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ON THE KERNEL OF THE MAGNUS REPRESENTATION OF THE TORELLI GROUP

MASAAKI SUZUKI

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ABSTRACT. From our previous paper, it is known that the Magnus representation of the Torelli group is not faithful. In this paper, we characterize the kernel of its representation for a certain kind of elements.

1. INTRODUCTION

The linearity of the mapping class group of a surface of genus $g \geq 2$ has been one of the well-known open problems. A group is called linear if it admits a finite-dimensional faithful representation. Recently, Korkmaz [K], Bigelow and Budney [B-B] proved that the mapping class group of a closed surface of genus 2 is linear. However, it still remains open for higher genera. Then it is significant to discuss whether some representations of the mapping class groups are faithful and to determine the kernel.

Let $\Sigma_{g,1}$ be an oriented surface obtained from a closed surface of genus g by removing an open disk. We denote by $\mathcal{M}_{g,1}$ the mapping class group of $\Sigma_{g,1}$ relative to the boundary, that is, the group of path components of the group of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ which restrict to the identity on the boundary. Let $\mathcal{I}_{g,1}$ be the Torelli group of $\Sigma_{g,1}$, namely the normal subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which act trivially on the first homology group of $\Sigma_{g,1}$.

The Magnus representations of various subgroups of the automorphism group of a free group are defined by making use of the Fox derivation [F]; see [Bir] for details. The Magnus representation for the Torelli group

$$r_1 : \mathcal{I}_{g,1} \to GL(2g; \mathbb{Z}[H])$$

was introduced in [M1], where $H = H_1(\Sigma_{g,1}; \mathbb{Z})$. From our previous paper [S1], the representation r_1 is not faithful for $g \geq 2$. Thus it makes sense to study the kernel of r_1. In this paper, we characterize the kernel of r_1 for the commutator of two BSCC maps, where the Dehn twist along a bounding simple closed curve is called a BSCC map. The following is one of the main results of this paper.

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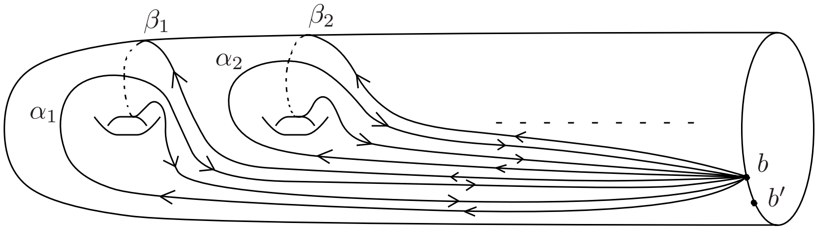

FIGURE 1. Generators of Γ_0 and base points b, b'

Corollary 4.4. *The commutator of two BSCC maps φ_1, φ_2 belongs to the kernel of r_1 if and only if the characteristic polynomial of the Magnus matrix of the product $\varphi_1\varphi_2$ is trivial. Here the Magnus matrix means the image of r_1 for a mapping class.*

In Section 2, we will recall the definitions of the Magnus representation of the mapping class group and the Torelli group.

In Section 3, we will give a certain pairing for two curves on $\Sigma_{g,1}$ and show the relationship with the pairing and the kernel of r_1.

In Section 4, we will introduce another pairing for two curves on $\Sigma_{g,1}$ in order to obtain additional information of the kernel of r_1.

2. DEFINITION OF THE MAGNUS REPRESENTATION OF THE TORELLI GROUP

In this section, we recall the definitions of the Magnus representation for the mapping class group and the Torelli group from [M1], [S1] and [S4].

Let $\mathbb{Z}[\Gamma_0]$ be the integral group ring of $\Gamma_0 = \pi_1(\Sigma_{g,1}, b)$. We fix a system of generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of the free group Γ_0 as shown in Figure 1. Let us simply write $\gamma_1, \ldots, \gamma_{2g}$ for them.

Definition 2.1. We call the mapping

$$\begin{array}{cccc} r: & \mathcal{M}_{g,1} & \longrightarrow & GL(2g;\mathbb{Z}[\Gamma_0]) \\ & \varphi & \longmapsto & \left(\overline{\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i}}\right)_{i,j} \end{array}$$

the Magnus representation for the mapping class group, where $\frac{\partial}{\partial \gamma_i} : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ is the Fox derivation and $^{-} : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ is the antiautomorphism induced by the mapping $\gamma \mapsto \gamma^{-1}$.

This mapping is not a homomorphism but a crossed homomorphism.

Proposition 2.2 (Morita [M1]). *For any two elements $\varphi, \psi \in \mathcal{M}_{g,1}$, we have*

$$r(\varphi\psi) = r(\varphi) \cdot {}^{\varphi}r(\psi)$$

where ${}^{\varphi}r(\psi)$ denotes the matrix obtained from $r(\psi)$ by applying the automorphism $\varphi : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ on each entry.

It follows that if this mapping r is restricted to the Torelli group $\mathcal{I}_{g,1}$ and if the coefficients are reduced to $\mathbb{Z}[H]$, then we obtain the following genuine representation:

$$r_1 : \mathcal{I}_{g,1} \longrightarrow GL(2g;\mathbb{Z}[H]).$$

Here the reduction is induced by the abelianization $\mathfrak{a} : \Gamma_0 \to H$, and r_1 denotes the composition $r^{\mathfrak{a}}$ of the mapping r by the abelianization $\mathfrak{a}$. We call r_1 the Magnus representation of the Torelli group.

We have another definition of this representation (see [S4]). Let $p : \widehat{\Sigma} \to \Sigma_{g,1}$ be the universal abelian covering, that is, the regular covering corresponding to the abelianization. An arbitrary element of the Torelli group induces an automorphism of $H_1(\widehat{\Sigma}, p^{-1}(b); \mathbb{Z})$ as a free $\mathbb{Z}[H]$-module of rank $2g$. Therefore we get the following representation:

$$r_1 : \mathcal{I}_{g,1} \longrightarrow GL(2g; \mathbb{Z}[H]).$$

3. A HIGHER INTERSECTION NUMBER OF TWO LOOPS AND THE KERNEL OF r_1

The non-triviality of the kernel of r_1 for $g \geq 2$ is proved in [S1]. Moreover, it is proved in [S2] that none of the terms of the lower central series of $\mathcal{I}_{g,1}$ is contained in the kernel. Then it is interesting to characterize and determine the kernel.

First, we define a pairing of two loops on $\Sigma_{g,1}$. This pairing is useful to give information about the kernel of r_1. Choose base points b and b' on $\partial\Sigma_{g,1}$ as depicted in Figure 1. Fix a point $\hat{b}$, which is a lift of b to the universal abelian covering $\widehat{\Sigma}$. The point $\hat{b}'$, which is a lift of b', is determined as follows. We denote by bb' the path on $\partial\Sigma_{g,1}$ from b to b' with an orientation opposite to that of $\Sigma_{g,1}$. Let $\widehat{bb'}$ be the lift of bb' to $\widehat{\Sigma}$ starting at $\hat{b}$. Then we set $\hat{b}'$ for the endpoint of $\widehat{bb'}$.

Definition 3.1. Let c_1, c_2 be two oriented loops on $\Sigma_{g,1}$ based at b, b' respectively. We define

$$\langle c_1, c_2 \rangle_H = \sum_{h \in H} (h\hat{c}_1, \hat{c}_2)\, h.$$

Here $\hat{c}_1$ is the lift of c_1 to $\widehat{\Sigma}$ starting at $\hat{b}$, $\hat{c}_2$ is the lift of c_2 to $\widehat{\Sigma}$ starting at $\hat{b}'$ and $(\cdot,\cdot)$ denotes the algebraic intersection number of two arcs. We write $h\hat{c}_1$ for the arc obtained by an element h of the covering transformation group H acting on $\hat{c}_1$.

Suppose that c_1 and c_2 are bounding simple closed curves on $\Sigma_{g,1}$, where bounding means 0-homologous. If we regard c_1, c_2 as *oriented* loops *based at* b, b' respectively, then we can compute the pairing $\langle c_1, c_2 \rangle_H$ up to multiplication by ± 1 and by an element of H. That is to say, the pairing $\langle c_1, c_2 \rangle_H$ depends on how c_1, c_2 are represented as loops. However, whether $\langle c_1, c_2 \rangle_H$ is zero or not does not depend on the choices, and we will use this fact.

Proposition 3.2. *Suppose that c_1 and c_2 are two bounding simple closed curves on $\Sigma_{g,1}$, and φ_1 and φ_2 the Dehn twists along c_1 and c_2 respectively. If $\langle c_1, c_2 \rangle_H = 0$, then $[\varphi_1, \varphi_2] \in \ker r_1$.*

Proof. We denote by $\widehat{\varphi}_*$ the automorphism of $H_1(\widehat{\Sigma}, p^{-1}(b); \mathbb{Z})$ induced by a diffeomorphism φ of $\Sigma_{g,1}$ representing an element of $\mathcal{M}_{g,1}$. Let $\hat{c}_1, \hat{c}_2$ be lifts of c_1, c_2 to $\widehat{\Sigma}$ respectively. Then $[\hat{c}_1], [\hat{c}_2]$ belong to $H_1(\widehat{\Sigma}, p^{-1}(b); \mathbb{Z})$. Since $\langle c_1, c_2 \rangle_H = 0$, the intersection number $(\hat{c}_1, \hat{c}_2)$ equals zero. For a loop c based at b, we denote by $\hat{c}$ a lift of c to $\widehat{\Sigma}$. Then we have an element $[\hat{c}]$ of $H_1(\widehat{\Sigma}, p^{-1}(b); \mathbb{Z})$ and

$$\widehat{\varphi}_{i*}([\hat{c}]) = [\hat{c}] + (\hat{c}_i, \hat{c})[\hat{c}_i], \qquad i = 1, 2.$$

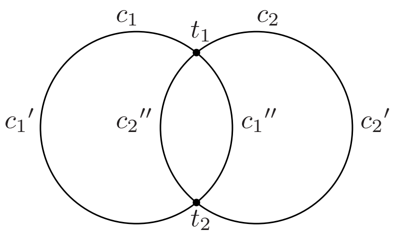

FIGURE 2. Geometric intersection number 2

Then we obtain

$$\begin{aligned}\widehat{\varphi_1}_* \circ \widehat{\varphi_2}_*([\hat{c}]) &= \widehat{\varphi_1}_*([\hat{c}] + (\hat{c}_2, \hat{c})[\hat{c}_2]) \\ &= [\hat{c}] + (\hat{c}_1, \hat{c})[\hat{c}_1] + (\hat{c}_2, \hat{c})[\hat{c}_2] \\ &= \widehat{\varphi_2}_* \circ \widehat{\varphi_1}_*([\hat{c}]).\end{aligned}$$

It follows that $\widehat{\varphi_1}_*$ commutes with $\widehat{\varphi_2}_*$, and this completes the proof. $\square$

Corollary 3.3. *Suppose that c_1 and c_2 are two bounding simple closed curves. If the geometric intersection number of c_1 and c_2 is two, then $[\varphi_1, \varphi_2] \in \ker r_1$.*

Proof. Let t_1, t_2 be the intersection points. Also, let ${c_i}'$ be the subarcs of c_i from t_1 to t_2, and the subarcs ${c_i}''$ from t_2 to t_1; see Figure 2. The number of the terms of $\langle c_1, c_2 \rangle_H$ is two. Each value of the terms is decided by the value at t_1 and t_2 respectively. We consider loops ${c_1}'{c_2}'', {c_1}'{{c_2}'}^{-1}, {c_2}'{c_1}''$ and ${c_2}''{{c_1}''}^{-1}$, where c^{-1} is the same arc as c with the opposite orientation. All of these are bounding simple closed curves. It follows that the value at t_1 is -1 times that of t_2. Then $\langle c_1, c_2 \rangle_H = 0$. By Proposition 3.2, this completes the proof. $\square$

4. ANOTHER PAIRING OF BOUNDING SIMPLE CLOSED CURVES AND THE KERNEL OF r_1

We define another pairing for two bounding simple closed curves:

$$\langle\langle c_1, c_2 \rangle\rangle = -\langle c_1, c_2 \rangle_H \cdot \langle c_2, c_1 \rangle_H.$$

The pairing $\langle \cdot, \cdot \rangle_H$ depends on the manner of assigning orientations and attaching basepoints to two bounding simple closed curves. However, the way in which this is done does not have an effect on the pairing $\langle\langle \cdot, \cdot \rangle\rangle$. That is, we obtain the following lemma.

Lemma 4.1. *Let c_1, c_2 be two bounding simple closed curves on $\Sigma_{g,1}$. Then we have*

(1) $\langle\langle c_1, c_2 \rangle\rangle = \langle\langle c_2, c_1 \rangle\rangle$,
(2) $\langle\langle \gamma c_1 \gamma^{-1}, c_2 \rangle\rangle = \langle\langle c_1, c_2 \rangle\rangle$,
(3) $\langle\langle {c_1}^{-1}, c_2 \rangle\rangle = \langle\langle c_1, c_2 \rangle\rangle$,

where γ is a loop based at b and ${c_1}^{-1}$ is the same loop as c_1 with the opposite orientation.

We recall the following before proving Lemma 4.1.

Theorem 4.2 (Morita [M1]). *There exists a matrix* $\widetilde{J}$ *such that for any element* $f \in \mathcal{M}_{g,1}$ *the following equality holds:*

$$\overline{{}^t r(f)}\, \widetilde{J}\, r(f) = {}^f \widetilde{J}.$$

This means that the Magnus representation of the mapping class group is symplectic in a sense. The explicit expression for $\widetilde{J}$ can be found in [M1] and [S4] and is not included in this paper.

In this section, $\overrightarrow{c}$ denotes ${}^t\left(\mathfrak{a}\left(\frac{\partial c}{\partial \gamma_1}\right), \ldots, \mathfrak{a}\left(\frac{\partial c}{\partial \gamma_{2g}}\right)\right)$.

Proof of Lemma 4.1. (1) This is obvious from the definition of the pairing $\langle\langle \cdot, \cdot \rangle\rangle$.

(2) We can consider γ as an element of Γ_0 naturally. Because

$$\begin{aligned}
\mathfrak{a}\left(\frac{\partial \gamma c_1 \gamma^{-1}}{\partial \gamma_i}\right) &= \mathfrak{a}\left(\frac{\partial \gamma}{\partial \gamma_i}\right) + \mathfrak{a}(\gamma)\mathfrak{a}\left(\frac{\partial c_1}{\partial \gamma_i}\right) + \mathfrak{a}(\gamma)\mathfrak{a}(c_1)\mathfrak{a}\left(\frac{\partial \gamma^{-1}}{\partial \gamma_i}\right) \\
&= \mathfrak{a}(\gamma)\mathfrak{a}\left(\frac{\partial c_1}{\partial \gamma_i}\right),
\end{aligned}$$

then we get

$$\overrightarrow{\gamma c_1 \gamma^{-1}} = \mathfrak{a}(\gamma)\overrightarrow{c_1}.$$

By [S4, Lemma 4.4], we have $\langle c_1, c_2 \rangle_H = -{}^t\overrightarrow{c_2}\, J_1\, \overline{\overrightarrow{c_1}}$, where $\mathfrak{a}(\widetilde{J}) = J_1$. Therefore

$$\begin{aligned}
\langle\langle \gamma c_1 \gamma^{-1}, c_2 \rangle\rangle &= -{}^t\overrightarrow{c_2}\, J_1\, \overline{\mathfrak{a}(\gamma)}\, \overline{\overrightarrow{c_1}}{}^t\mathfrak{a}(\gamma)\overrightarrow{c_1}\, J_1\, \overline{\overrightarrow{c_2}} \\
&= -{}^t\overrightarrow{c_2}\, J_1\, \overline{\overrightarrow{c_1}}{}^t\overrightarrow{c_1}\, J_1\, \overline{\overrightarrow{c_2}} \\
&= \langle\langle c_1, c_2 \rangle\rangle.
\end{aligned}$$

(3) Since

$$\mathfrak{a}\left(\frac{\partial {c_1}^{-1}}{\partial \gamma_i}\right) = \mathfrak{a}({c_1}^{-1})\mathfrak{a}\left(\frac{\partial c_1}{\partial \gamma_i}\right) = -\mathfrak{a}\left(\frac{\partial c_1}{\partial \gamma_i}\right),$$

we deduce this lemma. □

The relation between the pairing $\langle\langle \cdot, \cdot \rangle\rangle$ and the Magnus representation r_1 of the Torelli group can be expressed as the following formula.

Theorem 4.3. *Suppose that* c_1 *and* c_2 *are two bounding simple closed curves on* $\Sigma_{g,1}$. *Then we obtain*

$$\langle\langle c_1, c_2 \rangle\rangle = \operatorname{tr}(I_{2g} - r_1(\varphi_1\varphi_2)) = 2g - \operatorname{tr}(r_1(\varphi_1\varphi_2))$$

where φ_1, φ_2 *are the Dehn twists along* c_1, c_2 *respectively.*

Proof. Any bounding simple closed curve can be written as $f(d_k)$ for a certain element $f \in \mathcal{M}_{g,1}$ and for a bounding simple closed curve d_k, which is shown in Figure 3. First, we will prove the statement in the case $c_1 = f(d_i), c_2 = d_j$. That is, we will consider the case $\varphi_1 = f\psi_i f^{-1}, \varphi_2 = \psi_j$, where ψ_k is the Dehn twist along d_k. By Lemma 4.1, we can assume that c_1 and c_2 have expressions such as

$$c_1 = f([\beta_i, \alpha_i] \cdots [\beta_1, \alpha_1]), \quad c_2 = [\beta_j, \alpha_j] \cdots [\beta_1, \alpha_1].$$

We see from [S3] that

$$r_1(\psi_k) = I_{2g} + a_k b_k. \tag{4.1}$$

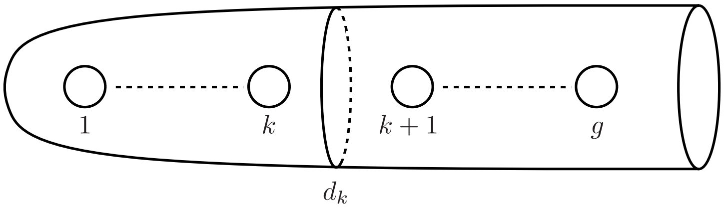

FIGURE 3. Bounding simple closed curve

Here

$$\begin{aligned} a_k &= {}^t(\bar{y}_1 - 1 \ \cdots \ \bar{y}_k - 1 \underbrace{0 \ \cdots \ 0}_{g-k \text{ times}} \ 1 - \bar{x}_1 \ \cdots \ 1 - \bar{x}_k \underbrace{0 \ \cdots \ 0}_{g-k\text{times}}), \\ b_k &= (1 - \bar{x}_1 \ \cdots \ 1 - \bar{x}_k \underbrace{0 \ \cdots \ 0}_{g-k \text{ times}} \ 1 - \bar{y}_1 \ \cdots \ 1 - \bar{y}_k \underbrace{0 \ \cdots \ 0}_{g-k \text{ times}}), \end{aligned}$$

and x_i, y_i are the homology classes of α_i, β_i respectively. Note that $\operatorname{tr}(a_k b_k) = b_k a_k = 0$. We denote by $r^{\mathfrak{a}}$ the composition of the mapping r by the abelianization $\mathfrak{a} : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[H]$. If we consider elements of the Torelli group, we write r_1 for $r^{\mathfrak{a}}$ as before. By the abelianization, Theorem 4.2 can be stated as

$$\overline{{}^t r^{\mathfrak{a}}(f)} \, J_1 \, r^{\mathfrak{a}}(f) = {}^f J_1. \tag{4.2}$$

The following equalities can be checked easily:

$$b_k {J_1}^{-1} = \overline{{}^t a_k}, \quad \overline{{}^t a_k} J_1 = b_k. \tag{4.3}$$

We will compute $\overline{\overrightarrow{c_1}}$ by an explicit calculation. Since

$$\begin{aligned} &\mathfrak{a}\left(\frac{\partial c_1}{\partial \gamma_l}\right) \\ &= \sum_{k=1}^{i} \mathfrak{a}\left(\frac{\partial f([\beta_k, \alpha_k])}{\partial \gamma_l}\right) \\ &= \sum_{k=1}^{i} \left\{ \mathfrak{a}\left(\frac{\partial f(\beta_k)}{\partial \gamma_l}\right) + \mathfrak{a}(f(\beta_k)) \cdot \mathfrak{a}\left(\frac{\partial f(\alpha_k)}{\partial \gamma_l}\right) \right. \\ &\qquad \left. + \mathfrak{a}(f(\beta_k)) \cdot \mathfrak{a}(f(\alpha_k)) \cdot \mathfrak{a}\left(\frac{\partial f({\beta_k}^{-1})}{\partial \gamma_l}\right) + \mathfrak{a}(f(\alpha_k)) \cdot \mathfrak{a}\left(\frac{\partial f({\alpha_k}^{-1})}{\partial \gamma_l}\right) \right\} \\ &= \sum_{k=1}^{i} \left\{ \mathfrak{a}\left(\frac{\partial f(\beta_k)}{\partial \gamma_l}\right) + f(y_k) \cdot \mathfrak{a}\left(\frac{\partial f(\alpha_k)}{\partial \gamma_l}\right) \right. \\ &\qquad \left. - f(x_k) \cdot \mathfrak{a}\left(\frac{\partial f(\beta_k)}{\partial \gamma_l}\right) - \mathfrak{a}\left(\frac{\partial f(\alpha_k)}{\partial \gamma_l}\right) \right\} \\ &= \sum_{k=1}^{i} \left\{ (f(y_k) - 1) \cdot \mathfrak{a}\left(\frac{\partial f(\alpha_k)}{\partial \gamma_l}\right) + (1 - f(x_k)) \cdot \mathfrak{a}\left(\frac{\partial f(\beta_k)}{\partial \gamma_l}\right) \right\}, \end{aligned}$$

we obtain

$$\overline{\overrightarrow{c_1}} = r^{\mathfrak{a}}(f) \cdot {}^f a_i. \tag{4.4}$$

Similarly, $\overrightarrow{c_2} = \overline{a_j}$. Therefore,

$$\begin{aligned}
&\operatorname{tr}(I_{2g} - r_1(\varphi_1\varphi_2))\\
&\quad = \operatorname{tr}(I_{2g} - r^{\mathfrak{a}}(f) \cdot {}^{f}r_1(\psi_i) \cdot r^{\mathfrak{a}}(f)^{-1} \cdot r_1(\psi_j))\\
&\quad = \operatorname{tr}(I_{2g} - r^{\mathfrak{a}}(f) \cdot (I_{2g} + {}^{f}a_i\, {}^{f}b_i) \cdot r^{\mathfrak{a}}(f)^{-1} \cdot (I_{2g} + a_j b_j)) \qquad \text{by (4.1)}\\
&\quad = \operatorname{tr}(-r^{\mathfrak{a}}(f) \cdot {}^{f}a_i\, {}^{f}b_i \cdot r^{\mathfrak{a}}(f)^{-1} - a_j b_j - r^{\mathfrak{a}}(f) \cdot {}^{f}a_i\, {}^{f}b_i \cdot r^{\mathfrak{a}}(f)^{-1} \cdot a_j b_j)\\
&\quad = -\operatorname{tr}(r^{\mathfrak{a}}(f) \cdot {}^{f}a_i\, {}^{f}b_i \cdot r^{\mathfrak{a}}(f)^{-1} \cdot a_j b_j)\\
&\quad = -\operatorname{tr}(r^{\mathfrak{a}}(f) \cdot {}^{f}a_i\, {}^{f}b_i \cdot {}^{f}J_1{}^{-1} \cdot \overline{{}^{t}r^{\mathfrak{a}}(f)} \cdot J_1 \cdot a_j b_j\,) \qquad \text{by (4.2)}\\
&\quad = -\operatorname{tr}(r^{\mathfrak{a}}(f) \cdot {}^{f}a_i\, \overline{{}^{tf}a_i} \cdot \overline{{}^{t}r^{\mathfrak{a}}(f)} \cdot J_1 \cdot a_j\, \overline{{}^{t}a_j} \cdot J_1) \qquad \text{by (4.3)}\\
&\quad = -\overline{{}^{tf}a_i} \cdot \overline{{}^{t}r^{\mathfrak{a}}(f)} \cdot J_1 \cdot a_j \cdot \operatorname{tr}(r^{\mathfrak{a}}(f) \cdot {}^{f}a_i\, \overline{{}^{t}a_j} \cdot J_1)\\
&\quad = -{}^{t}\overrightarrow{c_1}\, J_1\, \overline{\overrightarrow{c_2}} \cdot \overline{{}^{t}a_j}\, J_1\, r^{\mathfrak{a}}(f) \cdot {}^{f}a_i \qquad \text{by (4.4)}\\
&\quad = -{}^{t}\overrightarrow{c_1}\, J_1\, \overline{\overrightarrow{c_2}} \cdot {}^{t}\overrightarrow{c_2}\, J_1\, \overline{\overrightarrow{c_1}}\\
&\quad = \langle\langle c_2, c_1 \rangle\rangle = \langle\langle c_1, c_2 \rangle\rangle.
\end{aligned}$$

Next, we consider the general case $\varphi_1\varphi_2 = g\, f\psi_i f^{-1}\, \psi_j\, g^{-1}$ for $g \in \mathcal{M}_{g,1}$. The pairing $\langle\langle \cdot, \cdot \rangle\rangle$ is $\mathcal{M}_{g,1}$-equivariant by [S4, Lemma 4.3]; that is,

$$\langle\langle g(c_1), g(c_2) \rangle\rangle = g(\langle\langle c_1, c_2 \rangle\rangle).$$

Moreover, we see from [S3, Proposition 3.2] that

$$\operatorname{tr}(r_1(g\varphi_1\varphi_2 g^{-1})) = g(\operatorname{tr}(r_1(\varphi_1\varphi_2))).$$

This means that $\operatorname{tr}(r_1(\cdot))$ is also $\mathcal{M}_{g,1}$-equivariant. Therefore this completes the proof. □

The Dehn twist along a bounding simple closed curve is called a BSCC map. From our previous paper [S3], it is known that any BSCC map φ does not lie in the kernel of r_1, and the characteristic polynomial of the Magnus matrix of φ is trivial:

$$\det(\lambda I_{2g} - r_1(\varphi)) = (\lambda - 1)^{2g}.$$

It follows that $\mathcal{K}_{g,1}$ is not contained in the kernel of r_1, where $\mathcal{K}_{g,1}$ denotes the subgroup generated by the BSCC maps. We remark that the characteristic polynomial of the Magnus matrix on $\mathcal{K}_{g,1}$ is not always trivial (see [S4] for details).

Theorem 4.3 gives a characterization of the kernel of r_1 for the commutator of two BSCC maps.

Corollary 4.4. *The commutator of two BSCC maps φ_1, φ_2 belongs to the kernel of r_1 if and only if the characteristic polynomial of the Magnus matrix of the product $\varphi_1\varphi_2$ is trivial. Here the Magnus matrix means the image of r_1 for a mapping class.*

Proof. In general, if the characteristic polynomials of two matrices A, B are trivial and A commutes with B, then the characteristic polynomial of AB is also trivial.

Suppose that the commutator of two BSCC maps φ_1, φ_2 belongs to the kernel of r_1, that is, $r_1(\varphi_1)$ commutes with $r_1(\varphi_2)$. Because the characteristic polynomial of the Magnus matrix for any BSCC map is trivial, we get

$$\det(\lambda I_{2g} - r_1(\varphi_1\varphi_2)) = (\lambda - 1)^{2g}.$$

Conversely, suppose that the characteristic polynomial is trivial. Then we have

$$-\operatorname{tr}(r_1(\varphi_1\varphi_2)) = -2g.$$

By Theorem 4.3, we conclude that $\langle\langle c_1, c_2\rangle\rangle = 0$. This means that $\langle c_1, c_2\rangle_H = 0$ or $\langle c_2, c_1\rangle_H = 0$, because $\mathbb{Z}[H]$ is an integral domain. By virtue of Proposition 3.2, $\langle c_1, c_2\rangle_H = 0$ gives $[\varphi_1, \varphi_2] \in \ker r_1$ and $\langle c_2, c_1\rangle_H = 0$ gives $[\varphi_2, \varphi_1] \in \ker r_1$. This completes the proof. □

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GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO, TOKYO 153-8914, JAPAN
E-mail address: `macky@ms.u-tokyo.ac.jp`