TENSOR PRODUCTS
OF $\sigma$-WEAKLY CLOSED NEST ALGEBRA SUBMODULES

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(Communicated by David R. Larson)

Abstract. In this paper we prove that for any unital $\sigma$-weakly closed algebra $A$ which is $\sigma$-weakly generated by finite-rank operators in $A$, every $\sigma$-weakly closed $A$-submodule has Property $S_{\sigma}$. In the case of nest algebras, if $L_1, \cdots, L_n$ are nests, we obtain the following $n$-fold tensor product formula:

$$U_{\phi_1} \otimes \cdots \otimes U_{\phi_n} = U_{\phi_1 \otimes \cdots \otimes \phi_n},$$

where each $U_{\phi_i}$ is the $\sigma$-weakly closed $\text{Alg}L_i$-submodule determined by an order homomorphism $\phi_i$ from $L_i$ into itself.

1. Introduction

One of the central results in the theory of tensor products of von Neumann algebras is Tomita’s commutation formula:

$$M' \otimes N' = (M \otimes N)',$$

where $M$ and $N$ are von Neumann algebras. It was observed in [2] that if we let $L_1$ and $L_2$ denote the projection lattices of $M$ and $N$ respectively, then (1) can be rewritten as

$$\text{Alg}L_1 \otimes \text{Alg}L_2 = \text{Alg}(L_1 \otimes L_2).$$

This version of Tomita’s theorem makes sense for any pair of reflexive algebras $\text{Alg}L_1$ and $\text{Alg}L_2$. It remains a deep open question whether the tensor product formula (2) is valid for general reflexive algebras, or even general CSL algebras. However, (2) has been verified in a number of special cases ([2], [4], [5], [6], [7]). In particular, it is known that if $L_1$ is a commutative subspace lattice that is either completely distributive [8] or of finite width [4], then (2) is valid for $L_1$ and any subspace lattice $L_2$.

The main purpose of this paper is to study tensor products of $\sigma$-weakly closed submodules of some reflexive algebras (in particular, of nest algebras). Section 1 of this paper is devoted to notation and preliminaries. In Section 2, we make use of slice maps to show that if $A$ is a $\sigma$-weakly closed algebra which is $\sigma$-weakly generated by finite-rank operators in $A$, then every $\sigma$-weakly closed $A$-submodule has Property $S_{\sigma}$. As a corollary, we obtain $U_{r_1} \otimes U_{r_2} = U_{r_1 \otimes r_2}$, where each $U_r$ is
a \( \sigma \)-weakly closed \( \text{Alg} \mathcal{L}_i \)-submodule and \( \mathcal{L}_i \) is a nest. However, the 2-fold tensor product formula cannot be generalized to the \( n \)-fold formula by induction (see the beginning of Section 3). So in Section 3, we use another method to prove the \( n \)-fold tensor product formula \( U_{A_{i_1} \otimes \cdots \otimes U_{A_{i_n}}} = U_{A_{i_1} \otimes \cdots \otimes A_{i_n}} \), where each \( U_{A_i} \) is a \( \sigma \)-weakly closed \( \text{Alg} \mathcal{L}_i \)-submodule and \( \mathcal{L}_i \) is a nest. The key to this proof is \([3 \text{ Theorem } 2]\) and \([2 \text{ Proposition } 2,4]\).

In this paper, all Hilbert spaces will be separable. Let \( \mathcal{B}(\mathcal{H}) \) be the algebra of bounded operators on \( \mathcal{H} \) and \( \mathcal{F}(\mathcal{H}) \) be the set of finite-rank operators on \( \mathcal{H} \). A sublattice \( \mathcal{L} \) of the projection lattice of \( \mathcal{B}(\mathcal{H}) \) is said to be a subspace lattice if it contains 0 and 1 and is strongly closed, where we identify projections with their ranges. If the elements of \( \mathcal{L} \) pairwise commute, \( \mathcal{L} \) is a commutative subspace lattice (CSL). A nest is a totally ordered subspace lattice. If \( \mathcal{L} \) is a subspace lattice, \( \text{Alg} \mathcal{L} \) denotes the set of operators in \( \mathcal{B}(\mathcal{H}) \) that leave the elements of \( \mathcal{L} \) invariant. Note that \( \text{Alg} \mathcal{L} \) is a \( \sigma \)-weakly closed subalgebra of \( \mathcal{B}(\mathcal{H}) \). If \( \mathcal{L} \) is a CSL, \( \text{Alg} \mathcal{L} \) is said to be a CSL algebra. If \( \mathcal{L} \) is a nest, \( \text{Alg} \mathcal{L} \) is said to be a nest algebra.

If \( \mathcal{A} \) is a subset of \( \mathcal{B}(\mathcal{H}) \), then \( \text{Lat} \mathcal{A} \), the set of projections left invariant by each element of \( \mathcal{A} \), is a subspace lattice. A subalgebra \( \mathcal{A} \) of \( \mathcal{B}(\mathcal{H}) \) is reflexive if \( \mathcal{A} = \text{Alg} \text{Lat} \mathcal{A} \). The reflexive algebras are precisely the algebras of the form \( \text{Alg} \mathcal{L} \), where \( \mathcal{L} \) is a subspace lattice. If \( \mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i) \) \( (i = 1, \cdots , n) \) are subspace lattices, \( \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n \) is the subspace lattice in \( \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n) \) generated by \( \{ P_1 \otimes \cdots \otimes P_n : P_i \in \mathcal{L}_i, i = 1, \cdots , n \} \). If \( \mathcal{S}_i \subseteq \mathcal{B}(\mathcal{H}_i) \) \( (i = 1, \cdots , n) \) are \( \sigma \)-weakly closed subspaces, then \( \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n \) denotes the \( \sigma \)-weakly closed linear span of \( \{ \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n : \mathcal{S}_i \subseteq \mathcal{S}_i \} \) in \( \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n) \).

The main technical tool in Section 2 is the use of slice maps. Slice maps were introduced by Tomiyama in \([11]\) and have been used extensively in the study of tensor products of \( C^* \)-algebras and tensor products of von Neumann algebras. We recall some definitions and results from \([7]\) and refer the reader to \([7]\) for further results and motivation. If \( \mathcal{M} \) and \( \mathcal{N} \) are von Neumann algebras, and \( \phi \) is in the predual \( \mathcal{M}_* \) of \( \mathcal{M} \), then the right slice map \( R_\phi \) is the unique \( \sigma \)-weakly continuous linear map from \( \mathcal{M} \otimes \mathcal{N} \to \mathcal{N} \) such that

\[
\langle X, \phi \otimes \psi \rangle = \langle R_\phi(X), \psi \rangle, \quad \forall X \in \mathcal{M} \otimes \mathcal{N}, \psi \in \mathcal{N}_*.
\]

If \( X = A \otimes B \) \( (A \in \mathcal{M}, B \in \mathcal{N}) \), then \( R_\phi(X) = \langle A, \phi \rangle B \). The left slice map \( L_\psi : \mathcal{M} \otimes \mathcal{N} \to \mathcal{M} \) \( \psi \in \mathcal{M}_* \), is similarly defined. If \( \mathcal{S} \subseteq \mathcal{M} \) and \( \mathcal{T} \subseteq \mathcal{N} \) are \( \sigma \)-weakly closed subspaces, let

\[
F(\mathcal{S}, \mathcal{T}) = \{ X \in \mathcal{M} \otimes \mathcal{N} : R_\phi(X) \in \mathcal{T} \text{ and } L_\psi(X) \in \mathcal{S}, \quad \forall \phi \in \mathcal{M}_*, \psi \in \mathcal{N}_* \}.
\]

As noted in \([7]\), we can replace \( \mathcal{M} \) by \( \mathcal{B}(\mathcal{H}_1) \) and \( \mathcal{N} \) by \( \mathcal{B}(\mathcal{H}_2) \) without affecting \( F(\mathcal{S}, \mathcal{T}) \). Moreover \( S \otimes T \subseteq F(\mathcal{S}, \mathcal{T}) \). Tomiyama proved in \([12]\) that if \( \mathcal{S} \) and \( \mathcal{T} \) are von Neumann algebras, then

\[
(3) \quad S \otimes T = F(\mathcal{S}, \mathcal{T}).
\]

His proof uses Tomita’s theorem and, in fact, Tomita’s theorem (1) is equivalent to the validity of (3) for von Neumann algebras. Hence (3) can be considered as a possible general version of Tomita’s theorem for \( \sigma \)-weakly closed subspaces.

A \( \sigma \)-weakly closed subspace \( \mathcal{S} \subseteq \mathcal{B}(\mathcal{H}) \) is said to have \textit{Property S}_\( \phi \) if

\[
\{ X \in S \otimes \mathcal{N} : R_\phi(X) \in \mathcal{T} \text{ for all } \phi \in \mathcal{B}(\mathcal{H})_* \} = S \otimes \mathcal{T}
\]
for all pairs \( \{ T, N \} \), where \( T \) is a \( \sigma \)-weakly closed subspace of a von Neumann algebra \( \mathcal{N} \). \( \mathcal{S} \) has Property \( S_\sigma \) if and only if \( F(\mathcal{S}, T) = S \otimes T \) for all \( \sigma \)-weakly closed subspaces \( T \) of each von Neumann algebra \( \mathcal{N} \) (\cite{1} Remark 1.5)).

2. Property \( S_\sigma \)

Let \( \mathcal{A} \) be a reflexive subalgebra of \( \mathcal{B}(\mathcal{H}) \). Suppose that \( E \rightarrow \tau(E) \) is an order homomorphism of \( \text{Lat} \mathcal{A} \) into itself (i.e., \( E \leq F \) implies \( \tau(E) \leq \tau(F) \)). Then the set \( \mathcal{U} = \{ T \in \mathcal{B}(\mathcal{H}) : (I - \tau(E))TE = 0, \forall E \in \text{Lat} \mathcal{A} \} \) is clearly a \( \sigma \)-weakly closed \( \mathcal{A} \)-submodule of \( \mathcal{B}(\mathcal{H}) \). We denote \( \mathcal{U} \) by \( \mathcal{U}_\tau \).

Erdos and Power in \cite{11} proved that any \( \sigma \)-weakly closed \( \mathcal{A} \)-submodule of \( \mathcal{B}(\mathcal{H}) \) for a nest algebra \( \mathcal{A} \) is of the above form. Here the following result is due to Han Deguang \cite{3}:

**Theorem H.** Let \( \mathcal{A} \) be a unital \( \sigma \)-weakly closed subalgebra which is \( \sigma \)-weakly generated by rank-one operators in \( \mathcal{A} \), and let \( \mathcal{U} \) be a \( \sigma \)-weakly closed \( \mathcal{A} \)-submodule of \( \mathcal{B}(\mathcal{H}) \). Then \( \mathcal{U} \) has the form

\[
\mathcal{U} = \{ T \in \mathcal{B}(\mathcal{H}) : (I - \tau(E))TE = 0, \forall E \in \text{Lat} \mathcal{A} \},
\]

where \( E \rightarrow \tau(E) = [\mathcal{U}E] \) is an order homomorphism of \( \text{Lat} \mathcal{A} \) into itself.

**Theorem 2.1.** Let \( \mathcal{A} \) be a unital \( \sigma \)-weakly closed subalgebra of \( \mathcal{B}(\mathcal{H}) \) with the property that the finite-rank operators of \( \mathcal{A} \) are \( \sigma \)-weakly dense in \( \mathcal{A} \). Then every \( \sigma \)-weakly closed \( \mathcal{A} \)-submodule has Property \( S_\sigma \).

**Proof.** Suppose that \( \mathcal{U} \) is a \( \sigma \)-weakly closed \( \mathcal{A} \)-submodule. Let \( T \) be a \( \sigma \)-weakly closed subspace of a von Neumann algebra \( \mathcal{N} \), and suppose that \( X \in \mathcal{U} \otimes \mathcal{N} \) and \( R_\phi(X) \in T \) for all \( \phi \in \mathcal{B}(\mathcal{H})_\sigma \). It suffices to show that \( X \in \mathcal{U} \otimes T \). Let \( \pi \) be the normal \( * \)-isomorphism of \( \mathcal{B}(\mathcal{H}) \) into \( \mathcal{B}(\mathcal{H}) \otimes \mathcal{N} \) defined by \( \pi(A) = A \otimes I \) for \( A \in \mathcal{B}(\mathcal{H}) \). If \( F_1, F_2 \in \mathcal{A} \cap \mathcal{F}(\mathcal{H}) \) and \( \phi \in \mathcal{B}(\mathcal{H})_\sigma \), a routine calculation shows that \( R_\phi(\pi(F_1)X\pi(F_2)) = R_\phi R_{\phi F_1}(X) \), where \( R_{\phi F_1} \in \mathcal{B}(\mathcal{H})_\sigma \) is defined by \( (A, F_2 \phi F_1) = \langle F_1 \mathcal{A} F_2, \phi \rangle, A \in \mathcal{B}(\mathcal{H}) \). Hence \( R_\phi(\pi(F_1)X\pi(F_2)) \) is in \( T \) for all \( \phi \in \mathcal{B}(\mathcal{H})_\sigma \). Since \( \pi(F_1)(\mathcal{U} \otimes \mathcal{N})\pi(F_2) = F_1 \mathcal{U} F_2 \otimes \mathcal{N} \) and \( F_1 \mathcal{U} F_2 \) has Property \( S_\sigma \) by [7 Proposition 1.7], \( \pi(F_1)X\pi(F_2) \) is in \( \mathcal{U} \otimes T \). But \( F_1 \mathcal{U} F_2 \subseteq \mathcal{U} \); thus \( \pi(F_1)X\pi(F_2) \in \mathcal{U} \otimes T \). Let \( \{ F_\alpha \} \) be a net in \( \mathcal{A} \cap \mathcal{F}(\mathcal{H}) \) converging \( \sigma \)-weakly to the identity map \( I \). Then \( \pi(F_\alpha)X\pi(F) \) converges \( \sigma \)-weakly to \( X\pi(F) \) for all \( F \in \mathcal{A} \cap \mathcal{F}(\mathcal{H}) \), and so \( X\pi(F) \in \mathcal{U} \otimes T \) for all \( F \in \mathcal{A} \cap \mathcal{F}(\mathcal{H}) \). Finally, \( X\pi(F_\alpha) \) converges \( \sigma \)-weakly to \( X \), and so \( X \in \mathcal{U} \otimes T \). Hence \( \mathcal{U} \) has Property \( S_\sigma \). \( \square \)

It is known from [11] that a commutative subspace lattice \( \mathcal{L} \) is completely distributive if and only if the rank-one subalgebra of \( \text{Alg} \mathcal{L} \) is \( \sigma \)-weakly dense in \( \text{Alg} \mathcal{L} \). Thus we have the following result:

**Corollary 2.2.** If \( \mathcal{L} \) is a completely distributive CSL, then every \( \sigma \)-weakly closed \( \text{Alg} \mathcal{L} \)-submodule has Property \( S_\sigma \).

If \( \mathcal{L} \) is a completely distributive CSL, it follows from Theorem H that every \( \sigma \)-weakly closed \( \text{Alg} \mathcal{L} \)-submodule is of the form \( \mathcal{U}_\tau \), where \( E \rightarrow \tau(E) \) is an order homomorphism of \( \mathcal{L} \) into itself.

**Corollary 2.3.** Suppose that \( \mathcal{L}_i \) (\( i = 1, 2 \)) are completely distributive CSLs, and that \( \mathcal{U}_{\tau_i} \) (\( i = 1, 2 \)) are \( \sigma \)-weakly closed \( \text{Alg} \mathcal{L}_i \)-submodules respectively. Then \( \mathcal{U}_{\tau_1} \overline{\mathcal{U}_{\tau_2}} = F(\mathcal{U}_{\tau_1}, \mathcal{U}_{\tau_2}) \).
Proof. A \(\sigma\)-weakly closed subspace \(S\) has Property \(S_\sigma\) if and only if \(S \otimes T = F(S, T)\) for all \(\sigma\)-weakly closed subspaces \(T\) \([\ref{2}\) Remark 1.5\)]. Thus the corollary follows from Corollary 2.2.

In the case of nest algebras, we can say more about tensor products of \(\sigma\)-weakly closed nest algebra submodules. In the rest of this paper, we suppose that \(L_i\) \((i = 1, 2, \cdots, n)\) are nests on separable complex Hilbert spaces \(H_i\) and \(\tau_i\) are order homomorphisms of \(L_i\) into \(L_i\).

If \(L \in L_1 \otimes \cdots \otimes L_n\), it follows from \([\ref{2}]\) Proposition 2.4 that
\[
L = \vee \{E_1 \otimes \cdots \otimes E_n : E_1 \otimes \cdots \otimes E_n \leq L\}.
\]
Thus we can define
\[
(\tau_1 \otimes \cdots \otimes \tau_n)(L) = \vee \{\tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n) : E_1 \otimes \cdots \otimes E_n \leq L\}.
\]
Obviously, \((\tau_1 \otimes \cdots \otimes \tau_n)(E_1 \otimes \cdots \otimes E_n) = \tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n)\). Thus \(\tau_1 \otimes \cdots \otimes \tau_n\) is a well-defined order homomorphism of \(L_1 \otimes \cdots \otimes L_n\) into itself and \(U_{\tau_1 \otimes \cdots \otimes \tau_n}\) is a \(\sigma\)-weakly closed \(\text{Alg}(L_1 \otimes \cdots \otimes L_n)\)-submodule. Hence the equality \(\text{Alg}L_1 \otimes \cdots \otimes \text{Alg}L_n = \text{Alg}(L_1 \otimes \cdots \otimes L_n)\) of \([\ref{2}]\) Theorem 2.6\] can be rewritten as
\[
U_{\tau_1} \otimes \cdots \otimes U_{\tau_n} = U_{\tau_1 \otimes \cdots \otimes \tau_n},
\]
where \(I_i\) is the identity map of \(L_i\) into \(L_i\).

**Lemma 2.4.** Let \(L_i\) \((i = 1, 2)\) be nests on separable Hilbert spaces \(H_i\) and \(\tau_i\) \((i = 1, 2)\) be order homomorphisms of \(L_i\) into \(L_i\). Then \(U_{\tau_1 \otimes \tau_2} = F(U_{\tau_1}, U_{\tau_2})\).

**Proof.** Suppose that \(X \in U_{\tau_1 \otimes \tau_2} \subseteq B(H_1 \otimes H_2)\). If \(E_2 \in L_2\) and \(\phi \in B(H_1)_*\), it follows from \([\ref{7}]\) (1.3) that
\[
\tau_2(E_2)R_\phi(X)E_2 = R_\phi((I_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2)) = R_\phi((I_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2)) = R_\phi(X(I_1 \otimes E_2)) = R_\phi(X)E_2.
\]
So \(R_\phi(X) \in U_{\tau_2}\). Similarly, \(L_\phi(X) \in U_{\tau_1}\) for all \(\psi \in B(H_2)_*\). Hence by the definition of \(F(U_{\tau_1}, U_{\tau_2})\), we have \(U_{\tau_1 \otimes \tau_2} \subseteq F(U_{\tau_1}, U_{\tau_2})\).

Conversely, suppose that \(X \in F(U_{\tau_1}, U_{\tau_2})\). If \(E_2 \in L_2\) and \(\phi \in B(H_1)_*\), then \(\tau_2(E_2)R_\phi(X)E_2 = R_\phi(X)E_2\). Thus \(R_\phi((I_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2)) = R_\phi(X(I_1 \otimes E_2))\) for all \(\phi \in B(H_1)_*\). It follows from \([\ref{7}]\) (1.5) that
\[
(I_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2) = X(I_1 \otimes E_2).
\]
Similarly, if \(E_1 \in L_1\), we have that \(X(E_1 \otimes I_2) = (\tau_1(E_1) \otimes I_2)X(E_1 \otimes I_2)\). Therefore,
\[
X(E_1 \otimes E_2) = X(E_1 \otimes I_2)(I_1 \otimes E_2) = (\tau_1(E_1) \otimes I_2)X(I_1 \otimes E_2)(E_1 \otimes I_2) = (\tau_1(E_1) \otimes I_2)(I_1 \otimes \tau_2(E_2))X(E_1 \otimes E_2).
\]
Thus, by virtue of \([\ref{2}]\) Proposition 2.4, it is easy to show that \(XL \subseteq (\tau_1 \otimes \tau_2)(L)\) for each \(L \in L_1 \otimes L_2\). Hence \(X \in U_{\tau_1 \otimes \tau_2}\) and \(U_{\tau_1 \otimes \tau_2} = F(U_{\tau_1}, U_{\tau_2})\).

**Theorem 2.5.** Let \(L_i\) and \(\tau_i\) be as in the preceding lemma. Then \(U_{\tau_1} \overline{\otimes} U_{\tau_2} = U_{\tau_1 \otimes \tau_2}\).

**Proof.** Since every nest is a completely distributive CSL, the theorem follows from Corollary 2.3 and Lemma 2.4, obviously.
Lemma 3.1. For each \( i = 1, \ldots, n \), let \( E_i \in \mathcal{L}_i \) and \( f_i \in E_i \) such that \([([\text{Alg} \mathcal{L}_i] f_i) = E_i\). Then \([([\text{Alg} \mathcal{L}_1] \circ \cdots \circ [\text{Alg} \mathcal{L}_n]) (f_1 \circ \cdots \circ f_n) = [\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n] \subseteq \mathcal{U}_1 \circ \cdots \circ \mathcal{U}_n\).

Proof. Since \( \mathcal{U}_i \circ \text{Alg} \mathcal{L}_1 = \mathcal{U}_1, \mathcal{U}_i \circ \mathcal{U}_1 = [\mathcal{U}_1, \mathcal{U}_i] \). By virtue of [2] Lemma 2.2,

\[
E_1 \circ \cdots \circ E_n = ([\text{Alg} \mathcal{L}_1 \circ \cdots \circ \text{Alg} \mathcal{L}_n] (f_1 \circ \cdots \circ f_n)).
\]

Thus, since \([\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n] (\text{Alg} \mathcal{L}_1 \circ \cdots \circ \text{Alg} \mathcal{L}_n) = \mathcal{U}_1 \circ \cdots \circ \mathcal{U}_n\),

\[
[([\text{Alg} \mathcal{L}_1] \circ \cdots \circ [\text{Alg} \mathcal{L}_n]) (f_1 \circ \cdots \circ f_n)] = [([\text{Alg} \mathcal{L}_1] \circ \cdots \circ [\text{Alg} \mathcal{L}_n]) (f_1 \circ \cdots \circ f_n)].
\]

Hence it suffices to prove that

\[
([\text{Alg} \mathcal{L}_1] \circ \cdots \circ [\text{Alg} \mathcal{L}_n]) (f_1 \circ \cdots \circ f_n) = [\mathcal{U}_1, f_1] \circ \cdots \circ [\mathcal{U}_n, f_n].
\]

If \( g_i \) is any vector in \([\mathcal{U}_1, f_1] \), then \( g_i \) can be norm approximated by vectors of the form \( T_j f_i \), where \( T_j \in \mathcal{U}_j \). Hence \( g_1 \circ \cdots \circ g_n \) can be approximated by vectors of the form \( T_1 f_1 \circ \cdots \circ T_n f_n = (T_1 \circ \cdots \circ T_n) (f_1 \circ \cdots \circ f_n) \). Thus any vector of the form \( g_1 \circ \cdots \circ g_n \) with \( g_i \in [\mathcal{U}_1, f_1] \) lies in \([([\text{Alg} \mathcal{L}_1] \circ \cdots \circ [\text{Alg} \mathcal{L}_n]) (f_1 \circ \cdots \circ f_n)]\).

Since such vectors generate \([\text{Alg} \mathcal{L}_1] \circ \cdots \circ [\text{Alg} \mathcal{L}_n] \circ \text{Alg} \mathcal{L}_n)\), we have \([\mathcal{U}_1, f_1] \circ \cdots \circ [\mathcal{U}_n, f_n] \subseteq \mathcal{U}_1 \circ \cdots \circ \mathcal{U}_n\) and \([\mathcal{U}_1, f_1] \circ \cdots \circ [\mathcal{U}_n, f_n] \subseteq [\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n] (f_1 \circ \cdots \circ f_n)\).

To prove the reverse inequality, for any \( T_i \in \mathcal{U}_i \), we have that

\[
([\mathcal{U}_1, f_1] \circ \cdots \circ [\mathcal{U}_n, f_n]) (T_1 \circ \cdots \circ T_n) (E_1 \circ \cdots \circ E_n) = ([\mathcal{U}_1, f_1] \circ \cdots \circ [\mathcal{U}_n, f_n]) (T_1 \circ \cdots \circ T_n) (E_1 \circ \cdots \circ E_n) = (T_1 f_1) (T_1 \circ \cdots \circ T_n) (E_1 \circ \cdots \circ E_n) = T_1 E_1 \circ \cdots \circ T_n E_n = (T_1 \circ \cdots \circ T_n) (E_1 \circ \cdots \circ E_n).
\]

This shows that

\[
([\mathcal{U}_1, f_1] \circ \cdots \circ [\mathcal{U}_n, f_n]) ([\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n] (E_1 \circ \cdots \circ E_n)) = [([\text{Alg} \mathcal{L}_1] \circ \cdots \circ [\text{Alg} \mathcal{L}_n]) (E_1 \circ \cdots \circ E_n)].
\]

Thus

\[
([\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n] (f_1 \circ \cdots \circ f_n)) = ([\mathcal{U}_1, f_1] \circ \cdots \circ [\mathcal{U}_n, f_n]).
\]

Therefore

\[
([\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n] (f_1 \circ \cdots \circ f_n)) = [\mathcal{U}_1, f_1] \circ \cdots \circ [\mathcal{U}_n, f_n].
\]

This completes the proof. \(\square\)
Theorem 3.2. Let $\mathcal{U}_i (i = 1, \ldots, n)$ be $\sigma$-weakly closed Alg $\mathcal{L}_i$-submodules and $\tau_i(E) = [\mathcal{U}_i E]$ for any $E \in \mathcal{L}_i$. Then $\mathcal{U}_1 \bigotimes \cdots \bigotimes \mathcal{U}_n = \mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$.

Proof. It is obvious that $\mathcal{U}_1 \bigotimes \cdots \bigotimes \mathcal{U}_n$ is a $\sigma$-weakly closed Alg $\mathcal{L}_1 \bigotimes \cdots \bigotimes \mathcal{L}_n$-submodule. By virtue of [2] Theorem 2.6, $\mathcal{U}_1 \bigotimes \cdots \bigotimes \mathcal{U}_n$ is a $\sigma$-weakly closed Alg($\mathcal{L}_1 \bigotimes \cdots \bigotimes \mathcal{L}_n$)-submodule. It follows from [2] Proposition 2.7 that $\mathcal{L}_1 \bigotimes \cdots \bigotimes \mathcal{L}_n$ is a completely distributive CSL. Thus, Theorem H shows that $\mathcal{U}_1 \bigotimes \cdots \bigotimes \mathcal{U}_n$ is determined by the order homomorphism $L \rightarrow [(\mathcal{U}_1 \bigotimes \cdots \bigotimes \mathcal{U}_n)L]$ of $\mathcal{L}_1 \bigotimes \cdots \bigotimes \mathcal{L}_n$ into itself.

Now suppose that $E_i \in \mathcal{L}_i$. For each $i$, choose a vector $v_i \in E_i$ such that $[(\text{Alg} \mathcal{L}_i)v_i] = E_i$ (the proof of the existence of such $v_i$ is routine). It follows from Lemma 3.1 that

$$[(\mathcal{U}_1 \bigotimes \cdots \bigotimes \mathcal{U}_n)(E_1 \otimes \cdots \otimes E_n)] = [\mathcal{U}_1 E_1] \otimes \cdots \otimes [\mathcal{U}_n E_n] = \tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n) = (\tau_1 \otimes \cdots \otimes \tau_n)(E_1 \otimes \cdots \otimes E_n).$$

If $L \in \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$, [2] Proposition 2.4] shows that

$$L = \vee\{E_1 \otimes \cdots \otimes E_n : E_1 \otimes \cdots \otimes E_n \leq L\}.$$

Thus,

$$[(\mathcal{U}_1 \bigotimes \cdots \bigotimes \mathcal{U}_n)L] = \vee\{(\mathcal{U}_1 \bigotimes \cdots \bigotimes \mathcal{U}_n)(E_1 \otimes \cdots \otimes E_n) : E_1 \otimes \cdots \otimes E_n \leq L\} = \vee\{(\tau_1 \otimes \cdots \otimes \tau_n)(E_1 \otimes \cdots \otimes E_n) : E_1 \otimes \cdots \otimes E_n \leq L\} = (\tau_1 \otimes \cdots \otimes \tau_n)(L).$$

Hence $\mathcal{U}_1 \bigotimes \cdots \bigotimes \mathcal{U}_n$ and $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$ are $\sigma$-weakly closed Alg($\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$)-submodules determined by the same order homomorphism. This shows that

$$\mathcal{U}_1 \bigotimes \cdots \bigotimes \mathcal{U}_n = \mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n.$$ 

\[\square\]

Given general order homomorphisms $\phi_i$ from $\mathcal{L}_i$ into $\mathcal{L}_i$, we will consider the relation between $\mathcal{U}_\phi \bigotimes \cdots \bigotimes \mathcal{U}_\phi$ and $\mathcal{U}_\phi \otimes \cdots \otimes \mathcal{U}_\phi$. We need some lemmas at first.

For non-zero vectors $x, y \in \mathcal{H}$, the rank-one operator $xy^*$ is defined by the equation

$$(xy^*)(z) = (z, y)x, \quad \forall z \in \mathcal{H}.$$ 

Lemma 3.3. Suppose that $\mathcal{L}$ is a subspace lattice, and that $\mathcal{U}_\phi$ is the $\sigma$-weakly closed Alg $\mathcal{L}$-submodule determined by an order homomorphism $\phi$ from $\mathcal{L}$ into itself. Then a rank-one operator $xy^* \in \mathcal{U}_\phi$ if and only if there exists an element $N \in \mathcal{L}$ such that $x \in N$ and $y \in \phi_\left(N^\perp\right)$, where $\phi_\left(N^\perp\right) = \vee\{G \in \mathcal{L} : \phi(G) \not\leq N\}$.

Proof. The proof is routine. We leave the details to the interested readers. \[\square\]

Lemma 3.4. Let $\mathcal{L}_i$ be a nest and $\phi_i$ be an order homomorphism from $\mathcal{L}_i$ into itself. Define $\psi_i : I_1 \otimes \cdots \otimes I_i \otimes \cdots \otimes I_n \rightarrow I_1 \otimes \cdots \otimes I_i \otimes \cdots \otimes I_n$ by

$$\psi_i(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n) = I_1 \otimes \cdots \otimes \phi_i(N_i) \otimes \cdots \otimes I_n, \quad \forall N_i \in \mathcal{L}_i.$$ 

Then the rank-one operator $xy^* \in \mathcal{U}_\phi$ if and only if there exists an element $N_i \in \mathcal{L}_i$ such that $x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n$ and $y \in I_1 \otimes \cdots \otimes \phi_\left(N_i^\perp\right)$ $\otimes \cdots \otimes I_n$.
**Proof.** Certainly $\psi_i$ is an order homomorphism from $I_1 \otimes \cdots \otimes I_n$ into itself, and $U_{\psi_i}$ is the $\sigma$-weakly closed $\text{Alg}(I_1 \otimes \cdots \otimes I_n)$-submodule determined by $\psi_i$. By virtue of Lemma 3.3, a rank-one operator $xy^* \in U_{\psi_i}$ if and only if there exists an element $N_i \in L_i$ such that $x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n$ and $y \in \psi_{i\sim}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)^\perp$. In the following, we compute

$$
\psi_{i\sim}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)^\perp.
$$

By the definition of $\psi_{i\sim}$, we have

$$
\psi_{i\sim}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)
= \bigvee\{I_1 \otimes \cdots \otimes G_i \otimes \cdots \otimes I_n : \psi_i(I_1 \otimes \cdots \otimes G_i \otimes \cdots \otimes I_n) \not\geq I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n \}
= \bigvee\{I_1 \otimes \cdots \otimes G_i \otimes \cdots \otimes I_n : \psi_i(G_i) \not\geq I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n \}
= \bigvee\{I_1 \otimes \cdots \otimes \phi_i(N_i) \otimes \cdots \otimes I_n, \forall N_i \in L_i, \text{ and so } \phi_{i\sim}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)^\perp = I_1 \otimes \cdots \otimes \phi_{i\sim}(N_i)^\perp \otimes \cdots \otimes I_n.
$$

**Proposition 3.5.** Let $L_i (i = 1, \ldots, n)$ be nests and $\phi_i$ be order homomorphisms from $L_i$ into itself. Then a rank-one operator $xy^* \in U_{\phi_1 \otimes \cdots \otimes \phi_n}$ if and only if there exist $N_i \in L_i$ such that $x \in N_1 \otimes \cdots \otimes N_n$ and $y \in \phi_{i\sim}(N_1)^\perp \otimes \cdots \otimes \phi_{i\sim}(N_n)^\perp$.

**Proof.** Set $F_i = I_1 \otimes \cdots \otimes L_i \otimes \cdots \otimes I_n$, and define $\psi_i : F_i \to F_i$ by

$$
\psi_i(I_1 \otimes \cdots \otimes I_n \otimes \cdots \otimes I_n) = I_1 \otimes \cdots \otimes \phi_i(N_i) \otimes \cdots \otimes I_n, \forall N_i \in L_i.
$$

Each $\psi_i$ is an order homomorphism from $F_i$ into itself and $U_{\psi_i}$ is the $\sigma$-weakly closed $\text{Alg}(F_i)$-submodules determined by $\psi_i$. Thus we have the equation $U_{\phi_1 \otimes \cdots \otimes \phi_n} = U_{\psi_1} \cap \cdots \cap U_{\psi_n}$. In fact, by virtue of [2, Proposition 2.4],

$$
L = \bigvee\{N_1 \otimes \cdots \otimes N_n : N_1 \otimes \cdots \otimes N_n \leq L \} \text{ for any } L \in L_1 \otimes \cdots \otimes L_n.
$$

Thus it is easy to show that

$$
U_{\phi_1 \otimes \cdots \otimes \phi_n} = \{T \in B(H_1 \otimes \cdots \otimes H_n) : T(N_1 \otimes \cdots \otimes N_n) \subseteq \phi_1(N_1) \otimes \cdots \otimes \phi_n(N_n), \forall N_i \in L_i \},
$$

and so $U_{\phi_1 \otimes \cdots \otimes \phi_n} \subseteq U_{\psi_1} \cap \cdots \cap U_{\psi_n}$. For any $T \in U_{\psi_1} \cap \cdots \cap U_{\psi_n}$, we have that for any $N_i \in L_i, T(N_1 \otimes \cdots \otimes N_n) \subseteq T(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n) \subseteq I_1 \otimes \cdots \otimes \phi_i(N_i) \otimes \cdots \otimes I_n, \forall 1 \leq i \leq n$. Thus $T(N_1 \otimes \cdots \otimes N_n) \subseteq \phi_1(N_1) \otimes \cdots \otimes \phi_n(N_n)$ and $T \in U_{\phi_1 \otimes \cdots \otimes \phi_n}$. Hence $U_{\phi_1 \otimes \cdots \otimes \phi_n} = U_{\psi_1} \cap \cdots \cap U_{\psi_n}$. From Lemma 3.4 it follows that for any $1 \leq i \leq n$, a rank-one operator $xy^* \in U_{\psi_i}$ if and only if there exists $N_i \in L_i$ such that $x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n$ and $y \in I_1 \otimes \cdots \otimes \phi_{i\sim}(N_i)^\perp \otimes \cdots \otimes I_n$. Therefore a rank-one operator $xy^* \in U_{\psi_1} \cap \cdots \cap U_{\psi_n}$ if and only if there exists $N_i \in L_i (1 \leq i \leq n)$ such that $x \in N_1 \otimes \cdots \otimes N_n$ and $y \in \phi_{i\sim}(N_i)^\perp \otimes \cdots \otimes \phi_{i\sim}(N_n)^\perp$. \(\square\)

**Lemma 3.6.** Suppose that $L$ is a subspace lattice and that $U_{\phi}$ is the $\sigma$-weakly closed $\text{Alg} L$-submodule determined by an order homomorphism $\phi$ from $L$ into itself. Then $\tau \leq \phi$ and $\tau_{i\sim} = \phi_{i\sim}$, where $\tau(E) = [U_{\phi} E]$ for any $E \in L$. \(\square\)
Proof. It follows from the definition of $U_\phi$ that
\[ \tau(E) = [U_\phi E] \leq \phi(E) \quad \text{for any } E \in \mathcal{L}. \]
So $\tau \leq \phi$.

Since $\tau \leq \phi$, we have $\tau_\omega \geq \phi_\omega$. So it suffices to show that $\tau_\omega \leq \phi_\omega$. If not, there exists $E \in \mathcal{L}$ such that $\tau_\omega(E) \not\leq \phi_\omega(E)$. It follows from the definition of $\tau_\omega$ that there exists $F \in \mathcal{L}$ such that $\tau(F) \not\leq E$ and $F \not\leq \phi_\omega(E)$. Thus we can choose non-zero vectors $x, y$ such that $x \in E$ and $x \not\in \tau(F)$, $y \in \phi_\omega(E)^\perp$ and $y \not\in F^\perp$. From Lemma 3.3, it follows that $x \otimes y \in U_\phi$. Since $(I - \tau(F))(x \otimes y)F \neq 0$, $x \otimes y \not\in U_\phi$. However it follows from the proof of Theorem H that $U_\tau = U_\phi$. This is a contradiction. Accordingly, $\tau_\omega \leq \phi_\omega$ and $\tau_\omega = \phi_\omega$. $\square$

Now we are in the position to show the general tensor product formula of $\sigma$-weakly closed $\mathcal{L}_i$-submodules.

Theorem 3.7. Let $\mathcal{L}_i$ $(1 = 1, \cdots, n)$ be nests and $\phi_i$ be order homomorphisms from $\mathcal{L}_i$ into itself. Then $U_{\phi_1} \otimes \cdots \otimes U_{\phi_n} = U_{\phi_1 \otimes \cdots \otimes \phi_n}$.

Proof. It follows from Theorem H that $U_{\phi_i} = U_{\tau_i}$, where $\tau_i(E) = [U_{\phi_i} E]$ for any $E \in \mathcal{L}_i$. Thus by virtue of Theorem 3.2, we have that
\[ U_{\phi_1} \otimes \cdots \otimes U_{\phi_n} = U_{\tau_1} \otimes \cdots \otimes U_{\tau_n}. \]
So it suffices to show $U_{\tau_1} \otimes \cdots \otimes U_{\tau_n} = U_{\phi_1 \otimes \cdots \otimes \phi_n}$. Since $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ is a completely distributive CSL ([2, Proposition 2.7]), it follows from [10] Theorem 3 that the rank-one operators of $\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ are $\sigma$-weakly dense in $\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$. So it is routine to show that the linear spans of rank-one operators in $U_{\tau_1} \otimes \cdots \otimes U_{\tau_n}$ and $U_{\phi_1 \otimes \cdots \otimes \phi_n}$ are $\sigma$-weakly dense in $U_{\tau_1} \otimes \cdots \otimes U_{\tau_n}$ and $U_{\phi_1 \otimes \cdots \otimes \phi_n}$ respectively. From Proposition 3.5 and Lemma 3.6, it follows that $U_{\tau_1} \otimes \cdots \otimes U_{\tau_n}$ and $U_{\phi_1 \otimes \cdots \otimes \phi_n}$ have the same rank-one operators. Therefore $U_{\tau_1} \otimes \cdots \otimes U_{\tau_n} = U_{\phi_1 \otimes \cdots \otimes \phi_n}$ and $U_{\phi_1} \otimes \cdots \otimes U_{\phi_n} = U_{\phi_1 \otimes \cdots \otimes \phi_n}$. $\square$

Remark 3.8. Theorem 2.5 is a particular case of Theorem 3.2. In [3], Theorem 2.2 shows that $U_{\mu_i}$ $(i = 1, \cdots, n)$ are reflexive subspaces. Combining the above result, we know that the tensor product of $U_{\mu_i}$ is also reflexive. It is natural to ask whether the tensor product of reflexive subspaces is also reflexive. This seems a challenging problem.

References
σ-WEAKLY CLOSED NEST ALGEBRA SUBMODULES


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