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TENSOR PRODUCTS OF σ-WEAKLY CLOSED NEST ALGEBRA SUBMODULES

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Abstract. In this paper we prove that for any unital σ-weakly closed algebra $\mathcal{A}$ which is σ-weakly generated by finite-rank operators in $\mathcal{A}$, every σ-weakly closed $\mathcal{A}$-submodule has *Property* S_σ. In the case of nest algebras, if $\mathcal{L}_1, \cdots, \mathcal{L}_n$ are nests, we obtain the following n-fold tensor product formula:

$$\mathcal{U}_{\phi_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\phi_n}=\mathcal{U}_{\phi_1\otimes\cdots\otimes\phi_n},$$

where each $\mathcal{U}_{\phi_i}$ is the σ-weakly closed $\mathrm{Alg}\mathcal{L}_i$-submodule determined by an order homomorphism ϕ_i from $\mathcal{L}_i$ into itself.

1. Introduction

One of the central results in the theory of tensor products of von Neumann algebras is Tomita's commutation formula:

$$\mathcal{M}'\overline{\otimes}\mathcal{N}'=(\mathcal{M}\overline{\otimes}\mathcal{N})', \tag{1}$$

where $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras. It was observed in [2] that if we let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the projection lattices of $\mathcal{M}$ and $\mathcal{N}$ respectively, then (1) can be rewritten as

$$\mathrm{Alg}\mathcal{L}_1\overline{\otimes}\,\mathrm{Alg}\,\mathcal{L}_2=\mathrm{Alg}(\mathcal{L}_1\otimes\mathcal{L}_2). \tag{2}$$

This version of Tomita's theorem makes sense for any pair of reflexive algebras $\mathrm{Alg}\,\mathcal{L}_1$ and $\mathrm{Alg}\,\mathcal{L}_2$. It remains a deep open question whether the tensor product formula (2) is valid for general reflexive algebras, or even general CSL algebras. However, (2) has been verified in a number of special cases ([2], [4], [5], [6], [7]). In particular, it is known that if $\mathcal{L}_1$ is a commutative subspace lattice that is either completely distributive [8] or of finite width [4], then (2) is valid for $\mathcal{L}_1$ and any subspace lattice $\mathcal{L}_2$.

The main purpose of this paper is to study tensor products of σ-weakly closed submodules of some reflexive algebras (in particular, of nest algebras). Section 1 of this paper is devoted to notation and preliminaries. In Section 2, we make use of slice maps to show that if $\mathcal{A}$ is a σ-weakly closed algebra which is σ-weakly generated by finite-rank operators in $\mathcal{A}$, then every σ-weakly closed $\mathcal{A}$-submodule has *Property* S_σ. As a corollary, we obtain $\mathcal{U}_{\tau_1}\overline{\otimes}\mathcal{U}_{\tau_2}=\mathcal{U}_{\tau_1\otimes\tau_2}$, where each $\mathcal{U}_{\tau_i}$ is

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a σ-weakly closed $\operatorname{Alg}\mathcal{L}_i$-submodule and $\mathcal{L}_i$ is a nest. However, the 2-fold tensor product formula cannot be generalized to the n-fold formula by induction (see the beginning of Section 3). So in Section 3, we use another method to prove the n-fold tensor product formula $\mathcal{U}_{\phi_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\phi_n}=\mathcal{U}_{\phi_1\otimes\cdots\otimes\phi_n}$, where each $\mathcal{U}_{\phi_i}$ is a σ-weakly closed $\operatorname{Alg}\mathcal{L}_i$-submodule and $\mathcal{L}_i$ is a nest. The key to this proof is [3, Theorem 2] and [2, Proposition 2.4].

In this paper, all Hilbert spaces will be separable. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$ and $\mathcal{F}(\mathcal{H})$ be the set of finite-rank operators on $\mathcal{H}$. A sublattice $\mathcal{L}$ of the projection lattice of $\mathcal{B}(\mathcal{H})$ is said to be a subspace lattice if it contains 0 and I and is strongly closed, where we identify projections with their ranges. If the elements of $\mathcal{L}$ pairwise commute, $\mathcal{L}$ is a commutative subspace lattice (CSL). A nest is a totally ordered subspace lattice. If $\mathcal{L}$ is a subspace lattice, $\operatorname{Alg}\mathcal{L}$ denotes the set of operators in $\mathcal{B}(\mathcal{H})$ that leave the elements of $\mathcal{L}$ invariant. Note that $\operatorname{Alg}\mathcal{L}$ is a σ-weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$. If $\mathcal{L}$ is a CSL, $\operatorname{Alg}\mathcal{L}$ is said to be a CSL algebra. If $\mathcal{L}$ is a nest, $\operatorname{Alg}\mathcal{L}$ is said to be a nest algebra.

If $\mathcal{A}$ is a subset of $\mathcal{B}(\mathcal{H})$, then $\operatorname{Lat}\mathcal{A}$, the set of projections left invariant by each element of $\mathcal{A}$, is a subspace lattice. A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is reflexive if $\mathcal{A}=\operatorname{Alg}\operatorname{Lat}\mathcal{A}$. The reflexive algebras are precisely the algebras of the form $\operatorname{Alg}\mathcal{L}$, where $\mathcal{L}$ is a subspace lattice. If $\mathcal{L}_i\subseteq\mathcal{B}(\mathcal{H}_i)$ $(i=1,\cdots,n)$ are subspace lattices, $\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n$ is the subspace lattice in $\mathcal{B}(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n)$ generated by $\{P_1\otimes\cdots\otimes P_n : P_i\in\mathcal{L}_i, i=1,\cdots,n\}$. If $\mathcal{S}_i\subseteq\mathcal{B}(\mathcal{H}_i)$ $(i=1,\cdots,n)$ are σ-weakly closed subspaces, then $\mathcal{S}_1\overline{\otimes}\cdots\overline{\otimes}\mathcal{S}_n$ denotes the σ-weakly closed linear span of $\{S_1\otimes\cdots\otimes S_n : S_i\in\mathcal{S}_i\}$ in $\mathcal{B}(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n)$.

The main technical tool in Section 2 is the use of slice maps. Slice maps were introduced by Tomiyama in [11] and have been used extensively in the study of tensor products of C^*-algebras and tensor products of von Neumann algebras. We recall some definitions and results from [7] and refer the reader to [7] for further results and motivation. If $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras, and ϕ is in the predual $\mathcal{M}_*$ of $\mathcal{M}$, then the right slice map R_ϕ is the unique σ-weakly continuous linear map from $\mathcal{M}\overline{\otimes}\mathcal{N}\to\mathcal{N}$ such that

$$\langle X,\phi\otimes\psi\rangle=\langle R_\phi(X),\psi\rangle,\qquad \forall X\in\mathcal{M}\overline{\otimes}\mathcal{N},\psi\in\mathcal{N}_*.$$

If $X=A\otimes B$ $(A\in\mathcal{M},B\in\mathcal{N})$, then $R_\phi(X)=\langle A,\phi\rangle B$. The left slice map $L_\psi:\mathcal{M}\overline{\otimes}\mathcal{N}\to\mathcal{M},\psi\in\mathcal{N}_*$, is similarly defined. If $\mathcal{S}\subseteq\mathcal{M}$ and $\mathcal{T}\subseteq\mathcal{N}$ are σ-weakly closed subspaces, let

$$F(\mathcal{S},\mathcal{T})=\{X\in\mathcal{M}\overline{\otimes}\mathcal{N} : R_\phi(X)\in\mathcal{T}\,\text{and}\,L_\psi(X)\in\mathcal{S},\quad\forall\phi\in\mathcal{M}_*,\psi\in\mathcal{N}_*\}.$$

As noted in [7], we can replace $\mathcal{M}$ by $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{N}$ by $\mathcal{B}(\mathcal{H}_2)$ without affecting $F(\mathcal{S},\mathcal{T})$. Moreover $\mathcal{S}\overline{\otimes}\mathcal{T}\subseteq F(\mathcal{S},\mathcal{T})$. Tomiyama proved in [12] that if $\mathcal{S}$ and $\mathcal{T}$ are von Neumann algebras, then

$$\mathcal{S}\overline{\otimes}\mathcal{T}=F(\mathcal{S},\mathcal{T}).\tag{3}$$

His proof uses Tomita's theorem and, in fact, Tomita's theorem (1) is equivalent to the validity of (3) for von Neumann algebras. Hence (3) can be considered as a possible general version of Tomita's theorem for σ-weakly closed subspaces.

A σ-weakly closed subspace $\mathcal{S}\subseteq\mathcal{B}(\mathcal{H})$ is said to have *Property* S_σ if

$$\{X\in\mathcal{S}\overline{\otimes}\mathcal{N} : R_\phi(X)\in\mathcal{T}\,\text{for all}\,\phi\in\mathcal{B}(\mathcal{H})_*\}=\mathcal{S}\overline{\otimes}\mathcal{T}$$

for all pairs $\{\mathcal{T}, \mathcal{N}\}$, where $\mathcal{T}$ is a σ-weakly closed subspace of a von Neumann algebra $\mathcal{N}$. $\mathcal{S}$ has *Property* S_σ if and only if $F(\mathcal{S}, \mathcal{T}) = \mathcal{S}\overline{\otimes}\mathcal{T}$ for all σ-weakly closed subspaces $\mathcal{T}$ of each von Neumann algebra $\mathcal{N}$ ([7, Remark 1.5]).

2. PROPERTY S_σ

Let $\mathcal{A}$ be a reflexive subalgebra of $\mathcal{B}(\mathcal{H})$. Suppose that $E \to \tau(E)$ is an order homomorphism of $\operatorname{Lat}\mathcal{A}$ into itself (i.e., $E \leq F$ implies $\tau(E) \leq \tau(F)$). Then the set $\mathcal{U} = \{T \in \mathcal{B}(\mathcal{H}) : (I - \tau(E))TE = 0, \forall E \in \operatorname{Lat}\mathcal{A}\}$ is clearly a σ-weakly closed $\mathcal{A}$-submodule of $\mathcal{B}(\mathcal{H})$. We denote $\mathcal{U}$ by $\mathcal{U}_\tau$.

Erdos and Power in [1] proved that any σ-weakly closed $\mathcal{A}$-submodule of $\mathcal{B}(\mathcal{H})$ for a nest algebra $\mathcal{A}$ is of the above form. Here the following result is due to Han Deguang [3]:

Theorem H. *Let $\mathcal{A}$ be a unital σ-weakly closed subalgebra which is σ-weakly generated by rank-one operators in $\mathcal{A}$, and let $\mathcal{U}$ be a σ-weakly closed $\mathcal{A}$-submodule of $\mathcal{B}(\mathcal{H})$. Then $\mathcal{U}$ has the form*

$$\mathcal{U} = \{T \in \mathcal{B}(\mathcal{H}) : (I - \tau(E))TE = 0, \forall E \in \operatorname{Lat}\mathcal{A}\},$$

where $E \to \tau(E) = [\mathcal{U}E]$ is an order homomorphism of $\operatorname{Lat}\mathcal{A}$ *into itself.*

Theorem 2.1. *Let $\mathcal{A}$ be a unital σ-weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ with the property that the finite-rank operators of $\mathcal{A}$ are σ-weakly dense in $\mathcal{A}$. Then every σ-weakly closed $\mathcal{A}$-submodule has Property S_σ.*

Proof. Suppose that $\mathcal{U}$ is a σ-weakly closed $\mathcal{A}$-submodule. Let $\mathcal{T}$ be a σ-weakly closed subspace of a von Neumann algebra $\mathcal{N}$, and suppose that $X \in \mathcal{U}\overline{\otimes}\mathcal{N}$ and $R_\phi(X) \in \mathcal{T}$ for all $\phi \in \mathcal{B}(\mathcal{H})_*$. It suffices to show that $X \in \mathcal{U}\overline{\otimes}\mathcal{T}$. Let π be the normal $*$-isomorphism of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})\overline{\otimes}\mathcal{N}$ defined by $\pi(A) = A \otimes I$ for $A \in \mathcal{B}(\mathcal{H})$. If $F_1, F_2 \in \mathcal{A} \cap \mathcal{F}(\mathcal{H})$ and $\phi \in \mathcal{B}(\mathcal{H})_*$, a routine calculation shows that $R_\phi(\pi(F_1)X\pi(F_2)) = R_{F_2\phi F_1}(X)$, where $F_2\phi F_1 \in \mathcal{B}(\mathcal{H})_*$ is defined by $\langle A, F_2\phi F_1\rangle = \langle F_1AF_2, \phi\rangle$, $A \in \mathcal{B}(\mathcal{H})$. Hence $R_\phi(\pi(F_1)X\pi(F_2))$ is in $\mathcal{T}$ for all $\phi \in \mathcal{B}(\mathcal{H})_*$. Since $\pi(F_1)(\mathcal{U}\overline{\otimes}\mathcal{N})\pi(F_2) = F_1\mathcal{U}F_2\overline{\otimes}\mathcal{N}$ and $F_1\mathcal{U}F_2$ has *Property* S_σ by [7, Proposition 1.7], $\pi(F_1)X\pi(F_2)$ is in $F_1\mathcal{U}F_2\overline{\otimes}\mathcal{T}$. But $F_1\mathcal{U}F_2 \subseteq \mathcal{U}$; thus $\pi(F_1)X\pi(F_2) \in \mathcal{U}\overline{\otimes}\mathcal{T}$. Let $\{F_\alpha\}$ be a net in $\mathcal{A} \cap \mathcal{F}(\mathcal{H})$ converging σ-weakly to the identity map I. Then $\pi(F_\alpha)X\pi(F)$ converges σ-weakly to $X\pi(F)$ for all $F \in \mathcal{A}\cap\mathcal{F}(\mathcal{H})$, and so $X\pi(F) \in \mathcal{U}\overline{\otimes}\mathcal{T}$ for all $F \in \mathcal{A} \cap \mathcal{F}(\mathcal{H})$. Finally, $X\pi(F_\alpha)$ converges σ-weakly to X, and so $X \in \mathcal{U}\overline{\otimes}\mathcal{T}$. Hence $\mathcal{U}$ has *Property* S_σ. □

It is known from [10] that a commutative subspace lattice $\mathcal{L}$ is completely distributive if and only if the rank-one subalgebra of $\operatorname{Alg}\mathcal{L}$ is σ-weakly dense in $\operatorname{Alg}\mathcal{L}$. Thus we have the following result:

Corollary 2.2. *If $\mathcal{L}$ is a completely distributive CSL, then every σ-weakly closed* $\operatorname{Alg}\mathcal{L}$*-submodule has Property S_σ.*

If $\mathcal{L}$ is a completely distributive CSL, it follows from Theorem H that every σ-weakly closed $\operatorname{Alg}\mathcal{L}$-submodule is of the form $\mathcal{U}_\tau$, where $E \to \tau(E)$ is an order homomorphism of $\mathcal{L}$ into itself.

Corollary 2.3. *Suppose that $\mathcal{L}_i$ $(i = 1, 2)$ are completely distributive CSLs, and that $\mathcal{U}_{\tau_i}$ $(i = 1, 2)$ are σ-weakly closed* $\operatorname{Alg}\mathcal{L}_i$*-submodules respectively. Then $\mathcal{U}_{\tau_1}\overline{\otimes}\mathcal{U}_{\tau_2} = F(\mathcal{U}_{\tau_1}, \mathcal{U}_{\tau_2})$.*

Proof. A σ-weakly closed subspace $\mathcal{S}$ has *Property* S_σ if and only if $\mathcal{S}\overline{\otimes}\mathcal{T} = F(\mathcal{S},\mathcal{T})$ for all σ-weakly closed subspaces $\mathcal{T}$ ([7, Remark 1.5]). Thus the corollary follows from Corollary 2.2. $\square$

In the case of nest algebras, we can say more about tensor products of σ-weakly closed nest algebra submodules. In the rest of this paper, we suppose that $\mathcal{L}_i$ ($i = 1, 2, \cdots, n$) are nests on separable complex Hilbert spaces $\mathcal{H}_i$ and τ_i are order homomorphisms of $\mathcal{L}_i$ into $\mathcal{L}_i$.

If $L \in \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$, it follows from [2, Proposition 2.4] that

$$L = \vee\{E_1 \otimes \cdots \otimes E_n : E_1 \otimes \cdots \otimes E_n \leq L\}.$$

Thus we can define

$$(\tau_1 \otimes \cdots \otimes \tau_n)(L) = \vee\{\tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n) : E_1 \otimes \cdots \otimes E_n \leq L\}.$$

Obviously, $(\tau_1 \otimes \cdots \otimes \tau_n)(E_1 \otimes \cdots \otimes E_n) = \tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n)$. Thus $\tau_1 \otimes \cdots \otimes \tau_n$ is a well-defined order homomorphism of $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ into itself and $\mathcal{U}_{\tau_1\otimes\cdots\otimes\tau_n}$ is a σ-weakly closed $\mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$-submodule. Hence the equality $\mathrm{Alg}\,\mathcal{L}_1\overline{\otimes}\cdots\overline{\otimes}\,\mathrm{Alg}\,\mathcal{L}_n = \mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ of [2, Theorem 2.6] can be rewritten as

$$\mathcal{U}_{I_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{I_n} = \mathcal{U}_{I_1\otimes\cdots\otimes I_n},$$

where I_i is the identity map of $\mathcal{L}_i$ into $\mathcal{L}_i$.

Lemma 2.4. *Let* $\mathcal{L}_i$ $(i = 1, 2)$ *be nests on separable Hilbert spaces* $\mathcal{H}_i$ *and* τ_i $(i = 1, 2)$ *be order homomorphisms of* $\mathcal{L}_i$ *into* $\mathcal{L}_i$. *Then* $\mathcal{U}_{\tau_1\otimes\tau_2} = F(\mathcal{U}_{\tau_1}, \mathcal{U}_{\tau_2})$.

Proof. Suppose that $X \in \mathcal{U}_{\tau_1\otimes\tau_2} \subseteq \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. If $E_2 \in \mathcal{L}_2$ and $\phi \in \mathcal{B}(\mathcal{H}_1)_*$, it follows from [7] (1.3) that

$$\begin{aligned}
\tau_2(E_2)R_\phi(X)E_2 &= R_\phi((I_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2)) \\
&= R_\phi((I_1 \otimes \tau_2(E_2)(\tau_1(I_1) \otimes \tau_2(E_2))X(I_1 \otimes E_2)) \\
&= R_\phi((\tau_1(I_1) \otimes \tau_2(E_2))X(I_1 \otimes E_2)) \\
&= R_\phi(X(I_1 \otimes E_2)) = R_\phi(X)E_2.
\end{aligned}$$

So $R_\phi(X) \in \mathcal{U}_{\tau_2}$. Similary, $L_\psi(X) \in \mathcal{U}_{\tau_1}$ for all $\psi \in \mathcal{B}(\mathcal{H}_2)_*$. Hence by the definition of $F(\mathcal{U}_{\tau_1}, \mathcal{U}_{\tau_2})$, we have $\mathcal{U}_{\tau_1\otimes\tau_2} \subseteq F(\mathcal{U}_{\tau_1}, \mathcal{U}_{\tau_2})$.

Conversely, suppose that $X \in F(\mathcal{U}_{\tau_1}, \mathcal{U}_{\tau_2})$. If $E_2 \in \mathcal{L}_2$ and $\phi \in \mathcal{B}(\mathcal{H}_1)_*$, then $\tau_2(E_2)R_\phi(X)E_2 = R_\phi(X)E_2$. Thus $R_\phi((I_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2)) = R_\phi(X(I_1 \otimes E_2))$ for all $\phi \in \mathcal{B}(\mathcal{H}_1)_*$. It follows from [7] (1.5) that

$$(I_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2) = X(I_1 \otimes E_2).$$

Similarly, if $E_1 \in \mathcal{L}_1$, we have that $X(E_1 \otimes I_2) = (\tau_1(E_1) \otimes I_2)X(E_1 \otimes I_2)$. Therefore,

$$\begin{aligned}
X(E_1 \otimes E_2) &= X(E_1 \otimes I_2)(I_1 \otimes E_2) \\
&= (\tau_1(E_1) \otimes I_2)X(I_1 \otimes E_2)(E_1 \otimes I_2) \\
&= (\tau_1(E_1) \otimes I_2)(I_1 \otimes \tau_2(E_2))X(E_1 \otimes E_2) \\
&= (\tau_1(E_1) \otimes \tau_2(E_2))X(E_1 \otimes E_2).
\end{aligned}$$

Thus, by virtue of [2, Proposition 2.4], it is easy to show that $XL \subseteq (\tau_1 \otimes \tau_2)(L)$ for each $L \in \mathcal{L}_1 \otimes \mathcal{L}_2$. Hence $X \in \mathcal{U}_{\tau_1\otimes\tau_2}$ and $\mathcal{U}_{\tau_1\otimes\tau_2} = F(\mathcal{U}_{\tau_1}, \mathcal{U}_{\tau_2})$. $\square$

Theorem 2.5. *Let* $\mathcal{L}_i$ *and* τ_i *be as in the preceding lemma. Then* $\mathcal{U}_{\tau_1}\overline{\otimes}\mathcal{U}_{\tau_2} = \mathcal{U}_{\tau_1\otimes\tau_2}$.

Proof. Since every nest is a completely distributive CSL, the theorem follows from Corollary 2.3 and Lemma 2.4, obviously. $\square$

3. THE n-FOLD TENSOR PRODUCT FORMULA

Since $\mathcal{L}_1 \otimes \mathcal{L}_2$ is not totally ordered in general, we cannot deduce the tensor product formula $\mathcal{U}_{\tau_1\otimes\tau_2\otimes\tau_3} = \mathcal{U}_{\tau_1}\overline{\otimes}\mathcal{U}_{\tau_2}\overline{\otimes}\mathcal{U}_{\tau_3}$ by

$$\begin{aligned}\mathcal{U}_{\tau_1\otimes\tau_2\otimes\tau_3} &= \mathcal{U}_{(\tau_1\otimes\tau_2)\otimes\tau_3}\\ &= \mathcal{U}_{\tau_1\otimes\tau_2}\overline{\otimes}\mathcal{U}_{\tau_3} = \mathcal{U}_{\tau_1}\overline{\otimes}\mathcal{U}_{\tau_2}\overline{\otimes}\mathcal{U}_{\tau_3}.\end{aligned}$$

(In order to use Theorem 2.5, the second equality needs the totally ordered property of $\mathcal{L}_1 \otimes \mathcal{L}_2$.) So we cannot generalize Theorem 2.5 to n-fold tensor products for $n > 2$ by induction. In this section, instead of the slice maps, we shall use Theorem H to prove the n-fold tensor product formula. Let $\mathcal{L}_i$ $(i = 1, \cdots, n)$ be nests and let $\mathcal{U}_i$ be σ-weakly closed $\operatorname{Alg}\mathcal{L}_i$-submodules. From Theorem H, it follows from $\mathcal{U}_i = \mathcal{U}_{\tau_i}$, where $\tau_i(E) = [\mathcal{U}_i E]$ for any $E \in \mathcal{L}_i$. In the rest of this section, we always use τ_i to denote these special order homomorphisms.

Lemma 3.1. *For each $i = 1, \cdots, n$, let $E_i \in \mathcal{L}_i$ and $f_i \in E_i$ such that $[(\operatorname{Alg}\mathcal{L}_i)f_i] = E_i$. Then $[(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(E_1 \otimes \cdots \otimes E_n)] = [\mathcal{U}_{\tau_1}E_1] \otimes \cdots \otimes [\mathcal{U}_{\tau_n}E_n]$.*

Proof. Since $\mathcal{U}_{\tau_i} \cdot \operatorname{Alg}\mathcal{L}_i = \mathcal{U}_{\tau_i}$, $[\mathcal{U}_{\tau_i}E_i] = [\mathcal{U}_{\tau_i}f_i]$. By virtue of [2] Lemma 2.2,

$$E_1 \otimes \cdots \otimes E_n = [(\operatorname{Alg}\mathcal{L}_1\overline{\otimes}\cdots\overline{\otimes}\operatorname{Alg}\mathcal{L}_n)(f_1 \otimes \cdots \otimes f_n)].$$

Thus, since $(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(\operatorname{Alg}\mathcal{L}_1\overline{\otimes}\cdots\overline{\otimes}\operatorname{Alg}\mathcal{L}_n) = \mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n}$,

$$[(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(E_1 \otimes \cdots \otimes E_n)] = [(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(f_1 \otimes \cdots \otimes f_n)].$$

Hence it suffices to prove that

$$[(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(f_1 \otimes \cdots \otimes f_n)] = [\mathcal{U}_{\tau_1}f_1] \otimes \cdots \otimes [\mathcal{U}_{\tau_n}f_n].$$

If g_i is any vector in $[\mathcal{U}_{\tau_i}f_i]$, then g_i can be norm approximated by vectors of the form T_if_i, where $T_i \in \mathcal{U}_{\tau_i}$. Hence $g_1 \otimes \cdots \otimes g_n$ can be approximated by vectors of the form $T_1f_1 \otimes \cdots \otimes T_nf_n = (T_1 \otimes \cdots \otimes T_n)(f_1 \otimes \cdots \otimes f_n)$. Thus any vector of the form $g_1 \otimes \cdots \otimes g_n$ with $g_i \in [\mathcal{U}_{\tau_i}f_i]$ lies in $[(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(f_1 \otimes \cdots \otimes f_n)]$. Since such vectors generate $[\mathcal{U}_{\tau_1}f_1] \otimes \cdots \otimes [\mathcal{U}_{\tau_n}f_n]$, we have $[\mathcal{U}_{\tau_1}f_1] \otimes \cdots \otimes [\mathcal{U}_{\tau_n}f_n] \subseteq [(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(f_1 \otimes \cdots \otimes f_n)]$.

To prove the reverse inequality, for any $T_i \in \mathcal{U}_{\tau_i}$, we have that

$$\begin{aligned}&([\mathcal{U}_{\tau_1}f_1] \otimes \cdots \otimes [\mathcal{U}_{\tau_n}f_n])(T_1 \otimes \cdots \otimes T_n)(E_1 \otimes \cdots \otimes E_n)\\ &= ([\mathcal{U}_{\tau_1}E_1] \otimes \cdots \otimes [\mathcal{U}_{\tau_n}E_n])(T_1 \otimes \cdots \otimes T_n)(E_1 \otimes \cdots \otimes E_n)\\ &= (\tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n))(T_1 \otimes \cdots \otimes T_n)(E_1 \otimes \cdots \otimes E_n)\\ &= \tau_1(E_1)T_1E_1 \otimes \cdots \otimes \tau_n(E_n)T_nE_n\\ &= T_1E_1 \otimes \cdots \otimes T_nE_n\\ &= (T_1 \otimes \cdots \otimes T_n)(E_1 \otimes \cdots \otimes E_n).\end{aligned}$$

This shows that

$$\begin{aligned}&([\mathcal{U}_{\tau_1}f_1] \otimes \cdots \otimes [\mathcal{U}_{\tau_n}f_n])[(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(E_1 \otimes \cdots \otimes E_n)]\\ &= [(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(E_1 \otimes \cdots \otimes E_n)].\end{aligned}$$

Thus

$$\begin{aligned}[(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(f_1 \otimes \cdots \otimes f_n)] &= [(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(E_1 \otimes \cdots \otimes E_n)]\\ &\leq [\mathcal{U}_{\tau_1}f_1] \otimes \cdots \otimes [\mathcal{U}_{\tau_n}f_n].\end{aligned}$$

Therefore

$$[(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(f_1 \otimes \cdots \otimes f_n)] = [\mathcal{U}_{\tau_1}f_1] \otimes \cdots \otimes [\mathcal{U}_{\tau_n}f_n].$$

This completes the proof. □

Theorem 3.2. *Let* $\mathcal{U}_i$ $(i = 1, \cdots, n)$ *be* σ*-weakly closed* $\operatorname{Alg}\mathcal{L}_i$*-submodules and* $\tau_i(E) = [\mathcal{U}_i E]$ *for any* $E \in \mathcal{L}_i$*. Then* $\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n} = \mathcal{U}_{\tau_1\otimes\cdots\otimes\tau_n}$*.*

Proof. It is obvious that $\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n}$ is a σ-weakly closed $\operatorname{Alg}\mathcal{L}_1\overline{\otimes}\cdots\overline{\otimes}\operatorname{Alg}\mathcal{L}_n$-submodule. By virtue of [2, Theorem 2.6], $\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n}$ is a σ-weakly closed $\operatorname{Alg}(\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n)$- submodule. It follows from [2, Proposition 2.7] that $\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n$ is a completely distributive CSL. Thus, Theorem H shows that $\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n}$ is determined by the order homomorphism $L \to [(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})L]$ of $\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n$ into itself.

Now suppose that $E_i \in \mathcal{L}_i$. For each i, choose a vector $v_i \in E_i$ such that $[(\operatorname{Alg}\mathcal{L}_i)v_i] = E_i$ (the proof of the existence of such v_i is routine). It follows from Lemma 3.1 that

$$\begin{aligned}
[(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(E_1\otimes\cdots\otimes E_n)] &= [\mathcal{U}_{\tau_1}E_1]\otimes\cdots\otimes[\mathcal{U}_{\tau_n}E_n]\\
&= \tau_1(E_1)\otimes\cdots\otimes\tau_n(E_n)\\
&= (\tau_1\otimes\cdots\otimes\tau_n)(E_1\otimes\cdots\otimes E_n).
\end{aligned}$$

If $L \in \mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n$, [2, Proposition 2.4] shows that

$$L = \vee\{E_1\otimes\cdots\otimes E_n : E_1\otimes\cdots\otimes E_n \le L\}.$$

Thus,

$$\begin{aligned}
&[(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})L]\\
&= \vee\{[(\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n})(E_1\otimes\cdots\otimes E_n)] : E_1\otimes\cdots\otimes E_n \le L\}\\
&= \vee\{(\tau_1\otimes\cdots\otimes\tau_n)(E_1\otimes\cdots\otimes E_n) : E_1\otimes\cdots\otimes E_n \le L\}\\
&= (\tau_1\otimes\cdots\otimes\tau_n)(L).
\end{aligned}$$

Hence $\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n}$ and $\mathcal{U}_{\tau_1\otimes\cdots\otimes\tau_n}$ are σ-weakly closed $\operatorname{Alg}(\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n)$-submodules determined by the same order homomorphism. This shows that

$$\mathcal{U}_{\tau_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\tau_n} = \mathcal{U}_{\tau_1\otimes\cdots\otimes\tau_n}.$$

□

Given general order homomorphisms ϕ_i from $\mathcal{L}_i$ into $\mathcal{L}_i$, we will consider the relation between $\mathcal{U}_{\phi_1}\overline{\otimes}\cdots\overline{\otimes}\mathcal{U}_{\phi_n}$ and $\mathcal{U}_{\phi_1\otimes\cdots\otimes\phi_n}$. We need some lemmas at first.

For non-zero vectors $x, y \in \mathcal{H}$, the rank-one operator xy^* is defined by the equation

$$(xy^*)(z) = \langle z, y\rangle x, \quad \forall z \in \mathcal{H}.$$

Lemma 3.3. *Suppose that* $\mathcal{L}$ *is a subspace lattice, and that* $\mathcal{U}_\phi$ *is the* σ*-weakly closed* $\operatorname{Alg}\mathcal{L}$*-submodule determined by an order homomorphism* ϕ *from* $\mathcal{L}$ *into itself. Then a rank-one operator* $xy^* \in \mathcal{U}_\phi$ *if and only if there exists an element* $N \in \mathcal{L}$ *such that* $x \in N$ *and* $y \in \phi_\sim(N)^\perp$*, where* $\phi_\sim(N) = \vee\{G \in \mathcal{L} : \phi(G) \not\geq N\}$*.*

Proof. The proof is routine. We leave the details to the interested readers. □

Lemma 3.4. *Let* $\mathcal{L}_i$ *be a nest and* ϕ_i *be an order homomorphism from* $\mathcal{L}_i$ *into itself. Define* $\psi_i : I_1\otimes\cdots\otimes\mathcal{L}_i\otimes\cdots\otimes I_n \to I_1\otimes\cdots\otimes\mathcal{L}_i\otimes\cdots\otimes I_n$ *by*

$$\psi_i(I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n) = I_1\otimes\cdots\otimes\phi_i(N_i)\otimes\cdots\otimes I_n, \quad \forall N_i \in \mathcal{L}_i.$$

Then the rank-one operator $xy^* \in \mathcal{U}_{\psi_i}$ *if and only if there exists an element* $N_i \in \mathcal{L}_i$ *such that* $x \in I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n$ *and* $y \in I_1\otimes\cdots\otimes\phi_{i\sim}(N_i)^\perp\otimes\cdots\otimes I_n$*.*

Proof. Certainly ψ_i is an order homomorphism from $I_1\otimes\cdots\otimes\mathcal{L}_i\otimes\cdots\otimes I_n$ into itself, and $\mathcal{U}_{\psi_i}$ is the σ-weakly closed $\mathrm{Alg}(I_1\otimes\cdots\otimes\mathcal{L}_i\otimes\cdots\otimes I_n)$-submodule determined by ψ_i. By virtue of Lemma 3.3, a rank-one operator $xy^*\in\mathcal{U}_{\psi_i}$ if and only if there exists an element $N_i\in\mathcal{L}_i$ such that $x\in I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n$ and $y\in\psi_{i\sim}(I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n)^\perp$. In the following, we compute

$$\psi_{i\sim}(I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n)^\perp.$$

By the definition of $\psi_{i\sim}$, we have

$$\begin{aligned}
&\psi_{i\sim}(I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n)\\
&=\vee\{I_1\otimes\cdots\otimes G_i\otimes\cdots\otimes I_n:\psi_i(I_1\otimes\cdots\otimes G_i\otimes\cdots\otimes I_n)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\not\geq I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n\}\\
&=\vee\{I_1\otimes\cdots\otimes G_i\otimes\cdots\otimes I_n:I_1\otimes\cdots\otimes\phi_i(G_i)\otimes\cdots\otimes I_n\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\not\geq I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n\}\\
&=\vee\{I_1\otimes\cdots\otimes G_i\otimes\cdots\otimes I_n:\phi_i(G_i)\not\geq N_i\}\\
&=I_1\otimes\cdots\otimes(\vee\{G_i:\phi_i(G_i)\not\geq N_i\})\otimes\cdots\otimes I_n\\
&=I_1\otimes\cdots\otimes\phi_{i\sim}(N_i)\otimes\cdots\otimes I_n,
\end{aligned}$$

and so $\phi_{i\sim}(I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n)^\perp=I_1\otimes\cdots\otimes\phi_{i\sim}(N_i)^\perp\otimes\cdots\otimes I_n$. ☐

Proposition 3.5. *Let $\mathcal{L}_i$ $(i=1,\cdots,n)$ be nests and ϕ_i be order homomorphisms from $\mathcal{L}_i$ into itself. Then a rank-one operator $xy^*\in\mathcal{U}_{\phi_1\otimes\cdots\otimes\phi_n}$ if and only if there exist $N_i\in\mathcal{L}_i$ such that $x\in N_1\otimes\cdots\otimes N_n$ and $y\in\phi_{1\sim}(N_1)^\perp\otimes\cdots\otimes\phi_{n\sim}(N_n)^\perp$.*

Proof. Set $\mathcal{F}_i=I_1\otimes\cdots\otimes\mathcal{L}_i\otimes\cdots\otimes I_n$, and define $\psi_i:\mathcal{F}_i\to\mathcal{F}_i$ by

$$\psi_i(I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n)=I_1\otimes\cdots\otimes\phi_i(N_i)\otimes\cdots\otimes I_n,\quad\forall N_i\in\mathcal{L}_i.$$

Each ψ_i is an order homomorphism from $\mathcal{F}_i$ into itself and $\mathcal{U}_{\psi_i}$ is the σ-weakly closed $\mathrm{Alg}(\mathcal{F}_i)$-submodules determined by ψ_i. Thus we have the equation $\mathcal{U}_{\phi_1\otimes\cdots\otimes\phi_n}=\mathcal{U}_{\psi_1}\cap\cdots\cap\mathcal{U}_{\psi_n}$. In fact, by virtue of [2, Proposition 2.4],

$$L=\vee\{N_1\otimes\cdots\otimes N_n:N_1\otimes\cdots\otimes N_n\leq L\}\quad\text{for any}\quad L\in\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n.$$

Thus it is easy to show that

$$\begin{aligned}\mathcal{U}_{\phi_1\otimes\cdots\otimes\phi_n}=\{T\in\mathcal{B}(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n):T(N_1\otimes\cdots\otimes N_n)&\\
\subseteq\phi_1(N_1)\otimes\cdots\otimes\phi_n(N_n),\forall N_i\in\mathcal{L}_i\},&\end{aligned}$$

and so $\mathcal{U}_{\phi_1\otimes\cdots\otimes\phi_n}\subseteq\mathcal{U}_{\psi_1}\cap\cdots\cap\mathcal{U}_{\psi_n}$. For any $T\in\mathcal{U}_{\phi_1}\cap\cdots\mathcal{U}_{\phi_n}$, we have that for any $N_i\in\mathcal{L}_i$,

$$T(N_1\otimes\cdots\otimes N_n)\subseteq T(I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n)\subseteq I_1\otimes\cdots\otimes\phi_i(N_i)\otimes\cdots\otimes I_n,\quad\forall 1\leq i\leq n.$$

Thus $T(N_1\otimes\cdots\otimes N_n)\subseteq\phi_1(N_1)\otimes\cdots\otimes\phi_n(N_n)$ and $T\in\mathcal{U}_{\phi_1\otimes\cdots\otimes\phi_n}$. Hence $\mathcal{U}_{\phi_1\otimes\cdots\otimes\phi_n}=\mathcal{U}_{\psi_1}\cap\cdots\cap\mathcal{U}_{\psi_n}$. From Lemma 3.4 it follows that for any $1\leq i\leq n$, a rank-one operator $xy^*\in\mathcal{U}_{\psi_i}$ if and only if there exists $N_i\in\mathcal{L}_i$ such that $x\in I_1\otimes\cdots\otimes N_i\otimes\cdots\otimes I_n$ and $y\in I_1\otimes\cdots\otimes\phi_{i\sim}(N_i)^\perp\otimes\cdots\otimes I_n$. Therefore a rank-one operator $xy^*\in\mathcal{U}_{\psi_1}\cap\cdots\cap\mathcal{U}_{\psi_n}=\mathcal{U}_{\phi_1\otimes\cdots\otimes\phi_n}$ if and only if there exists $N_i\in\mathcal{L}_i$ $(1\leq i\leq n)$ such that $x\in N_1\otimes\cdots\otimes N_n$ and $y\in\phi_{1\sim}(N_1)^\perp\otimes\cdots\otimes\phi_{n\sim}(N_n)^\perp$. ☐

Lemma 3.6. *Suppose that $\mathcal{L}$ is a subspace lattice and that $\mathcal{U}_\phi$ is the σ-weakly closed* $\mathrm{Alg}\,\mathcal{L}$*-submodule determined by an order homomorphism ϕ from $\mathcal{L}$ into itself. Then $\tau\leq\phi$ and $\tau_\sim=\phi_\sim$, where $\tau(E)=[\mathcal{U}_\phi E]$ for any $E\in\mathcal{L}$.*

Proof. It follows from the definition of $\mathcal{U}_\phi$ that

$$\tau(E) = [\mathcal{U}_\phi E] \leq \phi(E) \quad \text{for any} \quad E \in \mathcal{L}.$$

So $\tau \leq \phi$.

Since $\tau \leq \phi$, we have $\tau_\sim \geq \phi_\sim$. So it suffices to show that $\tau_\sim \leq \phi_\sim$. If not, there exists $E \in \mathcal{L}$ such that $\tau_\sim(E) \not\leq \phi_\sim(E)$. It follows from the definition of $\tau_\sim$ that there exists $F \in \mathcal{L}$ such that $\tau(F) \not\geq E$ and $F \not\leq \phi_\sim(E)$. Thus we can choose non-zero vectors x, y such that $x \in E$ and $x \notin \tau(F)$, $y \in \phi_\sim(E)^\perp$ and $y \notin F^\perp$. From Lemma 3.3, it follows that $x \otimes y \in \mathcal{U}_\phi$. Since $(I - \tau(F))(x \otimes y)F \neq 0$, $x \otimes y \notin \mathcal{U}_\tau$. However it follows from the proof of Theorem H that $\mathcal{U}_\tau = \mathcal{U}_\phi$. This is a contradiction. Accordingly, $\tau_\sim \leq \phi_\sim$ and $\tau_\sim = \phi_\sim$. □

Now we are in the position to show the general tenor product formula of σ-weakly closed Alg $\mathcal{L}_i$-submodules.

Theorem 3.7. *Let $\mathcal{L}_i$ $(1 = 1, \cdots, n)$ be nests and ϕ_i be order homomorphisms from $\mathcal{L}_i$ into itself. Then $\mathcal{U}_{\phi_1} \overline{\otimes} \cdots \overline{\otimes} \mathcal{U}_{\phi_n} = \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$.*

Proof. It follows from Theorem H that $\mathcal{U}_{\phi_i} = \mathcal{U}_{\tau_i}$, where $\tau_i(E) = [\mathcal{U}_{\phi_i} E]$ for any $E \in \mathcal{L}_i$. Thus by virtue of Theorem 3.2, we have that

$$\mathcal{U}_{\phi_1} \overline{\otimes} \cdots \overline{\otimes} \mathcal{U}_{\phi_n} = \mathcal{U}_{\tau_1} \overline{\otimes} \cdots \overline{\otimes} \mathcal{U}_{\tau_n} = \mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n}.$$

So it suffices to show $\mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n} = \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$. Since $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ is a completely distributive CSL ([2, Proposition 2.7]), it follows from [10] Theorem 3 that the rank-one operators of $\mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ are σ-weakly dense in $\mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$. So it is routine to show that the linear spans of rank-one operators in $\mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n}$ and $\mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$ are σ-weakly dense in $\mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n}$ and $\mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$ respectively. From Proposition 3.5 and Lemma 3.6, it follows that $\mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n}$ and $\mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$ have the same rank-one operators. Therefore $\mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n} = \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$ and $\mathcal{U}_{\phi_1} \overline{\otimes} \cdots \overline{\otimes} \mathcal{U}_{\phi_n} = \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$. □

Remark 3.8. Theorem 2.5 is a particular case of Theorem 3.2. In [9], Theorem 2.2 shows that $\mathcal{U}_{\tau_i}$ $(i = 1, \cdots, n)$ are reflexive subspaces. Combining the above result, we know that the tensor product of $\mathcal{U}_{\tau_i}$ is also reflexive. It is natural to ask whether the tensor product of reflexive subspaces is also reflexive. This seems a challenging problem.

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