y_r is an interior point of $\nabla b(\Omega_r)$, and hence in particular also an interior point of $\nabla b(\mathbb{R}^n)$, leading to the contradiction that $n = \dim_{y_r} \nabla b(\mathbb{R}^n) < 2$.

The proof is therefore complete if we show that $x \mapsto b(x) - y_r \cdot x$ is constant on $\partial \Omega_r$ for some $r \in (0, R)$. We shall do this by use of Proposition 2 after we compute, for suitable r , the $(n - 1/2)$ -dimensional Minkowski content of $\partial \Omega_r$.

Define $h(r) = \mathcal{L}^n(\Omega_r)$, $r \in (0, R)$. Then h is a non-decreasing function and $h(r) \leq \mathcal{L}^n(\Omega_R) < \infty$ for $r \in (0, R)$. The function h is therefore in particular

differentiable almost everywhere; hence we may select $r_0 \in (0, R)$ and an $\varepsilon_0 > 0$ so small that with $m = 2h'(r_0) + 1 < \infty$ we have

$$(3.4) \quad h(r_0 + \varepsilon) - h(r_0 - \varepsilon) < m\varepsilon \quad \text{for } \varepsilon \in (0, \varepsilon_0).$$

Fix $\varepsilon > 0$ with $\omega(\varepsilon) < r_0/2$. Let $x \in \mathcal{U}_\varepsilon(\partial\Omega_{r_0})$, i.e. there exists $x' \in \partial\Omega_{r_0}$ so that $\|x - x'\| < \varepsilon$. Since $\nabla b(x') = y_{r_0} \in \partial B(y_0, r_0)$ and $\|\nabla b(x') - \nabla b(x'')\| \leq \omega(\varepsilon)$ for all $x'' \in B(x', \varepsilon)$, we must have

$$\nabla b(B(x', \varepsilon)) \subset B(y_{r_0}, 2\omega(\varepsilon)) \subset B(y_0, r_0 + 2\omega(\varepsilon)) \setminus B[y_0, r_0 - 2\omega(\varepsilon)].$$

Hence $B(x', \varepsilon)$ does not intersect the boundaries $\partial\Omega_{r_0+2\omega(\varepsilon)}$, $\partial\Omega_{r_0-2\omega(\varepsilon)}$, and because $x' \in \partial\Omega_{r_0} \subset \Omega_{r_0+2\omega(\varepsilon)} \setminus \overline{\Omega_{r_0-2\omega(\varepsilon)}}$ and $B(x', \varepsilon)$ is connected, it follows that $B(x', \varepsilon) \subseteq \Omega_{r_0+2\omega(\varepsilon)} \setminus \overline{\Omega_{r_0-2\omega(\varepsilon)}}$. Consequently,

$$\mathcal{U}_\varepsilon(\partial\Omega_{r_0}) \subseteq \Omega_{r_0+2\omega(\varepsilon)} \setminus \overline{\Omega_{r_0-2\omega(\varepsilon)}}$$

when $\varepsilon > 0$ and $\omega(\varepsilon) < r_0/2$. Now for $\varepsilon > 0$ so small that $\omega(\varepsilon) < \min\{r_0, \varepsilon_0\}/2$ we get in view of (3.4) that $\mathcal{L}^n(\mathcal{U}_\varepsilon(\partial\Omega_{r_0})) \leq 2m\omega(\varepsilon)$, and therefore

$$\mathcal{M}_*^{n-\frac{1}{2}}(\partial\Omega_{r_0}) \leq \liminf_{\varepsilon \searrow 0} \frac{2m\omega(\varepsilon)}{\sqrt{2\varepsilon}} = 0.$$

Finally, we may use Proposition 2 with $s = 1/2$ to conclude that $g(x) = b(x) - y_{r_0} \cdot x$ is constant on the irreducible separator $\partial\Omega_{r_0}$. □

Proof of Proposition 2. By the assumption, C is an irreducible separator between two points x and y . Let Ω and Λ be the components of $\mathbb{R}^n \setminus C$ containing the points x and y , respectively. At least one of them is bounded and we may assume that it is Ω . By Lemma 4 (2), $\partial\Omega = C = \partial\Lambda$.

Let $d(x) = \text{dist}(x, \partial\Omega) = \inf\{\|x - y\| : y \in \partial\Omega\}$ denote the distance from x to the boundary $\partial\Omega$. There exists a function $\delta: \Omega \rightarrow \mathbb{R}$ that is \mathcal{C}^∞ on Ω and has the properties $d(x)/c_0 \leq \delta(x) \leq c_0 d(x)$, $\|\nabla\delta(x)\| \leq 1$ and $|\partial^\alpha\delta(x)| \leq c_{|\alpha|}d(x)^{1-|\alpha|}$ for all $x \in \Omega$ and any multi-index α , where $c_{|\alpha|} \in [1, \infty)$ are constants. We refer to [8], Th. 2, p. 171, for an explicit construction of such a regularized distance δ .

For each $t > 0$, define the auxiliary set $\Omega^{(t)} = \{x \in \Omega : \delta(x) > t\}$. In view of the (classical) Morse-Sard theorem, $\mathcal{L}^1(\delta(\{x \in \Omega : \nabla\delta(x) = 0\})) = 0$, so $\nabla\delta \neq 0$ on $\partial\Omega^{(t)}$ for almost all t . It follows that $\Omega^{(t)}$ has a \mathcal{C}^1 boundary with exterior unit normal $N = -\nabla\delta/\|\nabla\delta\|$ for almost all t .

Obviously, $\partial\Omega^{(t)} \subseteq \mathcal{U}_{2c_0t}(\partial\Omega)$. By use of the coarea formula we estimate

$$\mathcal{L}^n(\mathcal{U}_\varepsilon(\partial\Omega)) \geq \frac{1}{nc_1} \int_0^{\frac{\varepsilon}{2c_0}} \mathcal{H}^{n-1}(\partial\Omega^{(t)}) dt,$$

whereby it follows that we can select $t = t(\varepsilon) \in (0, \frac{\varepsilon}{2c_0})$, such that $\Omega^{(t)}$ has a \mathcal{C}^1 boundary and

$$(3.5) \quad \mathcal{H}^{n-1}(\partial\Omega^{(t)}) \leq \frac{2nc_0c_1 \mathcal{L}^n(\mathcal{U}_\varepsilon(\partial\Omega))}{\varepsilon}.$$

We now consider the vector field

$$V = \begin{cases} \nabla f & \text{in } \Omega, \\ 0 & \text{else.} \end{cases}$$

Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ denote a smooth and compactly supported test function and compute for distinct i, j from $\{1, \dots, n\}$:

$$\langle V_{,j}^i - V_{,i}^j, \varphi \rangle = \int_{\Omega} (f_{,j}\varphi_{,i} - f_{,i}\varphi_{,j}).$$

By the dominated convergence theorem, $\int_{\Omega^{(t)}} (f_{,j}\varphi_{,i} - f_{,i}\varphi_{,j}) \rightarrow \int_{\Omega} (f_{,j}\varphi_{,i} - f_{,i}\varphi_{,j})$ as $t \searrow 0$. For those t for which $\Omega^{(t)}$ has a \mathcal{C}^1 boundary we get, using a standard mollifier $\{\rho_s\}$ and integration by parts,

$$\begin{aligned} \int_{\Omega^{(t)}} (f_{,i}\varphi_{,j} - f_{,j}\varphi_{,i}) &= \lim_{s \searrow 0} \int_{\Omega^{(t)}} \operatorname{div}(\varphi(\rho_s \star f_{,i})e_j - \varphi(\rho_s \star f_{,j})e_i) \\ &= \lim_{s \searrow 0} \int_{\partial\Omega^{(t)}} (\varphi(\rho_s \star f_{,i})N^j - \varphi(\rho_s \star f_{,j})N^i) \\ &= \int_{\partial\Omega^{(t)}} \varphi(f_{,i}N^j - f_{,j}N^i), \end{aligned}$$

where $N = (N^1, \dots, N^n)$ denotes the exterior unit normal to $\partial\Omega^{(t)}$. This is in particular true if we take $t = t(\varepsilon)$. For this choice of t we estimate the last term using (3.5) (denoting $c = 2nc_0c_1$):

$$\left| \int_{\Omega^{(t)}} (f_{,j}\varphi_{,i} - f_{,i}\varphi_{,j}) \right| \leq \max |\varphi| \omega(2t) \mathcal{H}^{n-1}(\partial\Omega^{(t)}) \leq \max |\varphi| \frac{c \omega(\varepsilon) \mathcal{L}^n(\mathcal{U}_\varepsilon(\partial\Omega))}{\varepsilon},$$

where ω denotes the (nondecreasing) modulus of continuity of ∇f . By the assumption (either one of the two versions), the right-hand side $R(\varepsilon)$ has $\liminf_{\varepsilon \searrow 0} R(\varepsilon) = 0$, so we have shown that $\langle V_{,j}^i - V_{,i}^j, \varphi \rangle = 0$. It follows that V is curl-free on \mathbb{R}^n , and hence by virtue of Lemma 6, that f is constant on the irreducible separator $\partial\Omega$. The proof is finished. \square

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