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VECTOR BUNDLES ON A PRODUCT OF REAL ALGEBRAIC CURVES

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ABSTRACT. We study complex vector bundles on a product of nonsingular real algebraic curves.

Throughout this note the term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^N$, for some N, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions. Recall that each Zariski locally closed subset of real projective space $\mathbb{P}^N(\mathbb{R})$ can be regarded as a real algebraic variety [3, Propositions 3.2.10, 3.4.3]. Morphisms of real algebraic varieties will be called *regular maps*. Every real algebraic variety carries also the *Euclidean topology*, which is determined by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

Given a real algebraic variety X, an interesting problem is the one of comparison between algebraic and topological vector bundles on X. We deal here with $\mathbb{C}$-vector bundles, identifying $\mathbb{C}$ with $\mathbb{R}^2$ and viewing it as a real algebraic variety. A topological $\mathbb{C}$-vector bundle ξ on X is said to *admit an algebraic structure* if it is topologically isomorphic to an algebraic subbundle of the trivial vector bundle with total space $X \times \mathbb{C}^p$, for some p (equivalently, if ξ is topologically isomorphic to the $\mathbb{C}$-vector bundle corresponding, in the usual way [11, 12], to a finitely generated projective module over the ring of regular functions from X into $\mathbb{C}$) [3, 4]. It is known that every topological $\mathbb{C}$-vector bundle on the unit sphere

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\}$$

or projective space $\mathbb{P}^n(\mathbb{R})$ admits an algebraic structure [9], [3, p. 326], and [4, Example 2.3(a)]. However, the behavior of $\mathbb{C}$-vector bundles on other "simple" real algebraic varieties can be drastically different. For example, a topological $\mathbb{C}$-vector bundle on the k-fold product $S^1 \times \cdots \times S^1$ admits an algebraic structure if and only if it is stably trivial [3, Corollary 12.6.6, and p. 326]. More generally, for "generic" compact nonsingular real algebraic curves $C_1, \dots, C_k$, a topological $\mathbb{C}$-vector bundle on $C_1 \times \cdots \times C_k$ admits an algebraic structure if and only if it is stably trivial (cf. [5, 6, 7] for the precise meaning of the adjective "generic").

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We have the following:

Theorem. *Let M_i be a compact smooth submanifold (of class $\mathcal{C}^\infty$ and without boundary) of $\mathbb{P}^{n_i}(\mathbb{R})$, with $2\dim M_i + 1 \le n_i$ for $1 \le i \le k$. Then for each i there exists a smooth embedding $e_i : M_i \to \mathbb{P}^{n_i}(\mathbb{R})$, arbitrarily close in the $\mathcal{C}^\infty$ topology to the inclusion map $M_i \hookrightarrow \mathbb{P}^{n_i}(\mathbb{R})$, such that $X_i = e_i(M_i)$ is a nonsingular Zariski closed subvariety of $\mathbb{P}^{n_i}(\mathbb{R})$ and every topological $\mathbb{C}$-line bundle on $X_1 \times \cdots \times X_k$ admits an algebraic structure; assuming $\dim M_i = 1$ for $1 \le i \le k$, every topological $\mathbb{C}$-vector bundle on $X_1 \times \cdots \times X_k$ admits an algebraic structure. Furthermore, if $n_1 = \cdots = n_k$ and $M_1 = \cdots = M_k$, then one can take $e_1 = \cdots = e_k$ and hence $X_1 = \cdots = X_k$.*

Let us recall a result of R. Benedetti and A. Tognoli [2, Theorem 4.2]. For any compact smooth submanifold (without boundary) M of $\mathbb{P}^n(\mathbb{R})$, with $2\dim M + 1 \le n$, there exists a smooth embedding $e : M \hookrightarrow \mathbb{P}^n(\mathbb{R})$, arbitrarily close in the $\mathcal{C}^\infty$ topology to the inclusion map $M \hookrightarrow \mathbb{P}^n(\mathbb{R})$, such that $X = e(M)$ is a nonsingular Zariski closed subvariety of $\mathbb{P}^n(\mathbb{R})$ and every topological $\mathbb{C}$-vector bundle on X admits an algebraic structure (strictly speaking, [2] deals with $\mathbb{R}$-vector bundles and with $\mathbb{R}^n$ instead of $\mathbb{P}^n(\mathbb{R})$, but a straightforward modification of the proof yields the assertion just stated). This does not directly imply our theorem.

In preparation for the proof we shall review a certain construction. Let X be a nonsingular real algebraic variety. A nonsingular complexification of X is a pair (V, j), where V is a nonsingular complex quasiprojective variety defined over $\mathbb{R}$ and $j : X \to V$ is an injective map such that the set $V(\mathbb{R})$ of real points of V is Zariski dense in V, $j(X) = V(\mathbb{R})$, and j, viewed as a map from X onto $V(\mathbb{R})$, is a regular isomorphism of real algebraic varieties. Define $H^2_{\text{alg}}(V,\mathbb{Z})$ to be the subgroup of $H^2(V,\mathbb{Z})$ consisting of all cohomology classes of the form $c_1(L)$, where L is a complex algebraic line bundle on V and $c_1(-)$ stands for the first Chern class. Set

$$H^2_{\mathbb{C}-\text{alg}}(X,\mathbb{Z}) = j^*(H^2_{\text{alg}}(V,\mathbb{Z})),$$

where $j^* : H^*(V,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ is the homomorphism induced by j. One readily checks that $H^2_{\mathbb{C}-\text{alg}}(X,\mathbb{Z})$ does not depend on the choice of (V, j) [4]. Given a topological $\mathbb{C}$-line bundle λ on X,

$$\text{(1)} \qquad \lambda \text{ admits an algebraic structure } \Leftrightarrow c_1(\lambda) \in H^2_{\mathbb{C}-\text{alg}}(X,\mathbb{Z})$$

[4]. For any regular map $f : X \to Y$ between nonsingular real algebraic varieties, one has (cf. [4])

$$\text{(2)} \qquad f^*(H^2_{\mathbb{C}-\text{alg}}(Y,\mathbb{Z})) \subseteq H^2_{\mathbb{C}-\text{alg}}(X,\mathbb{Z}).$$

If $Y_1, \ldots, Y_n$ are connected nonsingular real algebraic varieties, then

$$H^2(Y_1 \times \cdots \times Y_n, \mathbb{Z}) = \sum_{i<j} p_{ij}^*(H^2(Y_i \times Y_j, \mathbb{Z})),$$

where $p_{ij} : Y_1 \times \cdots \times Y_n \to Y_i \times Y_j$ is the canonical projection, and hence (2) implies

$$\text{(3)} \qquad H^2_{\mathbb{C}-\text{alg}}(Y_1 \times \cdots \times Y_n, \mathbb{Z}) = H^2(Y_1 \times \cdots \times Y_n, \mathbb{Z}),$$

provided $H^2_{\mathbb{C}-\text{alg}}(Y_i \times Y_j, \mathbb{Z}) = H^2(Y_i \times Y_j, \mathbb{Z})$ for all $i < j$.

Note that

$$\sigma : \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}),\ \sigma(x, y) = (\bar{y}, \bar{x}),$$

where the bar indicates complex conjugation, is an antiholomorphic involution. Thus $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ can be regarded as a complex projective variety defined over $\mathbb{R}$, whose set of real points, denoted here by $\mathbb{P}^n(\mathbb{C})_{\mathbb{R}}$, coincides with the set of fixed points of the involution σ. Clearly, the map $\mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C})_{\mathbb{R}}$, $x \to \sigma(x, x) = (x, \bar{x})$, is a homeomorphism. Furthermore, by construction,

$$H^2_{\mathbb{C}-\mathrm{alg}}(\mathbb{P}^n(\mathbb{C})_{\mathbb{R}}, \mathbb{Z}) = H^2(\mathbb{P}^n(\mathbb{C})_{\mathbb{R}}, \mathbb{Z}). \tag{4}$$

Proof of the theorem. By [6, Corollary 1.6, Proposition 2.2], there exists a nonsingular algebraic curve C in $\mathbb{P}^2(\mathbb{R})$ such that C is connected and

$$H^2_{\mathbb{C}-\mathrm{alg}}(C \times C, \mathbb{Z}) = H^2(C \times C, \mathbb{Z}). \tag{5}$$

Choose a positive integer r with the property that the group $H^q(M_i, \mathbb{Z})$ can be generated by r elements for $1 \leq q \leq 2$ and $1 \leq i \leq k$. Let ℓ be a positive integer satisfying $\dim M_i + 1 \leq \ell$ for $1 \leq i \leq k$. Setting

$$B = (\mathbb{P}^\ell(\mathbb{C})_{\mathbb{R}})^r \times C^r,$$

we can find a smooth map $\varphi_i : M_i \to B$ for which the induced homomorphism

$$\varphi_i^* : H^q(B, \mathbb{Z}) \to H^q(M_i, \mathbb{Z})$$

is surjective for $1 \leq q \leq 2$ and $1 \leq i \leq k$ (see, for example, [8, Chap. V, 11.6, 11.9; Chap. VII, Sect. 12]).

Given a nonnegative integer p, we identify $\mathbb{P}^{n_i}(\mathbb{R})$ with the subset $\mathbb{P}^{n_i}(\mathbb{R}) \times \{0\}$ of $\mathbb{P}^{n_i}(\mathbb{R}) \times \mathbb{R}^p$; thus $\mathbb{P}^{n_i}(\mathbb{R}) \subseteq \mathbb{P}^{n_i}(\mathbb{R}) \times \mathbb{R}^p$. By [1, Theorem 2.8.4, Lemma 2.7.1], there exists a nonnegative integer p, a smooth embedding $\epsilon_i : M_i \to \mathbb{P}^{n_i}(\mathbb{R}) \times \mathbb{R}^p$, a nonsingular Zariski closed subvariety Y_i of $\mathbb{P}^{n_i} \times \mathbb{R}^p$, and a regular map $\psi_i : Y_i \to B$ such that $Y_i = \epsilon_i(M_i)$, ϵ_i is close in the $\mathcal{C}^\infty$ topology to the inclusion map $M_i \hookrightarrow \mathbb{P}^{n_i}(\mathbb{R}) \times \mathbb{R}^p$, and φ_i is homotopic to $\psi_i \circ \epsilon_i$ (the cited results of [1] are applicable since every homology class in $H_j(B, \mathbb{Z}/2)$, for $0 \leq j \leq \dim B$, can be represented by a Zariski closed subvariety of B). In particular, the induced homomorphism

$$\psi_i^* : H^q(B, \mathbb{Z}) \to H^q(Y_i, \mathbb{Z})$$

is surjective for $1 \leq q \leq 2$. Since $2 \dim Y_i + 1 \leq n_i$, one can find an algebraic embedding $\eta_i : Y_i \to \mathbb{P}^{n_i}(\mathbb{R})$ (this means that $\eta_i(Y_i)$ is a Zariski closed subvariety of $\mathbb{P}^{n_i}(\mathbb{R})$ and $\eta_i : Y_i \to \eta_i(Y_i)$ is a regular isomorphism), close to $\pi_i|Y_i$, where $\pi_i : \mathbb{P}^{n_i}(\mathbb{R}) \times \mathbb{R}^p \to \mathbb{P}^{n_i}(\mathbb{R})$ is the canonical projection; cf. [10, pp. 21, 22]. By construction, $e_i = \eta_i \circ \epsilon_i : M_i \to \mathbb{P}^{n_i}(\mathbb{R})$ is a smooth embedding, close in the $\mathcal{C}^\infty$ topology to the inclusion map $M_i \hookrightarrow \mathbb{P}^{n_i}(\mathbb{R})$, $X_i = e_i(M_i) = \eta_i(Y_i)$ is a nonsingular Zariski closed subvariety of $\mathbb{P}^{n_i}(\mathbb{R})$, and $f_i = \psi_i \circ \eta_i^{-1} : X_i \to B$ is a regular map for which the induced homomorphism

$$f_i^* : H^q(B, \mathbb{Z}) \to H^q(X_i, \mathbb{Z})$$

is surjective for $1 \leq q \leq 2$.

Consider the regular map

$$f = f_1 \times \cdots \times f_k : X_1 \times \cdots \times X_k \to B^k.$$

Clearly, the induced homomorphism

$$f^* : H^2(B^k, \mathbb{Z}) \to H^2(X_1 \times \cdots \times X_k, \mathbb{Z}) \tag{6}$$

is surjective. Furthermore, it easily follows from (3), (4), (5) that

$$H^2_{\mathbb{C}-\mathrm{alg}}(B^k, \mathbb{Z}) = H^2(B^k, \mathbb{Z}). \tag{7}$$

Combining (2), (6), (7), we obtain

$$H^2_{\mathbb{C}-\mathrm{alg}}(X_1 \times \cdots \times X_k, \mathbb{Z}) = H^2(X_1 \times \cdots \times X_k, \mathbb{Z}), \tag{8}$$

which in view of (1) implies that every topological $\mathbb{C}$-line bundle on $X_1 \times \cdots \times X_k$ admits an algebraic structure. Thus the first part of the theorem is proved.

Clearly, assuming $n_1 = \cdots = n_k$ and $M_1 = \cdots = M_k$, we can take $e_1 = \cdots = e_k$ and $X_1 = \cdots = X_k$.

Suppose now $\dim M_i = 1$ for $1 \leq i \leq k$. Without loss of generality, we may assume that each M_i is connected. Then the real algebraic variety $X = X_1 \times \cdots \times X_k$ is connected, $H^*(X, \mathbb{Z})$ has no torsion, and the ring $H^{\mathrm{even}}(X, \mathbb{Z}) = \bigoplus_{j \geq 0} H^{2j}(X, \mathbb{Z})$ is generated by $H^0(X, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$. Since, in addition to these facts, condition (8) holds, it follows from [5, Proposition 2.4] that every topological $\mathbb{C}$-vector bundle on X admits an algebraic structure. This completes the proof.

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