TENSOR PRODUCTS
OF $\sigma$-WEAKLY CLOSED NEST ALGEBRA SUBMODULES

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Abstract. In this paper we prove that for any unital $\sigma$-weakly closed algebra $A$ which is $\sigma$-weakly generated by finite-rank operators in $A$, every $\sigma$-weakly closed $A$-submodule has Property $S_\sigma$. In the case of nest algebras, if $L_1, \cdots, L_n$ are nests, we obtain the following $n$-fold tensor product formula:

$$U_{\phi_1} \otimes \cdots \otimes U_{\phi_n} = U_{\phi_1 \otimes \cdots \otimes \phi_n},$$

where each $U_{\phi_i}$ is the $\sigma$-weakly closed $A_{L_i}$-submodule determined by an order homomorphism $\phi_i$ from $L_i$ into itself.

1. Introduction

One of the central results in the theory of tensor products of von Neumann algebras is Tomita's commutation formula:

$$(1) \quad M' \otimes N' = (M \otimes N)',$$

where $M$ and $N$ are von Neumann algebras. It was observed in [2] that if we let $L_1$ and $L_2$ denote the projection lattices of $M$ and $N$ respectively, then (1) can be rewritten as

$$(2) \quad \text{Alg} L_1 \otimes \text{Alg} L_2 = \text{Alg} (L_1 \otimes L_2).$$

This version of Tomita's theorem makes sense for any pair of reflexive algebras $\text{Alg} L_1$ and $\text{Alg} L_2$. It remains a deep open question whether the tensor product formula (2) is valid for general reflexive algebras, or even general CSL algebras. However, (2) has been verified in a number of special cases ([2], [4], [5], [6], [7]). In particular, it is known that if $L_1$ is a commutative subspace lattice that is either completely distributive [8] or of finite width [4], then (2) is valid for $L_1$ and any subspace lattice $L_2$.

The main purpose of this paper is to study tensor products of $\sigma$-weakly closed submodules of some reflexive algebras (in particular, of nest algebras). Section 1 of this paper is devoted to notation and preliminaries. In Section 2, we make use of slice maps to show that if $A$ is a $\sigma$-weakly closed algebra which is $\sigma$-weakly generated by finite-rank operators in $A$, then every $\sigma$-weakly closed $A$-submodule has Property $S_\sigma$. As a corollary, we obtain $U_{\tau_1} \otimes U_{\tau_2} = U_{\tau_1 \otimes \tau_2}$, where each $U_{\tau_i}$ is

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a $\sigma$-weakly closed Alg $L_{i}$-submodule and $L_{i}$ is a nest. However, the 2-fold tensor product formula cannot be generalized to the $n$-fold formula by induction (see the beginning of Section 3). So in Section 3, we use another method to prove the $n$-fold tensor product formula $U_{\phi_{1}} \otimes \cdots \otimes U_{\phi_{n}} = U_{\phi_{1} \otimes \cdots \otimes \phi_{n}}$, where each $U_{\phi_{i}}$ is a $\sigma$-weakly closed Alg $L_{i}$-submodule and $L_{i}$ is a nest. The key to this proof is [3 Theorem 2] and [2 Proposition 2.4].

In this paper, all Hilbert spaces will be separable. Let $B(H)$ be the algebra of bounded operators on $H$ and $F(H)$ be the set of finite-rank operators on $H$. A sublattice $L$ of the projection lattice of $B(H)$ is said to be a subspace lattice if it contains 0 and 1 and is strongly closed, where we identify projections with their ranges. If the elements of $L$ pairwise commute, $L$ is a commutative subspace lattice (CSL). A nest is a totally ordered subspace lattice. If $L$ is a subspace lattice, Alg $L$ denotes the set of operators in $B(H)$ that leave the elements of $L$ invariant. Note that Alg $L$ is a $\sigma$-weakly closed subalgebra of $B(H)$. If $L$ is a CSL, Alg $L$ is said to be a CSL algebra. If $L$ is a nest, Alg $L$ is said to be a nest algebra.

If $A$ is a subset of $B(H)$, then Lat $A$, the set of projections left invariant by each element of $A$, is a subspace lattice. A subalgebra $A$ of $B(H)$ is reflexive if $A = \text{Alg Lat } A$. The reflexive algebras are precisely the algebras of the form Alg $L$, where $L$ is a subspace lattice. If $L_{i} \subseteq B(H_{i})$ ($i = 1, \cdots, n$) are subspace lattices, $L = \bigotimes_{i=1}^{n} L_{i}$ is the subspace lattice in $B(H_{1} \otimes \cdots \otimes H_{n})$ generated by $\{P_{1} \otimes \cdots \otimes P_{n} : P_{i} \in L_{i}, i = 1, \cdots, n\}$. If $S_{i} \subseteq B(H_{i})$ ($i = 1, \cdots, n$) are $\sigma$-weakly closed subspace lattices, then $S_{i} \otimes \cdots \otimes S_{n}$ denotes the $\sigma$-weakly closed linear span of $\{S_{1} \otimes \cdots \otimes S_{n} : S_{i} \in S_{i}\}$ in $B(H_{1} \otimes \cdots \otimes H_{n})$.

The main technical tool in Section 2 is the use of slice maps. Slice maps were introduced by Tomiyama in [11] and have been used extensively in the study of tensor products of $C^*$-algebras and tensor products of von Neumann algebras. We recall some definitions and results from [7] and refer the reader to [7] for further results and motivation. If $M$ and $N$ are von Neumann algebras, and $\phi$ is in the predual $M_{*}$ of $M$, then the right slice map $R_{\phi}$ is the unique $\sigma$-weakly continuous linear map from $M \overline{\otimes} N \to N$ such that

$$\langle X, \phi \otimes \psi \rangle = \langle R_{\phi}(X), \psi \rangle, \quad \forall X \in M \overline{\otimes} N, \psi \in N_{*}.$$  

If $X = A \otimes B$ ($A \in M, B \in N$), then $R_{\phi}(X) = \langle A, \phi \rangle B$. The left slice map $L_{\psi} : M \overline{\otimes} N \to M, \psi \in N_{*}$, is similarly defined. If $S \subseteq M$ and $T \subseteq N$ are $\sigma$-weakly closed subspaces, let

$$F(S, T) = \{X \in M \overline{\otimes} N : R_{\phi}(X) \in T \text{ and } L_{\psi}(X) \in S, \quad \forall \phi \in M_{*}, \psi \in N_{*}\}.$$  

As noted in [7], we can replace $M$ by $B(H_{1})$ and $N$ by $B(H_{2})$ without affecting $F(S, T)$. Moreover $S \overline{\otimes} T \subseteq F(S, T)$. Tomiyama proved in [12] that if $S$ and $T$ are von Neumann algebras, then

$$S \overline{\otimes} T = F(S, T).$$  

His proof uses Tomita’s theorem and, in fact, Tomita’s theorem (1) is equivalent to the validity of (3) for von Neumann algebras. Hence (3) can be considered as a possible general version of Tomita’s theorem for $\sigma$-weakly closed subspaces.

A $\sigma$-weakly closed subspace $S \subseteq B(H)$ is said to have Property $S_{\phi}$ if

$$\{X \in S \overline{\otimes} N : R_{\phi}(X) \in T \text{ for all } \phi \in B(H)_{*}\} = S \overline{\otimes} T$$
for all pairs \( \{ T, N \} \), where \( T \) is a \( \sigma \)-weakly closed subspace of a von Neumann algebra \( N \). \( S \) has Property \( S_\sigma \) if and only if \( F(S, T) = S \otimes T \) for all \( \sigma \)-weakly closed subspaces \( T \) of each von Neumann algebra \( N \) ([7, Remark 1.5]).

2. Property \( S_\sigma \)

Let \( A \) be a reflexive subalgebra of \( B(H) \). Suppose that \( E \rightarrow \tau(E) \) is an order homomorphism of \( \text{Lat} A \) into itself (i.e., \( E \leq F \) implies \( \tau(E) \leq \tau(F) \)). Then the set \( U = \{ T \in B(H) : (I - \tau(E))TE = 0, \forall E \in \text{Lat} A \} \) is clearly a \( \sigma \)-weakly closed \( A \)-submodule of \( B(H) \). We denote \( U \) by \( U_\tau \).

Erdos and Power in [1] proved that any \( \sigma \)-weakly closed \( A \)-submodule of \( B(H) \) for a nest algebra \( A \) is of the above form. Here the following result is due to Han Deguang [3]:

**Theorem H.** Let \( A \) be a unital \( \sigma \)-weakly closed subalgebra which is \( \sigma \)-weakly generated by rank-one operators in \( A \), and let \( U \) be a \( \sigma \)-weakly closed \( A \)-submodule of \( B(H) \). Then \( U \) has the form

\[
U = \{ T \in B(H) : (I - \tau(E))TE = 0, \forall E \in \text{Lat} A \},
\]

where \( E \rightarrow \tau(E) = [UE] \) is an order homomorphism of \( \text{Lat} A \) into itself.

**Theorem 2.1.** Let \( A \) be a unital \( \sigma \)-weakly closed subalgebra of \( B(H) \) with the property that the finite-rank operators of \( A \) are \( \sigma \)-weakly dense in \( A \). Then every \( \sigma \)-weakly closed \( A \)-submodule has Property \( S_\sigma \).

**Proof.** Suppose that \( U \) is a \( \sigma \)-weakly closed \( A \)-submodule. Let \( T \) be a \( \sigma \)-weakly closed subspace of a von Neumann algebra \( N \), and suppose that \( X \in U \otimes N \) and \( R_\phi(X) \in T \) for all \( \phi \in B(H)_+ \). It suffices to show that \( X \in U \otimes T \). Let \( \pi \) be the normal \(*\)-isomorphism of \( B(H) \) into \( B(H) \otimes N \) defined by \( \pi(A) = A \otimes I \) for \( A \in B(H) \). If \( F_1, F_2 \in A \cap F(H) \) and \( \phi \in B(H)_+ \), a routine calculation shows that \( R_\phi(\pi(F_1)X\pi(F_2)) = R_{F_2 \phi F_1}(X) \), where \( F_2 \phi F_1 \in B(H)_+ \) is defined by \( (A, F_2 \phi F_1) = (F_1 AF_2, \phi) \), \( A \in B(H) \). Hence \( R_\phi(\pi(F_1)X\pi(F_2)) \) is in \( T \) for all \( \phi \in B(H)_+ \). Since \( \pi(F_1)(U \otimes N)\pi(F_2) = F_1 UF_2 \otimes N \) and \( F_1 UF_2 \) has Property \( S_\sigma \) by [7, Proposition 1.7], \( \pi(F_1)X\pi(F_2) \) is in \( F_1 UF_2 \otimes T \). But \( F_1 UF_2 \subseteq U \); thus \( \pi(F_1)X\pi(F_2) \in U \otimes T \). Let \( \{ F_\alpha \} \) be a net in \( A \cap F(H) \) converging \( \sigma \)-weakly to the identity map \( I \). Then \( \pi(F_\alpha)X\pi(F) \) converges \( \sigma \)-weakly to \( X\pi(F) \) for all \( F \in A \cap F(H) \), and so \( X\pi(F) \in U \otimes T \) for all \( F \in A \cap F(H) \). Finally, \( X\pi(F_\alpha) \) converges \( \sigma \)-weakly to \( X \), and so \( X \in U \otimes T \). Hence \( U \) has Property \( S_\sigma \). \( \square \)

It is known from [11] that a commutative subspace lattice \( L \) is completely distributive if and only if the rank-one subalgebra of \( \text{Alg} L \) is \( \sigma \)-weakly dense in \( \text{Alg} L \). Thus we have the following result:

**Corollary 2.2.** If \( L \) is a completely distributive CSL, then every \( \sigma \)-weakly closed \( \text{Alg} L \)-submodule has Property \( S_\sigma \).

If \( L \) is a completely distributive CSL, it follows from Theorem H that every \( \sigma \)-weakly closed \( \text{Alg} L \)-submodule is of the form \( U_\tau \), where \( E \rightarrow \tau(E) \) is an order homomorphism of \( L \) into itself.

**Corollary 2.3.** Suppose that \( L_i \) (\( i = 1, 2 \)) are completely distributive CSLs, and that \( U_{\tau_i} \) (\( i = 1, 2 \)) are \( \sigma \)-weakly closed \( \text{Alg} L_i \)-submodules respectively. Then \( U_{\tau_1} \overline{\sigma U_{\tau_2}} = F(U_{\tau_1}, U_{\tau_2}) \).
Proof. A σ-weakly closed subspace $S$ has Property $S_\sigma$ if and only if $\mathcal{S} \overline{\mathcal{T}} = F(S, T)$ for all σ-weakly closed subspaces $T$ ([2] Remark 1.5). Thus the corollary follows from Corollary 2.2. □

In the case of nest algebras, we can say more about tensor products of σ-weakly closed nest algebra submodules. In the rest of this paper, we suppose that $L_i$ ($i = 1, 2, \cdots, n$) are nests on separable complex Hilbert spaces $\mathcal{H}_i$ and $\tau_i$ are order homomorphisms of $L_i$ into $L_i$.

If $L \in L_1 \otimes \cdots \otimes L_n$, it follows from [2] Proposition 2.4 that

$$L = \vee \{E_1 \otimes \cdots \otimes E_n : E_1 \otimes \cdots \otimes E_n \leq L\}.$$ 

Thus we can define

$$(\tau_1 \otimes \cdots \otimes \tau_n)(L) = \vee \{\tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n) : E_1 \otimes \cdots \otimes E_n \leq L\}.$$ 

Obviously, $(\tau_1 \otimes \cdots \otimes \tau_n)(E_1 \otimes \cdots \otimes E_n) = \tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n)$. Thus $\tau_1 \otimes \cdots \otimes \tau_n$ is a well-defined order homomorphism of $L_1 \otimes \cdots \otimes L_n$ into itself and $U_{\tau_1 \otimes \cdots \otimes \tau_n}$ is a σ-weakly closed $\text{Alg}(L_1 \otimes \cdots \otimes L_n)$-submodule. Hence the equality $\text{Alg}L_1 \otimes \cdots \otimes \text{Alg}L_n = \text{Alg}(L_1 \otimes \cdots \otimes L_n)$ of [2] Theorem 2.6] can be rewritten as

$$U_{\tau_1} \otimes \cdots \otimes U_{\tau_n} = U_{\tau_1 \otimes \cdots \otimes \tau_n},$$

where $I_i$ is the identity map of $L_i$ into $L_i$.

**Lemma 2.4.** Let $L_i$ ($i = 1, 2$) be nests on separable Hilbert spaces $\mathcal{H}_i$ and $\tau_i$ ($i = 1, 2$) be order homomorphisms of $L_i$ into $L_i$. Then $U_{\tau_1 \otimes \tau_2} = F(U_{\tau_1}, U_{\tau_2})$.

**Proof.** Suppose that $X \in U_{\tau_1 \otimes \tau_2} \subseteq \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. If $E_2 \in L_2$ and $\phi \in \mathcal{B}(\mathcal{H}_1)_*$, it follows from [7] (1.3) that

$$\tau_2(E_2)R_\phi(X)E_2 = R_\phi((I_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2)) = R_\phi((I_1 \otimes \tau_2(E_2))(\tau_1(I_1) \otimes \tau_2(E_2))X(I_1 \otimes E_2))$$

$$= R_\phi((\tau_1(I_1) \otimes \tau_2(E_2))X(I_1 \otimes E_2)) = R_\phi(X(I_1 \otimes E_2)).$$

So $R_\phi(X) \in U_{\tau_2}$. Similary, $L_\phi(X) \in U_{\tau_1}$ for all $\psi \in \mathcal{B}(\mathcal{H}_2)_*$. Hence by the definition of $F(U_{\tau_1}, U_{\tau_2})$, we have $U_{\tau_1 \otimes \tau_2} \subseteq F(U_{\tau_1}, U_{\tau_2})$.

Conversely, suppose that $X \in F(U_{\tau_1}, U_{\tau_2})$. If $E_2 \in L_2$ and $\phi \in \mathcal{B}(\mathcal{H}_1)_*$, then $\tau_2(E_2)R_\phi(X)E_2 = R_\phi(X)E_2$. Thus $R_\phi((I_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2)) = R_\phi(X(I_1 \otimes E_2))$ for all $\phi \in \mathcal{B}(\mathcal{H}_1)_*$. It follows from [7] (1.5) that

$$(I_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2) = X(I_1 \otimes E_2).$$

Similarly, if $E_1 \in L_1$, we have that $X(E_1 \otimes I_2) = (\tau_1(E_1) \otimes I_2)X(E_1 \otimes I_2)$. Therefore,

$$X(E_1 \otimes E_2) = X(E_1 \otimes I_2)(I_1 \otimes E_2)$$

$$= (\tau_1(E_1) \otimes I_2)(I_1 \otimes E_2)(E_1 \otimes I_2) = (\tau_1(E_1) \otimes I_2)(I_1 \otimes \tau_2(E_2))X(E_1 \otimes E_2)$$

$$= (\tau_1(E_1) \otimes \tau_2(E_2))X(E_1 \otimes E_2).$$

Thus, by virtue of [2] Proposition 2.4, it is easy to show that $XL \subseteq (\tau_1 \otimes \tau_2)(L)$ for each $L \in L_1 \otimes L_2$. Hence $X \in U_{\tau_1 \otimes \tau_2}$ and $U_{\tau_1 \otimes \tau_2} = F(U_{\tau_1}, U_{\tau_2})$. □

**Theorem 2.5.** Let $L_i$ and $\tau_i$ be as in the preceding lemma. Then $U_{\tau_1} \overline{\otimes} U_{\tau_2} = U_{\tau_1 \otimes \tau_2}$.

**Proof.** Since every nest is a completely distributive CSL, the theorem follows from Corollary 2.3 and Lemma 2.4, obviously. □
3. The \( n \)-fold tensor product formula

Since \( L_1 \otimes L_2 \) is not totally ordered in general, we cannot deduce the tensor product formula \( \mathcal{U}_{r_1} \otimes \mathcal{U}_{r_2} = \mathcal{U}_{r_1 \otimes r_2} \) by

\[
\mathcal{U}_{r_1} \otimes \mathcal{U}_{r_2} = \mathcal{U}_{(r_1 \otimes r_2)} = \mathcal{U}_{r_1 \otimes r_2} \otimes \mathcal{U}_{r_3}.
\]

(In order to use Theorem 2.5, the second equality needs the totally ordered property of \( L_1 \otimes L_2 \).) So we cannot generalize Theorem 2.5 to \( \mathcal{U} \)-fold tensor products for \( n > 2 \) by induction. In this section, instead of the slice maps, we shall use Theorem H to prove the \( \mathcal{U} \)-fold tensor product formula. Let \( L_i \) \((i = 1, \cdots, n)\) be nests and let \( \mathcal{U}_i \) be \( \mathcal{U} \)-weakly closed \( Alg L_i \)-submodules. From Theorem H, it follows from \( \mathcal{U}_i = \mathcal{U}_{r_i} \), where \( r_i(E) = [\mathcal{U}_i E] \) for any \( E \in L_i \). In the rest of this section, we always use \( r_i \) to denote these special order homomorphisms.

**Lemma 3.1.** For each \( i = 1, \cdots, n \), let \( E_i \in L_i \) and \( f_i \in E_i \) such that \([Alg L_i f_i] = E_i\). Then \([Alg L_i \otimes \cdots \otimes L_n f_n] = E_1 \otimes \cdots \otimes E_n\).

**Proof.** Since \( \mathcal{U}_i \cdot Alg L_i = \mathcal{U}_{r_i} \), \([\mathcal{U}_i, f] = [\mathcal{U}_i f] \). By virtue of \([2]\) Lemma 2.2,

\[
E_1 \otimes \cdots \otimes E_n = [(Alg L_1 \otimes \cdots \otimes Alg L_n)(f_1 \otimes \cdots \otimes f_n)].
\]

Thus, \((\mathcal{U}_{r_1} \otimes \cdots \otimes \mathcal{U}_{r_n})(Alg L_1 \otimes \cdots \otimes Alg L_n) = \mathcal{U}_{r_1} \otimes \cdots \otimes \mathcal{U}_{r_n},

\[
[\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n](f_1 \otimes \cdots \otimes f_n) = [\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n f_n] = [\mathcal{U}_1 f_1] \otimes \cdots \otimes [\mathcal{U}_n f_n].
\]

Hence it suffices to prove that

\[
[(\mathcal{U}_{r_1} \otimes \cdots \otimes \mathcal{U}_{r_n})(f_1 \otimes \cdots \otimes f_n)] = [\mathcal{U}_1 f_1] \otimes \cdots \otimes [\mathcal{U}_n f_n].
\]

If \( g_i \) is any vector in \([\mathcal{U}_i f_i] \), then \( g_i \) can be norm approximated by vectors of the form \( T_i f_i \), where \( T_i \in \mathcal{U}_{r_i} \). Hence \( g_1 \otimes \cdots \otimes g_n \) can be approximated by vectors of the form \( T_1 f_1 \otimes \cdots \otimes T_n f_n = (T_1 \otimes \cdots \otimes T_n)(f_1 \otimes \cdots \otimes f_n) \). Thus any vector of the form \( g_1 \otimes \cdots \otimes g_n \) with \( g_i \in [\mathcal{U}_i f_i] \) lies in \([\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n f_n]\).

Since such vectors generate \([\mathcal{U}_1 f_1] \otimes \cdots \otimes [\mathcal{U}_n f_n] \), we have \([\mathcal{U}_1 f_1] \otimes \cdots \otimes [\mathcal{U}_n f_n] \subseteq [\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n f_n]\).

To prove the reverse inequality, for any \( T_i \in \mathcal{U}_{r_i} \), we have that

\[
(\mathcal{U}_{r_1} f_1) \otimes \cdots \otimes (\mathcal{U}_{r_n} f_n) = (\mathcal{U}_1 f_1) \otimes \cdots \otimes (\mathcal{U}_n f_n) = (f_1 \otimes \cdots \otimes f_n) = T_1 E_1 \otimes \cdots \otimes T_n E_n = (T_1 \otimes \cdots \otimes T_n)(E_1 \otimes \cdots \otimes E_n).
\]

This shows that

\[
[\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n f_n] = [\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n](E_1 \otimes \cdots \otimes E_n) = [\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n](f_1 \otimes \cdots \otimes f_n).
\]

Thus

\[
[\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n f_n] = [\mathcal{U}_1 f_1] \otimes \cdots \otimes [\mathcal{U}_n f_n].
\]

Therefore

\[
[\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n f_n] = [\mathcal{U}_1 f_1] \otimes \cdots \otimes [\mathcal{U}_n f_n].
\]

This completes the proof. \( \square \)
**Theorem 3.2.** Let $\mathcal{U}_i$ ($i = 1, \cdots, n$) be $\sigma$-weakly closed $\text{Alg} \mathcal{L}_i$-submodules and $\tau_i(E) = [\mathcal{U}_i E]$ for any $E \in \mathcal{L}_i$. Then $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n = \mathcal{U}_{1 \otimes \cdots \otimes n}$.

**Proof.** It is obvious that $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$ is a $\sigma$-weakly closed $\text{Alg} \mathcal{L}_1 \otimes \cdots \otimes \text{Alg} \mathcal{L}_n$-submodule. By virtue of [2] Theorem 2.6, $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$ is a $\sigma$-weakly closed $\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$-submodule. It follows from [2] Proposition 2.7 that $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ is a completely distributive CSL. Thus, Theorem H shows that $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$ is determined by the order homomorphism $L \to ([\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n]L)$ of $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ into itself.

Now suppose that $E_i \in \mathcal{L}_i$. For each $i$, choose a vector $v_i \in E_i$ such that $[(\text{Alg} \mathcal{L}_i)v_i] = E_i$ (the proof of the existence of such $v_i$ is routine). It follows from Lemma 3.1 that

$$
([\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n](E_1 \otimes \cdots \otimes E_n)) = [\mathcal{U}_1 E_1] \otimes \cdots \otimes [\mathcal{U}_n E_n] = \tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n) = (\tau_1 \otimes \cdots \otimes \tau_n)(E_1 \otimes \cdots \otimes E_n).
$$

If $L \subseteq \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$, [2] Proposition 2.4 shows that

$$
L = \lor\{E_1 \otimes \cdots \otimes E_n \mid E_1 \otimes \cdots \otimes E_n \leq L\}.
$$

Thus,

$$
[\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n]L = \lor\{([\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n](E_1 \otimes \cdots \otimes E_n)) \mid E_1 \otimes \cdots \otimes E_n \leq L\} = \lor\{(\tau_1 \otimes \cdots \otimes \tau_n)(E_1 \otimes \cdots \otimes E_n) \mid E_1 \otimes \cdots \otimes E_n \leq L\} = (\tau_1 \otimes \cdots \otimes \tau_n)(L).
$$

Hence $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$ and $\mathcal{U}_{1 \otimes \cdots \otimes n}$ are $\sigma$-weakly closed $\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$-submodules determined by the same order homomorphism. This shows that

$$
\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n = \mathcal{U}_{1 \otimes \cdots \otimes n}.
$$

\[\square\]

Given general order homomorphisms $\phi_i$ from $\mathcal{L}_i$ into $\mathcal{L}_i$, we will consider the relation between $\mathcal{U}_\phi \otimes \cdots \otimes \mathcal{U}_{\phi_n}$ and $\mathcal{U}_{\phi_1 \otimes \cdots \otimes n}$. We need some lemmas at first.

For non-zero vectors $x, y \in \mathcal{H}$, the rank-one operator $xy^*$ is defined by the equation

$$(xy^*)(z) = (z, y)x, \quad \forall z \in \mathcal{H}.$$

**Lemma 3.3.** Suppose that $\mathcal{L}$ is a subspace lattice, and that $\mathcal{U}_\phi$ is the $\sigma$-weakly closed $\text{Alg} \mathcal{L}$-submodule determined by an order homomorphism $\phi$ from $\mathcal{L}$ into itself. Then a rank-one operator $xy^* \in \mathcal{U}_\phi$ if and only if there exists an element $N \in \mathcal{L}$ such that $x \in N$ and $y \in \phi_\sim(N)^+$, where $\phi_\sim(N) = \lor\{G \in \mathcal{L} : \phi(G) \geq N\}$.

**Proof.** The proof is routine. We leave the details to the interested readers. \[\square\]

**Lemma 3.4.** Let $\mathcal{L}_i$ be a nest and $\phi_i$ be an order homomorphism from $\mathcal{L}_i$ into itself. Define $\psi_i : I_1 \otimes \cdots \otimes I_i \otimes \cdots \otimes I_n \to I_1 \otimes \cdots \otimes I_i \otimes \cdots \otimes I_n$ by

$$
\psi_i(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n) = I_1 \otimes \cdots \otimes \phi_i(N_i) \otimes \cdots \otimes I_n, \quad \forall N_i \in \mathcal{L}_i.
$$

Then the rank-one operator $xy^* \in \mathcal{U}_{\phi_i}$ if and only if there exists an element $N_i \in \mathcal{L}_i$ such that $x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n$ and $y \in I_1 \otimes \cdots \otimes \phi_{i\sim}(N_i)^+ \otimes \cdots \otimes I_n$.\[\square\]
\textbf{Proof}. Certainly \( \psi_i \) is an order homomorphism from \( I_1 \otimes \cdots \otimes \mathcal{L}_i \otimes \cdots \otimes I_n \) into itself, and \( \mathcal{U}_{\psi_i} \) is the \( \sigma \)-weakly closed \( \text{Alg}(I_1 \otimes \cdots \otimes \mathcal{L}_i \otimes \cdots \otimes I_n) \)-submodule determined by \( \psi_i \). By virtue of Lemma 3.3, a rank-one operator \( xy^* \in \mathcal{U}_{\psi_i} \) if and only if there exists an element \( N_i \in \mathcal{L}_i \) such that \( x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n \) and \( y \in \psi_{i \sim}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)^\perp \). In the following, we compute
\[ \psi_{i \sim}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)^\perp. \]

By the definition of \( \psi_{i \sim} \), we have
\begin{align*}
\psi_{i \sim}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n) &= \{ I_1 \otimes \cdots \otimes G_i \otimes \cdots \otimes I_n : \psi_i(I_1 \otimes \cdots \otimes G_i \otimes \cdots \otimes I_n) \not\geq I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n \} \\
&= \{ I_1 \otimes \cdots \otimes G_i \otimes \cdots \otimes I_n : I_1 \otimes \cdots \otimes \psi_i(G_i) \otimes \cdots \otimes I_n \not\geq I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n \} \\
&= \{ I_1 \otimes \cdots \otimes (\psi_i(G_i) \not\geq N_i) \} \\
&= I_1 \otimes \cdots \otimes (\forall \in \mathcal{L}_i) \psi_{i \sim}(N_i) \otimes \cdots \otimes I_n,
\end{align*}
and so \( \phi_{i \sim}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)^\perp = I_1 \otimes \cdots \otimes \phi_{i \sim}(N_i)^\perp \otimes \cdots \otimes I_n. \quad \square
\]

\textbf{Proposition 3.5}. Let \( \mathcal{L}_i \) \((i = 1, \ldots, n)\) be nests and \( \phi_i \) be order homomorphisms from \( \mathcal{L}_i \) into itself. Then a rank-one operator \( xy^* \in \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n} \) if and only if there exist \( N_i \in \mathcal{L}_i \) such that \( x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n \) and \( y \in \phi_{i \sim}(N_1)^\perp \otimes \cdots \otimes \phi_{n \sim}(N_n)^\perp \).

\textbf{Proof}. Set \( \mathcal{F}_i = I_1 \otimes \cdots \otimes \mathcal{L}_i \otimes \cdots \otimes I_n \), and define \( \psi : \mathcal{F}_i \rightarrow \mathcal{F}_i \) by
\[ \psi(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n) = I_1 \otimes \cdots \otimes \phi_i(N_i) \otimes \cdots \otimes I_n, \quad \forall N_i \in \mathcal{L}_i. \]
Each \( \psi_i \) is an order homomorphism from \( \mathcal{F}_i \) into itself and \( \mathcal{U}_{\psi_i} \) is the \( \sigma \)-weakly closed \( \text{Alg}(\mathcal{F}_i) \)-submodules determined by \( \psi_i \). Thus we have the equation \( \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n} = \mathcal{U}_{\psi_1} \cap \cdots \cap \mathcal{U}_{\psi_n} \). In fact, by virtue of [2 Proposition 2.4],
\[ L = \{ N_1 \otimes \cdots \otimes N_n : N_1 \otimes \cdots \otimes N_n \leq L \} \quad \text{for any} \quad L \in \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n. \]
Thus it is easy to show that
\[ \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n} = \{ T \in \mathcal{B}(H_1 \otimes \cdots \otimes H_n) : T(N_1 \otimes \cdots \otimes N_n) \leq \phi_1(N_1) \otimes \cdots \otimes \phi_n(N_n), \forall N_i \in \mathcal{L}_i \}, \]
and so \( \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n} \subseteq \mathcal{U}_{\psi_1} \cap \cdots \cap \mathcal{U}_{\psi_n} \). For any \( T \in \mathcal{U}_{\psi_1} \cap \cdots \cap \mathcal{U}_{\psi_n} \), we have that for any \( N_i \in \mathcal{L}_i \),
\[ T(N_1 \otimes \cdots \otimes N_n) \subseteq T(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n) \subseteq I_1 \otimes \cdots \otimes \phi_i(N_i) \otimes \cdots \otimes I_n, \quad \forall 1 \leq i \leq n. \]
Thus \( T(N_1 \otimes \cdots \otimes N_n) \subseteq \phi_1(N_1) \otimes \cdots \otimes \phi_n(N_n) \) and \( T \in \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n} \). Hence \( \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n} = \mathcal{U}_{\psi_1} \cap \cdots \cap \mathcal{U}_{\psi_n} \). From Lemma 3.4 it follows that for any \( 1 \leq i \leq n \), a rank-one operator \( xy^* \in \mathcal{U}_{\psi_i} \) if and only if there exists \( N_i \in \mathcal{L}_i \) such that \( x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n \) and \( y \in \phi_{i \sim}(N_i)^\perp \otimes \cdots \otimes I_n \). Therefore a rank-one operator \( xy^* \in \mathcal{U}_{\psi_i} \cap \cdots \cap \mathcal{U}_{\psi_n} \) if and only if there exists \( N_i \in \mathcal{L}_i \) \((1 \leq i \leq n)\) such that \( x \in N_1 \otimes \cdots \otimes N_n \) and \( y \in \phi_{i \sim}(N_i)^\perp \otimes \cdots \otimes \phi_{n \sim}(N_n)^\perp \). \quad \square

\textbf{Lemma 3.6}. Suppose that \( \mathcal{L} \) is a subspace lattice and that \( \mathcal{U}_\phi \) is the \( \sigma \)-weakly closed \( \text{Alg} \mathcal{L} \)-submodule determined by an order homomorphism \( \phi \) from \( \mathcal{L} \) into itself. Then \( \tau \leq \phi \) and \( \tau_{\sim} = \phi_{\sim} \), where \( \tau(E) = [\mathcal{U}_\phi E] \) for any \( E \in \mathcal{L} \).
Proof. It follows from the definition of \( \mathcal{U}_\phi \) that
\[
\tau(E) = [\mathcal{U}_\phi E] \leq \phi(E) \quad \text{for any } E \in \mathcal{L}.
\]
So \( \tau \leq \phi \).

Since \( \tau \leq \phi \), we have \( \tau_\omega \geq \phi_\omega \). So it suffices to show that \( \tau_\omega \leq \phi_\omega \). If not, there exists \( E \in \mathcal{L} \) such that \( \tau_\omega(E) \notin \phi_\omega(E) \). It follows from the definition of \( \tau_\omega \) that there exists \( F \in \mathcal{L} \) such that \( \tau(F) \nleq E \) and \( F \nleq \phi(E) \). Thus we can choose non-zero vectors \( x, y \) such that \( x \in E \) and \( x \notin \tau(F) \), \( y \in \phi(E)^\perp \) and \( y \notin F^\perp \). From Lemma 3.3, it follows that \( x \otimes y \in \mathcal{U}_\phi \). Since \( (1 - \tau(F))(x \otimes y)F \neq 0 \), \( x \otimes y \notin \mathcal{U}_\tau \). However it follows from the proof of Theorem H that \( \mathcal{U}_\tau = \mathcal{U}_\phi \). This is a contradiction. Accordingly, \( \tau_\omega \leq \phi_\omega \). \( \square \)

Now we are in the position to show the general tensor product formula of \( \sigma \)-weakly closed \( \mathcal{L}_i \)-submodules.

Theorem 3.7. Let \( \mathcal{L}_i \ (1 = 1, \cdots, n) \) be nests and \( \phi_i \) be order homomorphisms from \( \mathcal{L}_i \) into itself. Then \( \mathcal{U}_{\phi_1} \otimes \cdots \otimes \mathcal{U}_{\phi_n} = \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n} \).

Proof. It follows from Theorem H that \( \mathcal{U}_{\phi_i} = \mathcal{U}_{\tau_i} \), where \( \tau_i(E) = [\mathcal{U}_{\phi_i} E] \) for any \( E \in \mathcal{L}_i \). Thus by virtue of Theorem 3.2, we have that
\[
\mathcal{U}_{\phi_1} \otimes \cdots \otimes \mathcal{U}_{\phi_n} = \mathcal{U}_{\tau_1} \otimes \cdots \otimes \mathcal{U}_{\tau_n}.
\]
So it suffices to show \( \mathcal{U}_{\tau_1} \otimes \cdots \otimes \tau_n = \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n} \). Since \( \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n \) is a completely distributive CSL (\cite{2}, Proposition 2.7), it follows from \cite{10} Theorem 3 that the rank-one operators of \( \text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) \) are \( \sigma \)-weakly dense in \( \text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) \). So it is routine to show that the linear spans of rank-one operators in \( \mathcal{U}_{\tau_1} \otimes \cdots \otimes \tau_n \) and \( \mathcal{U}_{\phi_1} \otimes \cdots \phi_n \) are \( \sigma \)-weakly dense in \( \mathcal{U}_{\tau_1} \otimes \cdots \otimes \tau_n \) and \( \mathcal{U}_{\phi_1} \otimes \cdots \phi_n \) respectively. From Proposition 3.5 and Lemma 3.6, it follows that \( \mathcal{U}_{\tau_1} \otimes \cdots \otimes \tau_n \) and \( \mathcal{U}_{\phi_1} \otimes \cdots \phi_n \) have the same rank-one operators. Therefore \( \mathcal{U}_{\tau_1} \otimes \cdots \otimes \tau_n = \mathcal{U}_{\phi_1} \otimes \cdots \phi_n \).

\( \square \)

Remark 3.8. Theorem 2.5 is a particular case of Theorem 3.2. In \cite{3}, Theorem 2.2 shows that \( \mathcal{U}_{\tau_i} \ (i = 1, \cdots, n) \) are reflexive subspaces. Combining the above result, we know that the tensor product of \( \mathcal{U}_{\tau} \) is also reflexive. It is natural to ask whether the tensor product of reflexive subspaces is also reflexive. This seems a challenging problem.
σ-WEEKLY CLOSED NEST ALGEBRA SUBMODULES


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