STEIN FILLABILITY AND THE REALIZATION OF CONTACT MANIFOLDS

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ABSTRACT. There is an intrinsic notion of what it means for a contact manifold to be the smooth boundary of a Stein manifold. The same concept has another more extrinsic formulation, which is often used as a convenient working hypothesis. We give a simple proof that the two are equivalent. Moreover it is shown that, even though a border always exists, its germ is not unique; nevertheless the germ of the Dolbeault cohomology of any border is unique. We also point out that any Stein fillable compact contact 3-manifold has a geometric realization in $\mathbb{C}^4$ via an embedding, or in $\mathbb{C}^3$ via an immersion.

Let $M$ be a smooth orientable compact real $(2n+1)$-dimensional manifold without boundary ($n = 1, 2, 3, \ldots$). Let $\Xi$ be a smooth orientable contact structure on $M$. The orientation of $\Xi$ is defined by a global contact form $\xi$ on $M$, with $\xi = \{ v \in TX \mid \xi(v) = 0 \}$, and which is strongly non-integrable, so that $\omega = \xi \wedge (d\xi)^n$ is $\neq 0$ at each $x \in M$, so defining an orientation of $M$. We shall always take $\omega$ as the orientation of $M$, and we shall say then that $M$ and $\Xi$ are equally oriented.

Assume that the contact manifold $(M, \Xi)$ is the smooth boundary of a Stein manifold $(X, J)$. Let us clarify this notion: Let $X$ be a complex manifold, of dimension $(n+1)$, with a smooth boundary $M$. Assuming that its complex structure $J$ is smooth up to the boundary $M$, it induces a smooth CR structure $(M, HM, J_M)$, $J_M : HM \to HM$, $J^2_M = -I$ of hypersurface type $(n, 1)$ on $M$. To say that a contact structure $\Xi$ on $M$ is induced by the CR structure of $M$ means that $\Xi = HM$ are the same distribution of $2n$-planes in $TM$. Since $M$ is a boundary, the contact structure $\Xi$ is orientable and a global contact form $\xi$ defines the Levi form of $M$:

\[ HM \ni v \to d\xi(J_M v, v) \in \mathbb{R}. \]

This is a Hermitian form on $HM$ for the complex structure $J_M$. The strong non-integrability condition $\xi \wedge (d\xi)^n \neq 0$, together with the formal integrability of the partial complex structure $J_M$, imply that for each $x \in M$ the Levi form $H_x M \ni v \to L(v) \in \mathbb{R}$ is non-degenerate, i.e. all its eigenvalues are different from zero.

In particular, when $M$ is the boundary of a Stein manifold $X$, the Levi form $L$ of $M$ is positive definite at every $x \in M$: in this case the induced CR structure is strongly pseudoconvex. In this situation it is customary to say that “the contact manifold $M$ is Stein fillable by $X$”.

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The purpose of this note is to delve into the issue of the meaning of the sentence in italics.

1. The intrinsic notion

Here is the issue: What is meant by saying that $M$ is the smooth boundary of a complex manifold $X$? If we are to enjoy the convenience and flexibility of a differential topologist, and want to work in the smooth ($C^\infty$) category, then the intrinsic notion is clear. It goes as follows:

(i) $\overline{X} = X \cup M$ has the structure of a $C^\infty$ manifold with a $C^\infty$ boundary $M$, $X$ being the interior of $\overline{X}$.
(ii) $X$ is endowed with a formally integrable almost complex structure $J : TX \to TX$, $J^2 = -I$, which is $C^\infty$ up to the boundary $M$.

[This much gives us a smooth induced almost-CR structure $J_M$ on $M$, which in turn induces a distribution of $2n$-planes $\Xi = HM$ on $M$. When $n = 1$, there are no integrability conditions and in fact the CR structure can be taken strictly pseudoconvex if the corresponding contact structure is strongly non-integrable.]

For Stein fillability we require in addition that

(iii) $X$ is a Stein manifold.

Remark 1.1. It follows from (ii) via the Newlander-Nirenberg theorem that $X$ has an atlas of interior holomorphic coordinate charts. But it does not follow immediately from the above definition that $X$ has an atlas of holomorphic coordinate charts [which would have to include boundary charts]. Nor does it immediately follow from the definition that $X$ can be regarded as the closure of a domain in some larger open complex manifold $\overline{X}$. See for example the discussion in [H1], [H2], [H3].

2. A working hypothesis

There has been considerable recent interest in compact contact manifolds which are Stein fillable, and many very interesting and significant results have been obtained, especially when $\dim_{\mathbb{R}} M = 3$ (see e.g. [E1], [E2], [E3], [G], [LIM]).

In these articles, however, the intrinsic notion is not always being used; what is being used instead is the following convenient working hypothesis:

1° The Stein manifold $X$ is an open set in a larger open complex manifold $Y$, with $X \Subset Y$.
2° There exists a real $C^\infty$ strictly plurisubharmonic function $\phi$ on $Y$.
3° $\overline{X} = X \cup M = \{ x \in Y \mid \phi(x) \leq 0 \}$ with $d\phi \mid_M \neq 0$.
4° $\phi$ is a Morse function on $Y$; i.e. $\phi$ has at most a finite number of critical points, all of which are nondegenerate.

This working hypothesis clearly implies the intrinsic notion, but it also involves a number of extrinsic elements. In §6 we give a simple proof that the intrinsic notion is equivalent to the convenient working hypothesis.

3. Existence and non-uniqueness of the border

In this section we do not need that $M$ be compact, nor that $X$ be Stein. But we will tacitly assume that all the manifolds are paracompact (i.e. countable at
infinity). Otherwise we place ourselves in the position of (i) and (ii) of the intrinsic notion.

**Theorem 3.1.** Assume that the contact manifold $M$ is the $C^\infty$ intrinsic boundary of a strictly pseudoconvex complex manifold $X$. Then:

(a) $X$ is a domain $X \subset \tilde{X}$, having interior $X$ and $C^\infty$ strictly pseudoconvex boundary $M$, in some open complex manifold $\tilde{X}$.

(b) Even though a border $\tilde{X} \setminus X$ exists by (a), its germ along $M$ is, in general, not unique.

**Proof.** (a) Since by (i) $\tilde{X}$ is a smooth manifold with a smooth boundary, there is a $C^\infty$ collar, so that we can consider $\tilde{X}$ as a domain in some open real $(2n + 2)$-dimensional smooth manifold $\Omega$. By (ii) there is a complex structure tensor $J$ on $X$ which is $C^\infty$ up to $M$, and hence induces the strictly pseudoconvex structure $J_M$ on $M$. Since $J$ is assumed in (ii) to be $C^\infty$ up to the boundary, we may consider its smooth extension $\tilde{J}$ to $\tilde{X}$, so $J_M = \tilde{J}|_{HM} = \tilde{J}|_{\Xi}$. Since Whitney sections over closed sets can be continued to smooth sections over open neighborhoods, we may, after possibly shrinking $\Omega$, extend $\tilde{J}$ to a smooth almost complex structure $J_\Omega$ on $\Omega$, such that $J_\Omega|_{\tilde{X}} = \tilde{J}$ satisfies the formal integrability conditions of the Newlander-Nirenberg theorem on $\tilde{X} \subset \Omega$. Now the statement (a) is the content of Theorem 1 in [HN1], where a detailed proof is given. It tells us that there is an open submanifold $X$, with $\tilde{X} \subset \tilde{X} \subset \Omega$, and a complex structure $\tilde{J}$ on $\tilde{X}$, such that $\tilde{J}|_{\tilde{X}} = \tilde{J}$. The proof of that theorem involves a tricky use of Zorn’s lemma, and employs an up-to-the-boundary version of the Newlander-Nirenberg theorem, which is valid here since $M$ is strictly pseudoconvex (see [HJ], [Ca]).

This completes the proof of (a).

**Remark 3.1.** When $M$ is compact, weakly pseudoconvex and of finite type in the sense of D’Angelo (see [DA]), the existence of $\tilde{X}$ was shown by [Ch] using a much more complicated argument. When $M$ is compact, strictly pseudoconvex, and is a boundary in the concrete sense (see [HJ]), the existence of $\tilde{X}$ was shown by [Oh] and [He]. Additional very interesting related results were obtained in [Le1], [Le2], [Le3].

(b) We give a simple counterexample to uniqueness of the germ of the border along $M$, even in the simple case where $X = B$ is an open ball in $\mathbb{C}^m$ ($m = 1, 2, 3, \ldots$) with boundary $\partial B = S^{2m-1}$. For convenience take $B$ to be the ball of radius $\frac{1}{2}$ centered at the point $\frac{1}{2}e_1$, where $e_1 = (1, 0, \ldots, 0)$. Let $D$ denote the open unit disc in $\mathbb{C}$, and $\omega$ denote a suitable open neighborhood of $\partial D$, to be chosen later. We set $U = \omega \times D^{m-1}$ and note that $U$ is an open neighborhood of $\partial B$ in $\mathbb{C}^m$. On $U$ we have the standard complex structure, which can be described by a single global holomorphic coordinate patch $(U; z_1, \ldots, z_m)$. We shall construct another complex structure on $U$, also described by a single global holomorphic coordinate patch of the form $(U; \phi(z_1), z_2, \ldots, z_m)$ such that:

1. the two complex structures coincide on $\partial D$,

2. the two complex structures cannot possibly coincide on any neighborhood in $U$ of the point $e_1 \in \partial B$. 

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This means that the standard complex structure on $\overline{D}$ can be extended in inequivalent ways to the border $U \setminus \overline{D}$.

Let $\alpha(z)$ denote the branch of $\sqrt{1-z}$ on $\mathbb{C} \setminus [1, \infty)$ which has positive real part. On the closure $\overline{D}$ we define

$$\phi(z) = \begin{cases} Az + \exp\left(-\frac{1}{\alpha(z)}\right), & z \neq 1, \\ A, & z = 1. \end{cases}$$

For every $A \in \mathbb{C}$ this defines a $C^\infty$ function on $\overline{D}$, in the sense of Whitney. For $|A|$ sufficiently large, it defines a biholomorphism of $D$ onto an open domain $G$ in $\mathbb{C}$. By Whitney’s theorem, for large $A$, $\phi$ extends to a smooth diffeomorphism $\tilde{\phi}$ of an open neighborhood $\omega$ of $\overline{D}$ in $\mathbb{C}$ onto a neighborhood $\Omega$ of $G$ in $\mathbb{C}$.

It follows from what was said above that the two complex structures are equivalent on $\overline{D}$, and hence on $\overline{B}$, yielding (1). It remains to establish (2): Consider the function $f(z_1, \ldots, z_m) = \tilde{\phi}(z_1)$ defined on $U$. Then $f|_{U}$ is holomorphic with respect to either of the two complex structures, and it is holomorphically extendable across $e_1$ with respect to the second one, since it is one of the holomorphic coordinate functions. But $f|_{U}$ is not holomorphically extendable across $e_1$ with respect to the standard complex structure, because if it were extendable across $e_1$, then $Az - \phi(z)$ would have a nonzero holomorphic extension to a neighborhood of 1 in $\mathbb{C}$, while at the same time being flat at 1; this gives a contradiction. □

4. Fundamental system of Stein neighborhoods

Now we return to the situation where $M$ is compact and $X$ is Stein. Theorem 3.1 supplies us with an open complex manifold $\tilde{X}$, in which $\tilde{X} = X \cup M$ appears as a compact domain with a smooth strictly pseudoconvex boundary.

**Theorem 4.1.** Assume that the compact contact manifold $M$ is the $C^\infty$ intrinsic boundary of a Stein manifold $X$. Then $\tilde{X}$ has a fundamental system of open Stein neighborhoods $\{Y\}$ with $\tilde{X} \in Y \in \tilde{X}$, for each $Y$.

**Proof.** This now follows from an old result that is proved using the bumping technique of [AG], applied to the strictly pseudoconvex domain $\tilde{X}$ in $\tilde{X}$: by employing a finite number of small smooth bumps, one can construct an arbitrarily small open neighborhood $Y$ of $\tilde{X}$, such that $\partial Y$ is smooth and remains strictly pseudoconvex. Then using local vanishing theorems for coherent analytic sheaves, and the Mayer-Vietoris sequence, applied a finite number of times, it can be shown that the restriction homomorphism

$$r : H^q(Y, \mathcal{F}) \to H^q(X, \mathcal{F}|_X)$$

is an isomorphism for $q > 0$, and any coherent analytic sheaf $\mathcal{F}$ on $Y$. We have that $H^q(Y, \mathcal{F}) \simeq H^q(X, \mathcal{F}|_X) = 0$ because $X$ is Stein. For more details, see Theorem 5 in [AH2], or consult [AG] Propositions 16, 17, 21, 22]. □

5. Geometric realization of Stein fillable contact structures

Let $n = 1$, so $\dim_{\mathbb{R}} M = 3$ and $\dim_{\mathbb{C}} X = 2$.

**Theorem 5.1.** Assume that the 3-dimensional compact contact manifold $M$ is the $C^\infty$ intrinsic boundary of a Stein manifold $X$. Then $M$ has a smooth CR embedding as a closed CR submanifold of $\mathbb{C}^4$ (or a closed CR immersion in $\mathbb{C}^3$).
Note that this means that the CR structure induced on $M$ from the embedding is the same as the one $M$ inherits from being the boundary of $X$. In particular: the contact structure on $M$ is achieved, via the embedding, by a complex tangent line at each point.

**Proof.** Choose one of the Stein manifolds $Y \ni X$. According to the embedding theorem for Stein manifolds (see [Bi], [Na]), $Y$ has a proper holomorphic embedding as a closed complex submanifold of $\mathbb{C}^5$. The restriction of this embedding to $M$ gives a $CR$ embedding of $M$ into $\mathbb{C}^5 \subset \mathbb{C}^5$. With $N = 5$ consider

$$M' = \{(p, r) \in M \times \mathbb{C}^5 | \overline{p} r \text{ is tangent to } M \text{ at } p\}.$$ 

Then $M'$ is a smooth submanifold of $M \times \mathbb{C}^5$ of real dimension 6, and $M' \ni (p, r) \rightarrow r \in \mathbb{C}^5$ is a smooth map. By Sard’s theorem its image has measure zero in $\mathbb{C}^5$, since $2N > 6$. By choosing a point $R_0 \notin \{\text{its range}\} \cup M$, and taking a holomorphic projection from $R_0$ to a hyperplane $\Sigma$ not containing $R_0$, we obtain a $CR$ closed immersion of $M$ into $\mathbb{C}^4$.

Next consider

$$M'' = \{(p, q, r) \mid (p, q) \in M \times M \setminus \Delta, r \in \mathbb{C}^5 \text{ and } p, q, r \text{ are collinear}\}.$$ 

Then $M''$ is a smooth manifold of real dimension 8, and $M'' \ni (p, q, r) \rightarrow r \in \mathbb{C}^5$ is a smooth map. Again by Sard’s theorem, its image has measure zero, because $2N > 8$. Thus it is possible to choose the point $R_0$ so that the $CR$ immersion obtained above is globally one-to-one. As a result we obtain a $CR$ embedding of $M$ into $\mathbb{C}^4$. To obtain a $CR$ immersion into $\mathbb{C}^3$, we repeat the above projection argument with $N = 4$, since then we still have $2N > 6$.

**Remark 5.1.** When $n = 2, 3, \ldots$, so that $\dim_{\mathbb{R}} M \geq 5$, the result analogous to Theorem 4.1 holds without any assumption of Stein fillability; one needs only the existence of a $CR$ structure on $M$ which is compatible with the contact structure: assume the $(2n + 1)$-dimensional compact orientable contact manifold $M$ has a smooth $CR$ structure of type $(n, 1)$ which induces the given contact structure and is strictly pseudoconvex. By a theorem of Boutet de Monvel [BM], $M$ has a smooth $CR$ embedding into $\mathbb{C}^5$, for some $N$. Then we can repeat the argument above and obtain that $M$ has a $CR$ embedding into $\mathbb{C}^{2n+2}$, or a $CR$ immersion into $\mathbb{C}^{2n+1}$. The contact structure on $M$ is then achieved, via the embedding, by a tangent affine $\mathbb{C}^n$ at each point.

For $CR$ manifolds which are not of hypersurface type, see [HN2].

6. **Equivalence of the intrinsic notion and the working hypothesis**

We return to the situation of §1 and §2.

**Theorem 6.1.** Assume that the compact contact manifold $M$ is the $C^\infty$ intrinsic boundary of a Stein manifold $X$. Then the working hypotheses 1°, 2°, 3°, 4° of §2 are satisfied.

**Proof.** Since a Stein manifold $Y$ has an exhaustion by a smooth strictly plurisubharmonic function, we obtain 1° and 2° from Theorems 3.1 and 4.1. To demonstrate 3° we proceed as follows: Fix a Stein neighborhood $Y$ of $\overline{X}$ in $\overline{X}$, a strictly plurisubharmonic function $\psi$ on $Y$, and a Hermitian metric on $Y$. Since $X$ is a domain in
Y, there exists a global defining function \( \rho \in C^\infty(Y) \) such that:
\[
\overline{X} = \{ x \in Y \mid \rho(x) \leq 0 \}, \quad d\rho|_M \neq 0
\]
(see [AH1 Proposition 1.1]). Since \( M \) is strictly pseudoconvex, the Levi form \( L(\rho) \)
is positive definite at each point of \( M \); i.e. has \( n \) positive eigenvalues. To obtain
\((n+1)\) positive eigenvalues for the complex Hessian \( i\partial\bar{\partial}\rho \) near \( M \), we replace \( \rho \) by
a modified global defining function
\[
\tilde{\rho} = \frac{1}{\lambda} \{ e^{\lambda \rho} - 1 \},
\]
with the constant \( \lambda > 0 \) chosen sufficiently large. It is easy to verify that there
is an open neighborhood \( U \) of \( M \) in \( Y \) in which \( \tilde{\rho} \) is strictly plurisubharmonic,
and \( d\tilde{\rho} \neq 0 \). Next we modify \( \tilde{\rho} \) to make it strictly plurisubharmonic in an open
neighborhood \( V \) of \( \overline{X} \), and establish \( 3^o \): Let \( \chi(\rho) \) be a smooth real convex function
of the real variable \( \rho \), such that \( \chi(\rho) = \rho \) for \( \rho \geq -\delta \) and \( \chi(\rho) = -2\delta \) for \( \rho \leq -3\delta \),
where \( \delta > 0 \) is chosen so small that \( \{ x \in \overline{X} \mid -3\delta \leq \tilde{\rho}(x) \leq 0 \} \subseteq U \). Let \( K \subset X \) be
a compact set such that \( K \supset \{ x \in X \mid \tilde{\rho}(x) \leq -\delta \} \). Choose a nonnegative smooth
cutoff function \( \mu \in C_0^\infty(X) \) such that \( \mu = 1 \) on a neighborhood of \( K \). Consider the
function:
\[
\phi = \chi(\tilde{\rho}) + \epsilon \mu \psi,
\]
with a small constant \( \epsilon > 0 \). Then \( d\phi|_M = d\tilde{\phi}|_M = d\rho|_M \neq 0 \) and \( \overline{X} = \{ x \in Y \mid \phi(x) \leq 0 \} \) for \( \epsilon > 0 \) taken sufficiently small. The function \( \chi(\tilde{\rho}) \) is smooth and
weakly plurisubharmonic on \( V = X \cup U \). The function \( \phi \) is strictly plurisubharmonic
in \( V \) for sufficiently small \( \epsilon > 0 \). This establishes \( 3^o \) without destroying \( 2^o \).

The function \( \psi \) can be chosen at the beginning to be a Morse function on \( Y \); see
[AF]. Hence by construction there is an \( \eta > 0 \) such that \( \phi \) has no critical points
on \( \{ x \in V \mid -\eta \leq \tilde{\rho}(x) \leq \eta \} \), and at most only a finite number of nondegenerate
critical points for \( \{ x \in V \mid \tilde{\rho}(x) \leq -3\delta \} \). To obtain \( 4^o \), we need to eliminate any
degenerate critical points of \( \phi \) in \( \{ x \in V \mid -3\delta < \tilde{\rho}(x) < -\eta \} \). Let \( \nu \in C_0^\infty(X) \),
\( 0 \leq \nu(x) \leq 1 \), be a smooth cutoff function with \( \nu = 1 \) on the set \( \{ x \mid \tilde{\rho}(x) \leq -\eta \} \).
By Sard’s theorem we can approximate \( \phi \), in the \( C^2 \)-norm on any compact subset
of \( V \), by a smooth function \( \tilde{\phi} \) which has only nondegenerate critical points; hence
\( \tilde{\phi} \) remains strictly plurisubharmonic. Set
\[
\phi_1 = \nu \tilde{\phi} + (1 - \nu) \phi.
\]
Then for \( \phi_1 - \tilde{\phi} = (1 - \nu)(\phi - \tilde{\phi}) \) there is an estimate
\[
\left| \phi_1 - \tilde{\phi} \right|_2 \leq \text{const} \left| \phi - \tilde{\phi} \right|_2,
\]
where the norms are \( C^2 \)-norms taken over some compact subset \( L \subset V \), with \( \overline{X} \subset L \).
So by taking a sufficiently good approximation \( \tilde{\phi} \) to \( \phi \), the function \( \phi_1 \) satisfies \( 1^o, 2^o, 3^o, 4^o \); hence the proof is complete.

7. Cohomology of the Border

In spite of the fact that the germ of the border \( X \setminus \overline{X} \) is not unique, it turns out
that the germ of its Dolbeault cohomology is unique.
Theorem 7.1. Assume that the compact contact manifold $M$ is the $C^\infty$ intrinsic boundary of a Stein manifold $X$. Then for any choice of the $\tilde{X}$, in which $\tilde{X}$ is a domain, and for any choice of the Stein neighborhood $Y$, $\tilde{X} \subset Y \subset \tilde{X}$, and for any $0 \leq p \leq n + 1$, we have:

1. $H^{p,q}(Y \setminus X) \simeq H^{p,q}(M) = 0$ for $0 < q < n,$
2. $H^{p,n}(Y \setminus X) \simeq H^{p,n}(M),$
3. $H^{p,n+1}(Y \setminus X) = 0,$
4. $\dim_{\mathbb{C}}H^{p,n}(Y \setminus X) = \infty.$

Here $H^{p,q}(Y \setminus X)$ denotes the Dolbeault cohomology of smooth $\bar{\partial}$-closed $(p,q)$-forms on $Y \setminus X$ modulo those which are $\bar{\partial}$ exact in $Y \setminus X$. Note that $Y \setminus X = (Y \setminus \tilde{X}) \cup M$ has smooth boundary $M$, and we are requiring here that the differential forms be $C^\infty$ up to $M$. $H^{p,q}(M)$ denotes the $\bar{\partial}_M$-cohomology of tangential $\bar{\partial}_M$-closed smooth $(p,q)$-forms on $M$, modulo those that are $\bar{\partial}_M$-exact on $M$.

The results (1), (2), (3), (4) are direct consequences of [AH1], [AH2], see Theorems 5 and 7, or see Theorem 7.2 in [HN3], and [La].

Remark 7.1. When $q = 0$ and $0 \leq p \leq n + 1$ we have that $H^{p,0}(Y \setminus X) \simeq H^{p,0}(Y)$ and $H^{p,0}(\tilde{X}) \simeq H^{p,0}(M)$; see [AH1].

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