

ON D. PETERSON'S COMPARISON FORMULA FOR GROMOV-WITTEN INVARIANTS OF G/P

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ABSTRACT. We prove a formula of Dale Peterson comparing Gromov-Witten (GW) invariants of G/P to those of G/B using canonical reductions of bundles.

An unpublished formula of Dale Peterson describes how 3-point, genus 0 Gromov-Witten invariants of G/P compare with those of G/B . Our purpose in this note is to describe an explanation, and in particular a proof, of this formula using ideas from moduli of principal bundles over curves. The quantum product with respect to the Schubert basis in G/B can be computed either recursively using Peterson's quantum Chevalley formula, proved in [7], or using polynomial representatives for the Schubert classes in the Givental-Kim presentation of the small quantum cohomology [5], [13]. Together these results give a practicable method for computing the small quantum cohomology in the Schubert basis for arbitrary G/P , although there are much more effective methods in many special cases [2, 3, 4, 11, 12, 15].

The idea of the proof is the following. Given a morphism φ of \mathbb{P}^1 to a partial flag variety X of a certain degree d , we can pull back the tautological bundles over X . Giving a lift φ' of degree d' of φ to a partial flag variety X' dominating X is equivalent to giving filtrations of the pull-back of the tautological bundles, by sub-bundles of ranks and degrees determined by the data X', d' . It turns out that for general φ one can determine the degree d' of the lift corresponding to the Harder-Narasimhan filtration. This produces a birational equivalence between the space of morphisms $\text{Hom}_d(\mathbb{P}^1, X)$ of degree d to X , and the space of morphisms of degree d' to X' . Playing a similar game with the Jordan-Hölder filtration relates this moduli space to a moduli space of morphisms of \mathbb{P}^1 to the full flag variety. The idea for arbitrary G/P is the same but uses the parabolic reductions of Atiyah-Bott and Ramanathan for principal bundles over curves, which generalize the Harder-Narasimhan and Jordan-Hölder filtrations for vector bundles.

We adopt the notation of our joint paper with W. Fulton [7]. In particular, G is a connected, simply connected, semisimple complex Lie group with Borel subgroup B , opposite Borel subgroup B^- , maximal torus T , and Weyl group W . Let w_o be the longest element of W . Let P be a standard parabolic subgroup, corresponding to a subset Δ_P of the simple roots. Let R_P^+ denote the set of roots that are combinations of elements of Δ_P . For any $u \in W/W_P$, the opposite Schubert

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variety is $Y(u) = \overline{B^{-1}uP/P}$. Its class in the integral cohomology ring $H^\bullet(G/P)$ is denoted by σ_u . The dual cohomology class is $\sigma^u := \sigma_{w_o u}$. Let $n \geq 3$ be an integer, $p_1, \dots, p_n \in \mathbb{P}^1$ distinct points, and $g_1, \dots, g_n \in G$ general elements. For any $u_1, \dots, u_n \in W/W_P$, define

$$\langle \sigma_{u_1}, \dots, \sigma_{u_n} \rangle_d = \#\{\varphi : \mathbb{P}^1 \rightarrow G/P, \text{deg}(\varphi) = d, \varphi(p_i) \in g_i Y(u_i) \text{ for } i = 1, \dots, n\}$$

if this number is finite, and zero otherwise. These invariants may also be defined as pairings in the Kontsevich-Manin moduli space $\overline{M}_{0,n}(G/P, d)$ of degree d n -pointed genus 0 stable maps. Namely, let

$$f : \overline{M}_{0,n}(G/P, d) \rightarrow \overline{M}_{0,n}, \quad e_i : \overline{M}_{0,n}(G/P, d) \rightarrow G/P$$

denote the forgetful morphism to the moduli space of stable n -pointed genus 0 curves, resp. the i -th evaluation map. Then $\langle \sigma_{u_1}, \dots, \sigma_{u_n} \rangle_d$ is the coefficient of the point class in $f_*(e_1^* \sigma_{u_1} \cdot \dots \cdot e_n^* \sigma_{u_n})$.

Define a deformation of the cohomology ring of G/P as follows. Let s_1, \dots, s_r be the simple reflections in W not in W_P . The classes $\sigma^{[s_1]}, \dots, \sigma^{[s_r]}$ form a basis for $H^{\dim(G/P)-2}(G/P)$ which we identify with $H_2(G/P)$. For any degree $d = \sum_{i=1}^r d_i \sigma^{[s_i]}$ set $q^d = q_1^{d_1} \cdot \dots \cdot q_r^{d_r}$ in $\mathbb{Z}[q] := \mathbb{Z}[q_1, \dots, q_r]$. The quantum multiplication formula

$$\sigma_{u_1} \star \dots \star \sigma_{u_{n-1}} = \sum_d q^d \sum_{u_n} \langle \sigma_{u_1}, \dots, \sigma_{u_n} \rangle_d \sigma^{u_n}$$

defines an associative, commutative, $\mathbb{Z}[q]$ -linear product on

$$QH^\bullet(G/P) = H^\bullet(G/P) \otimes_{\mathbb{Z}} \mathbb{Z}[q],$$

the *small quantum cohomology* ring of G/P . We call the structure coefficients $\langle \sigma_{u_1}, \dots, \sigma_{u_n} \rangle_d$ the small GW-invariants of G/P . These invariants should not be confused with the n -point GW-invariants of G/P that play a role in the large quantum cohomology and are less well understood.

Actually it is somewhat misleading to call the ring $QH^\bullet(G/P)$ cohomology, since it is not functorial: A morphism $h : X \rightarrow X'$ does not induce a morphism $QH^\bullet(X') \rightarrow QH^\bullet(X)$ unless h is an isomorphism. In particular, the projection $G/B \rightarrow G/P$ does not induce a morphism $QH^\bullet(G/P) \rightarrow QH^\bullet(G/B)$. Peterson's comparison formula (1) below fills this gap: it expresses the degree d_P invariants of G/P in terms of degree d_B invariants for G/B . Unfortunately the definition of d_B , which follows, is not very explicit. Let

$$\phi_{P/B} : G/B \rightarrow G/P$$

be the projection. For any weight μ , let $L(\mu)$ denote the corresponding line bundle over G/B and $c_1(L(\mu)) \in H^2(G/B)$ its first Chern class. We denote by $(\ , \)$ the pairing of homology and cohomology.

Lemma/Definition 1. *For any $d_P \in H_2(G/P)$, there exists a unique $d_B \in H_2(G/B)$ such that $(\phi_{P/B})_* d_B = d_P$ and*

$$(d_B, c_1(L(\alpha))) \in \{0, 1\}, \quad \forall \alpha \in R_P^+$$

Furthermore, if $\text{Hom}_{d_P}(\mathbb{P}^1, G/P)$ is non-empty, then so is $\text{Hom}_{d_B}(\mathbb{P}^1, G/B)$.

Proof. In short, this follows from the fact that the affine Weyl group for P acts on the set of lifts of d_P and this action has a Weyl alcove as fundamental domain. Denote by π_B^* the isomorphism from $H^2(G/B)$ to the weight lattice

$$\pi_B^* : H^2(G/B) \rightarrow \Lambda^*, \quad c_1(L(\mu)) \mapsto \mu$$

and by π_B the dual isomorphism $\pi_B : \Lambda \rightarrow H_2(G/B)$. For any parabolic subgroup $P \subset G$ we have similar isomorphisms

$$\pi_P^* : H^2(G/P) \rightarrow (\Lambda^*)^{W_P}, \quad \pi_P : \Lambda^P \rightarrow H_2(G/P)$$

where $\Lambda^P := ((\Lambda^*)^{W_P})^*$. Let $r_P : \Lambda \cong \Lambda^{**} \rightarrow \Lambda^P$ denote the map given by restriction. Let Λ_P denote the coweight lattice for the semi-simple part of the Levi factor of P , and $W_P^{\text{aff}} = W_P \ltimes \Lambda_P$ the affine Weyl group for P . The inverse image $r_P^{-1}(\lambda_P)$ is invariant under the action of W_P^{aff} , and

$$\mathfrak{A}_P = \{ \xi \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}, 0 \leq \alpha(\xi) \leq 1, \forall \alpha \in R_P^+ \}$$

is a fundamental domain for the action of W_P^{aff} ; see e.g. [9, p. 90]. So there is a lift λ_B of λ_P in \mathfrak{A}_P . Let $d_B = \pi_B(\lambda_B)$.

It follows from e.g. the discussion in [6] that $\text{Hom}_{d_P}(\mathbb{P}^1, G/P)$ is non-empty if and only if d_P is a non-negative combination of the classes $\sigma^{[s_i]}$ for $\alpha_i \in R_P^+$. Suppose that the latter holds. By e.g. localization [7, Lemma 2.1], $\sigma^{s_i} = \pi_B(-h_i)$, where h_i is the coroot of α_i . Write $\lambda_B = c_1 h_1 + \dots + c_n h_n$. We assume without loss of generality that the only positive coefficients are c_1, c_2, \dots, c_k , for some $k \leq n$. Let α denote the highest root for the parabolic subgroup defined by this subset. Then $(\alpha, h_j) \geq 0$ for $j \leq k$ and $(\alpha, h_j) \leq 0$ for $j > k$. Hence $(\lambda_B, \alpha) \geq 2$. We have $c_i = (\lambda_B, \omega_i) = (\lambda_P, \omega_i) \leq 0$ for $\alpha_i \notin \Delta_P$, where ω_i is the corresponding fundamental weight. Therefore the simple roots $\alpha_1, \dots, \alpha_k$ are in Δ_P and $\alpha \in R_P^+$ which contradicts the definition of λ_B . This shows that λ_B is a non-positive combination of the simple coroots, so d_B is a non-negative combination of the classes σ^{s_i} , so $\text{Hom}_{d_B}(\mathbb{P}^1, G/B)$ is non-empty. \square

In some cases one can find simple formulas for d_B :

Example 1. Suppose $G = SL(3)$, $P = P_{\omega_1}$. Then $G/P = \mathbb{P}^2$ and $H_2(G/P) \cong \mathbb{Z}$, with generator $\sigma^{[s_1]} = [\mathbb{P}^1]$. Let $h_1, h_2 \in \Lambda$ denote the simple coroots. Given a degree $d_P = d_1 \sigma^{[s_1]}$, we have $\lambda_P = -d_1 r_P(h_1)$. The lifts of λ_P are of the form $\lambda_B = -d_1 h_1 - d_2 h_2$. To find d_B , we solve for d_2 so that

$$(\alpha_2, \lambda_B) = d_1 - 2d_2 \in \{0, 1\}.$$

The solution is $d_2 = d_1/2$, if d_1 is even, and $(d_1 - 1)/2$, if d_1 is odd.

Define P' to be the parabolic subgroup of G so that $\Delta_{P'} = \{ \alpha \in \Delta_P, \alpha(\lambda_B) = 0 \}$. Let $d_{P'}$ denote the image of d_B under the projection $H_2(G/B) \rightarrow H_2(G/P')$ and $\lambda_{P'} = \pi_{P'}^{-1}(d_{P'})$. Let $w_{P'}$ denote the longest element of the Weyl group $W_{P'}$. For any $u \in W/W_P$, let $\tilde{u} \in W$ denote its minimal length lift.

Theorem 2 (Peterson's Comparison Formula). *Let $u_1, \dots, u_n \in W/W_P$. For any degree $d_P \in H_2(G/P)$ we have for the degree d_B defined by Lemma 1,*

$$(1) \quad \langle \sigma_{u_1}, \dots, \sigma_{u_n} \rangle_{d_P} = \langle \sigma_{\tilde{u}_1}, \dots, \sigma_{\tilde{u}_{n-1}}, \sigma_{\tilde{u}_n w_{P'}} \rangle_{d_B}.$$

Example 2. Let $G/P = SL(3)/P_{\omega_1} = \mathbb{P}^2$ and $d_P = \sigma^{[s_1]}$ be the generator of $H_2(\mathbb{P}^2)$. Then $\sigma_{[s_1]}$ is the cohomology class of a line, and $\sigma_{[s_2s_1]}$ is the class of a point. Since there is a unique line passing through a line and two points in general position in \mathbb{P}^2 , $\langle \sigma_{[s_1]}, \sigma_{[s_2s_1]}, \sigma_{[s_2s_1]} \rangle_{d_P} = 1$. The lift $d_B = \sigma^{s_1}$ in $H_2(G/B)$, by Example 1. Hence $P' = B$ and $w_{P'} = e$ is the identity in W . One can check that $\langle \sigma_{s_1}, \sigma_{s_2s_1}, \sigma_{s_2s_1} \rangle_{d_B} = 1$ using the Peterson’s quantum Chevalley formula [7], or explicitly as follows: The intersection $e_1^{-1}(Y(s_2s_1)) \cap e_2^{-1}(w_0Y(s_2s_1)) \subset \overline{M}_{0,3}(G/B, d_B)$ is proper, and maps isomorphically under e_3 onto $s_1Y(s_1s_2)$. The latter meets $Y(s_1)$ properly at $x(s_2s_1) \in G/P$, which implies that the GW-invariant is 1. Here $x(s_2s_1)$ denotes the T -fixed point corresponding to $s_2s_1 \in W$.

We prove Theorem 2 at the end of the paper using Theorem 3 below. Recall that the set $\text{Hom}_{d_P}(\mathbb{P}^1, G/P)$ of degree d_P morphisms $\mathbb{P}^1 \rightarrow G/P$ has the structure of a smooth, quasi-projective variety. Denote by $\phi_{P'/B}, \phi_{P/P'}$ the projections

$$(2) \quad \phi_{P'/B} : G/B \rightarrow G/P', \quad \phi_{P/P'} : G/P' \rightarrow G/P.$$

We denote by $\text{Hom}_{d_{P'}}(G/P') \times_{G/P'} G/B$ the fiber product over G/P' via evaluation at 0 and $\phi_{P'/B}$.

Theorem 3. *The morphism*

$$(3) \quad \text{Hom}_{d_B}(G/B) \rightarrow \text{Hom}_{d_{P'}}(G/P') \times_{G/P'} G/B, \quad \varphi \mapsto (\phi_{P'/B} \circ \varphi, \varphi(0))$$

is an open, dense immersion. The morphism

$$(4) \quad \text{Hom}_{d_{P'}}(G/P') \rightarrow \text{Hom}_{d_P}(G/P), \quad \varphi \mapsto \phi_{P/P'} \circ \varphi$$

is birational.

Theorems 2 and 3 were both stated in [14] without proof. We will prove them using basic facts on semistability of principal bundles over curves. Recall that a vector bundle $E \rightarrow X$ over a curve C is *semistable* if every sub-bundle $E' \subset E$ has *slope* $\mu(E') = \text{deg}(E')/\text{rank}(E')$ at most the slope $\mu(E)$ of E . If E is not semistable, there is a unique sub-bundle E' of maximal slope that is maximal rank among sub-bundles of slope $\mu(E')$. Applying this fact inductively leads to the Harder-Narasimhan filtration, which is the unique filtration with the given degrees and ranks.

In order to make what follows more readable, we will first prove the theorem for a simple example. Consider the case that $G = SL(3)$, $P = P_{\omega_1}$, and $\lambda_P = r_P(h_1)$ so that d_P is the degree of a line in $G/P = \mathbb{P}^2$. Over \mathbb{P}^2 we have the quotient vector bundle Q and the tautological bundle R , of ranks 2, 1 respectively, given by

$$R_{[z]} = [z], \quad Q_{[z]} = \mathbb{C}^3/[z], \quad [z] \in \mathbb{P}^2.$$

Any morphism $\varphi_P : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ of degree d_P maps \mathbb{P}^1 isomorphically onto a line in \mathbb{P}^2 . A theorem of Grothendieck states that any vector bundle splits over \mathbb{P}^1 ; in this example, $\varphi_P^*Q \cong \mathcal{O}(1) \oplus \mathcal{O}(0)$, $\varphi_P^*R \cong \mathcal{O}(-1)$. One way of seeing this is to note that $Q \otimes R^{-1}$ is the tangent bundle $T\mathbb{P}^2$ of \mathbb{P}^2 ; the pull-back $\varphi^*T\mathbb{P}^2$ is the sum of the tangent bundle $T\mathbb{P}^1 \cong \mathcal{O}(2)$ to \mathbb{P}^1 and the normal bundle $N\mathbb{P}^1 \cong \mathcal{O}(1)$. It follows that the Harder-Narasimhan filtration of φ_P^*Q has a single non-trivial term given by the line bundle S isomorphic to $\mathcal{O}(1)$. The choice of a line sub-bundle of Q defines a lift $\varphi_B : \mathbb{P}^1 \rightarrow G/B = \text{Flag}(\mathbb{C}^3)$ of φ_P as follows. Let $\pi_{[w]} : \mathbb{C}^3 \rightarrow \mathbb{C}^3/\varphi_P([w])$ denote the projection. Define

$$\varphi_B([w]) = \left(R_{[w]} \subset \pi_{[w]}^{-1}S_{[w]} \right).$$

A little yoga with the definition of degree shows that the element $\lambda_B = \pi_B^{-1}(\deg(\varphi_B))$ satisfies

$$(\lambda_B, \omega_1) = c_1(\varphi_P^* R) = -1, \quad (\lambda_B, \omega_2) = -c_1(\varphi_P^* R \oplus S) = 0,$$

which implies $\lambda_B = -h_1$. The fact that the Harder-Narasimhan filtration is the unique filtration with given degrees implies that φ_B is the unique lift of φ_P of degree $d_B = \pi_B(\lambda_B)$. Since this is true for any map $\varphi_B : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ of degree d_P , the map

$$\mathrm{Hom}(\mathbb{P}^1, \mathrm{Flag}(\mathbb{C}^3))_{d_B} \rightarrow \mathrm{Hom}(\mathbb{P}^1, \mathbb{P}^2)_{d_P}, \quad \varphi_B \mapsto \varphi_P := \phi_{P/B} \circ \varphi_B$$

is a bijection. Since both varieties are smooth it is an isomorphism; this is a special case of Theorem 3. In this example, $P' = B$, so (3) is a tautology. In general, the proof of (3) involves the Jordan-Hölder filtration, as we explain below.

In order to prove Theorem 3 in general, we need some terminology for principal G -bundles over a variety X . First, a principal G -bundle $\mathcal{E} \rightarrow X$ is a right G -variety over X that is locally trivial; in our situation we may assume local triviality in the Zariski topology. For any principal G -bundle $\mathcal{E} \rightarrow X$ and morphism $\varphi : X' \rightarrow X$, we denote by $\varphi^* \mathcal{E}$ the pull-back bundle. For any left G -variety F we denote the associated fiber bundle by $\mathcal{E}(F)$. Let $G' \subset G$ be a subgroup. A *reduction* of \mathcal{E} to G' is a section σ of the fiber bundle $\mathcal{E}(G/G')$. A special role is played by reductions to maximal parabolic subgroups $P \subset G$. In the case $G = GL(V)$, the maximal parabolic subgroups are the stabilizers of subspaces $V' \subset V$. A parabolic reduction $\sigma : X \rightarrow \mathcal{E}(G/P)$ is equivalent to a sub-bundle of the associated vector bundle $\mathcal{E}(V)$ with fiber V' .

Semistability of principal G -bundles is defined as follows. For any standard maximal parabolic P , let ω_P be the fundamental weight such that Δ_P is the set of simple roots vanishing on ω_P . A principal G -bundle $\mathcal{E} \rightarrow X$ is called semistable if and only if for any reduction $\sigma : C \rightarrow \mathcal{E}/P$ to a standard maximal parabolic P , the degree of the associated line bundle $\sigma^* \mathcal{E}(\omega_P)$ is non-positive. For $G = SL(n)$, semistability of \mathcal{E} is equivalent to semistability of the associated vector bundle (see Ramanathan [16] or Atiyah-Bott [1, Section 10]). For any G , semistability of \mathcal{E} is equivalent to semistability of the vector bundle $\mathcal{E}(\mathfrak{g})$ associated to the adjoint representation \mathfrak{g} . If \mathcal{E} is not semistable, there is a canonical *Atiyah-Bott* parabolic reduction $\sigma_{\mathcal{E}} : C \rightarrow \mathcal{E}/P_{\mathcal{E}}$, where the parabolic subgroup $P_{\mathcal{E}}$ has Lie algebra $\mathfrak{p}_{\mathcal{E}}$ isomorphic to the fiber of the degree-zero term $\mathcal{E}(\mathfrak{g})_0$ in the Harder-Narasimhan filtration of $\mathcal{E}(\mathfrak{g})$. The canonical reduction has a uniqueness property generalizing that of the Harder-Narasimhan filtration: For any reduction $\sigma : X \rightarrow \mathcal{E}/P$, define the slope μ of σ to be the homomorphism from characters χ of P to \mathbb{Z} given by mapping χ to the degree of the associated line bundle $\sigma^* \mathcal{E}(\chi)$.

Proposition 4 (see e.g. [17, pp.11-12]). *The reduction $\sigma_{\mathcal{E}}$ is the unique reduction of \mathcal{E} to $P_{\mathcal{E}}$ with slope $\mu_{\mathcal{E}}$.*

If a degree 0 vector bundle $E \rightarrow C$ is semistable, there is a Jordan-Hölder filtration on E characterized by the property that the associated graded bundle $\mathrm{Gr}(E)$ is semistable, and the filtration is maximal among filtrations of this type. The Jordan-Hölder filtration is not unique; however, $\mathrm{Gr}(E)$ is unique up to isomorphism. The corresponding notion for principal bundles was introduced by Ramanathan [16]: A reduction $\sigma : C \rightarrow \mathcal{E}/P$ is called *admissible* if σ has slope 0. Let $\pi_L : P \rightarrow L$

and $\iota_L : L \rightarrow G$ denote the homomorphisms given by projection and inclusion respectively for the standard Levi subgroup L of P .

Proposition 5 ([16, 3.5.11]). *Let $\sigma : C \rightarrow \mathcal{E}/P$ be an admissible reduction of \mathcal{E} . $(\iota_L)_*(\pi_L)_*\sigma^*\mathcal{E}$ is semistable if and only if \mathcal{E} is semistable.*

If σ is admissible and $(\pi_L)_*\sigma^*\mathcal{E}$ is stable, call σ a *Ramanathan* reduction. By [16, Proposition 3.12], Ramanathan reductions exist for any bundle \mathcal{E} . Define an equivalence relation on principal G -bundles by $\mathcal{E} \sim (\iota_L)_*(\pi_L)_*\sigma^*\mathcal{E}(G)$, where σ is a Ramanathan reduction. Ramanathan [16] constructs a coarse moduli space for equivalence classes of semistable principal bundles. In genus zero, the moduli problem is trivial, for the following reason, which is an easy consequence of Grothendieck’s theorem that any principal G -bundle over \mathbb{P}^1 admits a reduction to T [8]:

Theorem 6. *Any semistable principal G -bundle $\mathcal{E} \rightarrow \mathbb{P}^1$ is trivial: $\mathcal{E} \cong \mathbb{P}^1 \times G$.*

The results above have straightforward generalizations to the case that G is reductive.

We apply these results to pull-backs of bundles on G/P . Let $\varphi_P : X \rightarrow G/P$ be a morphism and \mathcal{E}_P the principal P -bundle $G \rightarrow G/P$. For any parabolic subgroup $P' \subseteq P$, lifts $\varphi_{P'} : X \rightarrow G/P'$ of $\varphi_P : X \rightarrow G/P$ are in one-to-one correspondence with reductions $\sigma_{P'} : X \rightarrow \varphi_P^*\mathcal{E}_P(P/P')$.

Our goal is to prove Theorem 3 by thinking of it as a statement about reductions of bundles. Let $P = LU$ and $P' = L'U'$ denote the standard Levi decompositions. We study the semistability of the principal L -bundle $(\pi_L)_*\varphi_P^*\mathcal{E}_P$.

Lemma 7. *Suppose there exists a lift $\varphi_B : \mathbb{P}^1 \rightarrow G/B$ of φ_P of degree d_B . Then the Atiyah-Bott canonical reduction of $(\pi_L)_*\varphi_P^*\mathcal{E}_P$ corresponds to the lift $\varphi_{P'} = \phi_{P'/B} \circ \varphi_B$.*

Proof. Let \mathcal{E}_B be the principal B -bundle over G/B with total space G . Because of the isomorphisms

$$\varphi_P^*\mathcal{E}_P(P/P') \rightarrow \varphi_B^*\mathcal{E}_B(P/P') \rightarrow \varphi_B^*\mathcal{E}_B(L/L \cap P'),$$

the map $\varphi_{P'}$ defines a reduction of $\varphi_B^*\mathcal{E}_B(L)$ to $L \cap P'$. The filtration $\mathfrak{l} \cap \mathfrak{u}' \subset \mathfrak{l} \cap \mathfrak{p}' \subset \mathfrak{l}$ is B -stable. We claim that

$$(5) \quad \varphi_B^*\mathcal{E}_B(\mathfrak{l} \cap \mathfrak{u}') \subseteq \varphi_B^*\mathcal{E}_B(\mathfrak{l} \cap \mathfrak{p}') \subseteq \varphi_B^*\mathcal{E}_B(\mathfrak{l})$$

is the Harder-Narasimhan filtration of $\varphi_B^*\mathcal{E}_B(\mathfrak{l})$. We have $\deg \varphi_B^*\mathcal{E}_B(\mathfrak{l}_\mu) = (\lambda_B, \mu)$. Using the definition of the Peterson lift, if μ is a positive (resp. negative) root of \mathfrak{l} that is not a root of \mathfrak{l}' , then $(\lambda_B, \mu) = 1$, resp. -1 ; otherwise $(\lambda_B, \mu) = 0$. It follows that the Harder-Narasimhan filtration is (5), and has slope-zero term $\varphi_B^*\mathcal{E}_B(\mathfrak{l} \cap \mathfrak{p}')$. □

Corollary 8. *Suppose that φ_P lifts to a map $\varphi_B : \mathbb{P}^1 \rightarrow G/B$ of degree d_B . Then the composition $\varphi_{P'}$ of φ_B with the projection to G/P' is the unique lift of φ_P to G/P' of degree $d_{P'}$.*

Proof. By Lemma 7 and Proposition 4. □

We now consider the comparison between G/P' and G/B . Let $\varphi_{P'} : \mathbb{P}^1 \rightarrow G/P'$ be a morphism of degree $d_{P'}$. Let $L' \subset P'$ be the standard Levi subgroup of P' , $Z(L')$ its center, and $L'_{\text{ss}} = L'/Z(L')$. Let $\pi_{L'_{\text{ss}}} : P' \rightarrow L'_{\text{ss}}$ denote the projection,

and B'_{ss} the image of $B \cap L'$ under $\pi_{L'_{ss}}$. Since both the standard unipotent subgroup $U' \subset P'$ and $Z(L')$ act trivially on P'/B , we have $\mathcal{E}_{P'}(P'/B) \cong \mathcal{E}_{P'}(L'_{ss}/B'_{ss})$.

Lemma 9. *Suppose that there exists a lift φ_B of $\varphi_{P'}$ to G/B of degree d_B . Then the corresponding reduction $\sigma_B : \mathbb{P}^1 \rightarrow \varphi_{P'}^* \mathcal{E}_{P'}(L'_{ss}/B'_{ss})$ is a Ramanathan reduction of $\varphi_{P'}^* \mathcal{E}_{P'}(L'_{ss})$.*

Proof. Any weight for L'_{ss} defines a weight μ for L' in the span of the roots of L' . Hence $(\lambda_B, \mu) = 0$ and the line bundle $\varphi_B^* L(\mu) \cong \sigma_B^* \varphi_{P'}^* \mathcal{E}_{P'}(\mu)$ is trivial. This implies that σ_B is admissible. □

Corollary 10. *If there exists a lift φ_B of $\varphi_{P'}$ to G/B of degree d_B , then the bundle $\varphi_{P'}^* \mathcal{E}_{P'}(P'/B)$ is trivial.*

Proof. By Lemma 9, Theorem 6 and Proposition 5. □

Now we prove Theorem 3. The morphism (3) is an injection. Indeed, by Lemma 9 any lift φ_B gives a Ramanathan reduction of $\varphi_B^* \mathcal{E}_B(L'_{ss})$. A Ramanathan reduction of the trivial bundle $\mathbb{P}^1 \times L'_{ss}$ is a constant morphism $\mathbb{P}^1 \rightarrow L'_{ss}/B'_{ss}$ and is therefore specified uniquely by its value at any point in \mathbb{P}^1 . The dimension of $\text{Hom}_{d_B}(G/B)$ is

$$\begin{aligned} \dim(\text{Hom}_{d_B}(\mathbb{P}^1, G/B)) &= \dim(G/B) + (c_1(G/B), d_B) \\ &= \dim(G/B) + \sum_{\alpha \in R^+} (\alpha, \lambda_B) \\ &= \dim(G/P') + \dim(P'/B) + \sum_{\alpha \in R^+ \setminus R_{P'}^+} (\alpha, \lambda_B) \\ &= \dim(G/P') + (c_1(G/P'), d_{P'}) + \dim(P'/B) \\ &= \dim(\text{Hom}_{d_{P'}}(\mathbb{P}^1, G/P')) + \dim(P'/B). \end{aligned}$$

It follows that (3) is injective. Since the domain and codomain are smooth, irreducible ([10], [18]) and the same dimension, (3) is an open, dense immersion.

Similarly, by Lemma 7, the morphism (4) is injective on the image of (3). The domain and codomain have the same dimension, since

$$\begin{aligned} \dim(\text{Hom}_{d_{P'}}(\mathbb{P}^1, G/P')) &= \dim(G/P') + \sum_{\alpha \in R_{P'}^+} (\alpha, \lambda_B) \\ &= \dim(G/P') + \sum_{\alpha \in R_P^+} (\alpha, \lambda_B) - \#R_P^+ \setminus R_{P'}^+ \\ &= \dim(G/P) + \sum_{\alpha \in R_P^+} (\alpha, \lambda_B) \\ &= \dim(\text{Hom}_{d_P}(\mathbb{P}^1, G/P)). \end{aligned}$$

Since the varieties are smooth and irreducible, (4) is an open, dense immersion on an open subset, and therefore birational.

Theorem 3 and Lemma 7 imply the following curious fact.

Proposition 11. *For general $\varphi_P \in \text{Hom}_{d_P}(\mathbb{P}^1, G/P)$, the pull-back $\varphi_P^* \mathcal{E}_P(L)$ is semistable if and only if λ_B is W_P -fixed, that is, $P = P'$.*

Example 3. Let $G = SL(3, \mathbb{C})$ and $P = P_{\omega_1}$. Under the correspondence between principal bundles and vector bundles, the bundle $\mathcal{E}_P(L)$ corresponds to $Q \oplus R$. Since (semi)stability is preserved by tensoring with line bundles, semistability of Q is equivalent to semistability of $T\mathbb{P}^2$. Therefore, a general degree d_P morphism $\varphi_P : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ has $\varphi_P^* T\mathbb{P}^2$ semistable if and only if d_P is even.

Now we prove Theorem 2. Recall the maps $\phi_{P'/B}, \phi_{P/P'}$ from (2). For any $u \in W/W_P$, we have the identities

$$(6) \quad (\phi_{P/B})^* \sigma_u = \sigma_{\tilde{u}}, \quad \sigma_u = (\phi_{P/B})_* \sigma_{\tilde{u}w_P}.$$

Composing with the projection and collapsing the unstable components produces morphisms

$$\begin{aligned} h_{P'/B} : \overline{M}_{0,n+1}(G/B, d_B) &\rightarrow \overline{M}_{0,n+1}(G/P', d_{P'}) \times_{G/P'} G/B, \\ h_{P/P'} : \overline{M}_{0,n+1}(G/P', d_{P'}) &\rightarrow \overline{M}_{0,n+1}(G/P, d_P). \end{aligned}$$

The existence of $h_{P/P'}$ is proved by the same arguments that construct the forgetful morphism f ; see [6]. Theorem 3 implies that these morphisms are birational. Let ϕ_1, ϕ_2 denote the projections so that

$$\phi_1 \times \phi_2 : \overline{M}_{0,n+1}(G/P', d_{P'}) \times_{G/P'} G/B \rightarrow \overline{M}_{0,n+1}(G/P', d_{P'}) \times G/B$$

is the canonical inclusion. Let $u'_j \in W/W_{P'}$ denote the coset of \tilde{u}_j . We denote by superscript B objects, maps, etc. for G/B , and by P' those for G/P' . From (6) and the identities

$$\phi_{P'/B} \circ e_i^B = e_i^{P'} \circ \phi_1 \circ h_{P'/B}, \quad f^B = f^{P'} \circ \phi_1 \circ h_{P'/B}$$

it follows that for any $w \in W$,

$$\begin{aligned} f_*^B ((e_1^B)^* \sigma_{\tilde{u}_1} \cdots (e_{n-1}^B)^* \sigma_{\tilde{u}_{n-1}} \cdot (e_n^B)^* \sigma_w) \\ = f_*^{P'} ((e_1^{P'})^* \sigma_{u'_1} \cdots (e_{n-1}^{P'})^* \sigma_{u'_{n-1}} \cdot (e_n^{P'})^* (\phi_{P'/B})_* \sigma_w). \end{aligned}$$

In particular,

$$f_*^B ((e_1^B)^* \sigma_{\tilde{u}_1} \cdots (e_{n-1}^B)^* \sigma_{\tilde{u}_{n-1}} \cdot (e_n^B)^* \sigma_{\tilde{u}_n w_{P'}}) = f_*^{P'} ((e_1^{P'})^* \sigma_{u'_1} \cdots (e_n^{P'})^* \sigma_{u'_n}).$$

Taking the coefficient of the point class in $H^\bullet(\overline{M}_{0,n})$ gives

$$\langle \sigma_{u'_1}, \dots, \sigma_{u'_n} \rangle_{d_{P'}} = \langle \sigma_{\tilde{u}_1}, \dots, \sigma_{\tilde{u}_{n-1}}, \sigma_{\tilde{u}_n w_{P'}} \rangle_{d_B}.$$

A similar but easier argument shows $\langle \sigma_{u_1}, \dots, \sigma_{u_n} \rangle_{d_P} = \langle \sigma_{u'_1}, \dots, \sigma_{u'_n} \rangle_{d_{P'}}$, which completes the proof.

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