

DETERMINING A SOUND-SOFT POLYHEDRAL SCATTERER BY A SINGLE FAR-FIELD MEASUREMENT

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ABSTRACT. We prove that a sound-soft polyhedral scatterer is uniquely determined by the far-field pattern corresponding to an incident plane wave at one given wavenumber and one given incident direction.

Lo duca e io per quel cammino ascoso
intrammo a ritornar nel chiaro mondo;
e senza cura aver d'alcun riposo,
salimmo su, el primo e io secondo,
tanto ch'i' vidi de le cose belle
che porta'l ciel, per un pertugio tondo;
e quindi uscimmo a riveder le stelle.

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1. INTRODUCTION

We consider the acoustic scattering problem with a sound-soft obstacle D . For simplicity of exposition, let us assume here that D is a bounded solid in \mathbb{R}^N , $N \geq 2$, that is that D is a connected compact set which coincides with the closure of its interior. We shall denote by G the exterior of D ,

$$(1.1) \quad G = \mathbb{R}^N \setminus D,$$

and we shall assume throughout that it is connected.

Let $\omega \in \mathbb{S}^{N-1}$ and let $k > 0$ be fixed. Let u be the complex valued solution to

$$(1.2) \quad \begin{cases} \Delta u + k^2 u = 0 & \text{in } G, \\ u(x) = u^s(x) + e^{ik\omega \cdot x} & x \in G, \\ u = 0 & \text{on } \partial G, \\ \lim_{r \rightarrow \infty} r^{(N-1)/2} \left(\frac{\partial u^s}{\partial r} - ik u^s \right) = 0 & r = \|x\|. \end{cases}$$

It is well known that the asymptotic behavior at infinity of the so-called scattered field $u^s(x) = u(x) - e^{ik\omega \cdot x}$ is governed by the formula

$$(1.3) \quad u^s(x) = \frac{e^{ik\|x\|}}{\|x\|^{(N-1)/2}} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{\|x\|}\right) \right\},$$

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as $\|x\|$ goes to ∞ , uniformly in all directions $\hat{x} = x/\|x\| \in \mathbb{S}^{N-1}$. The function u_∞ , which is defined on \mathbb{S}^{N-1} , is called the *far-field pattern* of u^s ; see for instance [2]. In this paper we prove that if $N = 2$ and D is a polygon, or if $N = 3$ and D is a polyhedron, then it is uniquely determined by the far-field pattern u_∞ for one wavenumber k and one incident direction ω ; see Theorem 2.2 below. Indeed, we prove this result independently of the dimension $N \geq 2$, and for this reason it is convenient to express our assumption on D by prescribing that it is an N -dimensional polyhedron, that is, a solid whose boundary is contained in the union of finitely many $(N - 1)$ -dimensional hyperplanes (more precisely, we should say a *polytope* (see for instance [3]), but for the sake of simplicity we prefer to stick to the 3-dimensional terminology). In fact, our result applies to a wider class of scatterers D , which need not to be solids, nor connected, but whose boundary is the finite union of the closures of open subsets of $(N - 1)$ -dimensional hyperplanes. See Section 2 below for a complete formulation.

We wish to mention here that in '94 C. Liu and A. Nachman [4] proved, among various results, that, for $N \geq 2$, u_∞ uniquely determines the convex hull of a polyhedral obstacle D . Their arguments involve a scattering theory analogue of a classical theorem of Polya on entire functions and the reflection principle for solutions of the Helmholtz equation across a flat boundary. In that preprint, they also presented some arguments towards a proof of the unique determination of a general polyhedral obstacle.

More recently, J. Cheng and M. Yamamoto [1], for the case $N = 2$, proved that the far-field pattern uniquely determines a polygonal obstacle D , provided D satisfies an additional geometrical condition, which, roughly speaking, is expressed in terms of the absence of trapped rays in its exterior G . The method of proof in [1] is mainly based on the use of the reflection principle and on the study of the behavior of the nodal line $\{u = 0\}$ of the solution u to (1.2) near the boundary ∂G .

Also in this paper we make use of such a reflection argument, but, rather than examining the boundary behavior of the nodal set we investigate the structure of the nodal set of u in the interior of G . In this respect, the main tool is summarized in the fact that if D is a polyhedron, then the nodal set of u in G does not contain any open portion of an $(N - 1)$ -dimensional hyperplane; see Theorem 2.4.

In Section 2 we set up our main hypotheses on the obstacle, we state the main results (Theorem 2.2 and Theorem 2.4) and prove Theorem 2.2.

In Section 3 we prove Theorem 2.4. The proof is preceded by a sequence of propositions and auxiliary lemmas regarding the study of the nodal sets of real-valued solutions to the Helmholtz equation (see Proposition 3.2) and the construction of a suitable path in G (*cammino ascoso* = *hidden path*) which connects a point in ∂D to infinity, avoiding the singular points in the nodal set of u and intersecting the nodal set orthogonally, Proposition 3.6.

2. THE UNIQUENESS RESULT

Definition 2.1. Let us define a *cell* as the closure of an open subset of an $(N - 1)$ -dimensional hyperplane. We shall say that D is a *polyhedral scatterer* if it is a compact subset of \mathbb{R}^N , such that

- (i) the exterior $G = \mathbb{R}^N \setminus D$ is connected,
- (ii) the boundary of G is given by the finite union of cells C_j .

Let us observe that an equivalent condition to (ii) is that D has the form

$$D = \left(\bigcup_{i=1}^m P_i\right) \cup \left(\bigcup_{j=1}^n S_j\right),$$

where each P_i is a polyhedron and each S_j is a cell; thus we are allowing the simultaneous presence of solid obstacles and of crack-type scatterers. Note also that, by this definition, a cell need not be an $(N - 1)$ -dimensional polyhedron.

We also recall that for any compact set D a weak solution $u \in W_{loc}^{1,2}(G)$ to (1.2) exists and is unique; see for instance [5]. As is well known, u is analytic in G , but, of course, due to the possible irregularity of the boundary of G , the Dirichlet boundary condition in (1.2) is, in general, satisfied in the weak sense only. On the other hand, one can see that, if $x^0 \in \partial G$ is an interior point of one of the cells forming ∂G , then it is a regular point for the Dirichlet problem in G , hence u is continuous up to x^0 and $u(x^0) = 0$.

Theorem 2.2. *Let us fix $\omega \in \mathbb{S}^{N-1}$ and $k > 0$. A polyhedral scatterer D is uniquely determined by the far-field pattern u_∞ .*

A proof of Theorem 2.2 will be obtained as a consequence of Theorem 2.4 below; the following definitions will be needed.

Definition 2.3. Let us denote by \mathcal{N}_u the nodal set of u in G , that is,

$$\mathcal{N}_u = \{x \in G : u(x) = 0\}.$$

We shall say that $x \in \mathcal{N}_u$ is a flat point if there exist a hyperplane Π through x and a positive number r such that $\Pi \cap B_r(x) \subset \mathcal{N}_u$.

Theorem 2.4. *Let D be a polyhedral scatterer. Then \mathcal{N}_u cannot contain any flat point.*

We postpone the proof of this result to Section 3 and we conclude the proof of Theorem 2.2.

Proof of Theorem 2.2. Let D and D' be two polyhedral scatterers and let u' be the solution to (1.2) when D is replaced with D' . Let us assume that for a given $\omega \in \mathbb{S}^{N-1}$ and $k > 0$, $u_\infty = u'_\infty$. We denote with \tilde{G} the connected component of $\mathbb{R}^N \setminus (D \cup D')$ which contains the exterior of a sufficiently large ball. By Rellich's Lemma (see for instance [2, Lemma 2.11]) and unique continuation we infer that $u = u'$ over \tilde{G} .

First, we notice that if $\partial\tilde{G} \subset D \cap D'$, then $D = D' = \mathbb{R}^N \setminus \tilde{G}$. This is due to the fact that both G and $G' = \mathbb{R}^N \setminus D'$ are connected.

Let us assume, by contradiction, that D is different from D' . Then, without loss of generality, we can assume that there exists a point $x' \in (\partial G' \setminus D) \cap \partial\tilde{G}$. We can also assume that x' belongs to the interior of one of the cells composing $\partial G'$, and therefore that there exist a hyperplane Π' and $r > 0$ such that $x' \in S' = \Pi' \cap B_r(x') \subset (\partial G' \setminus D) \cap \partial\tilde{G}$. Since $u = u'$ in \tilde{G} , by continuity we have that $u = u' = 0$ on S' , hence S' is contained into the nodal set of u , that is, $S' \subset \mathcal{N}_u$, and, consequently, x' is a flat point for \mathcal{N}_u . This contradicts Theorem 2.4. \square

3. THE *hidden path* AND THE PROOF OF THEOREM 2.4

We start with a well-known property of the nodal set of u .

Lemma 3.1. *The nodal set \mathcal{N}_u is bounded.*

Proof. By (1.3), we have that the scattered field $u^s(x)$ tends to zero, as $\|x\|$ tends to infinity, uniformly for all directions $\hat{x} = x/\|x\| \in \mathbb{S}^{N-1}$. Then the lemma immediately follows by observing that $|u(x)| = |u^s(x) + e^{ik\omega \cdot x}| \rightarrow 1$ uniformly as $\|x\| \rightarrow \infty$. \square

Next we discuss some properties of the nodal set of real-valued solutions to the Helmholtz equation. Let v be a nontrivial real-valued solution to

$$(3.1) \quad \Delta v + k^2 v = 0 \text{ in } G,$$

in a connected open set G . We denote the *nodal set* of v as

$$\mathcal{N}_v = \{x \in G : v(x) = 0\}$$

and we let \mathcal{C}_v be the set of *nodal critical points*, that is,

$$\mathcal{C}_v = \{x \in G : v(x) = 0 \text{ and } \nabla v(x) = 0\}.$$

We say that $\Sigma \subset \mathcal{N}_v$ is a *regular portion* of \mathcal{N}_v if it is an analytic open and connected hypersurface contained in $\mathcal{N}_v \setminus \mathcal{C}_v$. Let us denote by $A_1, A_2, \dots, A_n, \dots$ the *nodal domains* of v , that is, the connected components of the open set $\{x \in G : v(x) \neq 0\} = G \setminus \mathcal{N}_v$.

Proposition 3.2. *We can order the nodal domains $A_1, A_2, \dots, A_n, \dots$ in such a way that for any $j \geq 2$ there exist i , $1 \leq i < j$, and a regular portion Σ_j of \mathcal{N}_v such that*

$$(3.2) \quad \Sigma_j \subset \partial A_i \cap \partial A_j.$$

We subdivide the main steps of the proof of this proposition in the next two lemmas.

Lemma 3.3. *Let A_1, \dots, A_n be nodal domains and let $A = \overline{A_1 \cup \dots \cup A_n}^\circ$. If $x \in \partial A \cap G$, then for any $r > 0$ there exists $y \in (B_r(x) \cap G) \setminus \overline{A}$.*

Proof. We can assume, without loss of generality, that $r > 0$ is such that $B_r(x) \subset G$. Then, let us assume, by contradiction, that we have $B_r(x) \subset \overline{A}$. Then we infer that $x \in \overline{A}^\circ = A$ and this contradicts the fact that $x \in \partial A$. \square

Lemma 3.4. *Let A_1, \dots, A_n be nodal domains and let $A = \overline{A_1 \cup \dots \cup A_n}^\circ$. If $x \in \partial A \cap G$, then for any $r > 0$ there exists $y \in B_r(x) \cap \partial A \cap G$ such that $\nabla v(y) \neq 0$.*

Proof. Note that, if we a-priori knew that ∂A were smooth in a small neighborhood of x , then the thesis would be a straightforward consequence of Holmgren's Theorem. Lacking such an a-priori smoothness, we proceed as follows. We can assume, without loss of generality, that $r > 0$ is such that $B_r(x) \subset G$. Assume, by contradiction, that $\nabla v \equiv 0$ on $B_r(x) \cap \partial A$ and set $w = v$ in $B_r(x) \cap A$, $w = 0$ in $B_r(x) \setminus A$. One can easily verify that $w \in W^{2,\infty}(B_r(x))$ and also that w is a strong solution to the Helmholtz equation in $B_r(x)$. Now, by Lemma 3.3, $w \equiv 0$ on an open subset of $B_r(x)$ and hence by unique continuation $w \equiv 0$ in $B_r(x)$ which is impossible. \square

Proof of Proposition 3.2. We proceed by induction. We choose A_1 arbitrarily.

Let us assume that we have ordered A_1, \dots, A_n in such a way that there exist $\Sigma_2, \dots, \Sigma_n$ regular portions of \mathcal{N}_v such that (3.2) holds for any $j = 2, \dots, n$ and for some $i < j$.

Let $A = \overline{A_1 \cup \dots \cup A_n}$. If $A = G$, then we are done. Otherwise, since G is connected, we can find $x \in \partial A \cap G$. We apply Lemma 3.4 and we fix, for $r > 0$ small enough, a point $y \in B_r(x) \cap \partial A \cap G$ such that $\nabla v(y) \neq 0$. There exists a positive r_1 such that $B_{r_1}(y) \cap \partial A$ is a regular portion of \mathcal{N}_v and there exist exactly two nodal domains, $\tilde{A}_1 \subset A$ and \tilde{A}_2 with $\tilde{A}_2 \cap A = \emptyset$, whose intersections with $B_{r_1}(y)$ are not empty. It is clear that \tilde{A}_1 coincides with A_i for some $i = 1, \dots, n$, and if we pick $A_{n+1} = \tilde{A}_2$ and choose $\Sigma_{n+1} = B_{r_1}(y) \cap \mathcal{N}_v$, then (3.2) holds for $j = n + 1$, also. \square

We now show that we are able to connect points of $G \setminus \mathcal{N}_v$ with suitable regular curves contained in G which avoid the nodal critical points of v . Here and in the sequel we shall say that a curve $\gamma = \gamma(t)$ is regular if it is C^1 -smooth and $\frac{d}{dt}\gamma(t) \neq 0$ for every t .

Proposition 3.5. *Let x_1 and y_1 belong to $G \setminus \mathcal{N}_v$. Then there exists a regular curve γ contained in G and connecting x_1 with y_1 such that the following conditions are satisfied:*

$$(3.3) \quad \gamma \cap \mathcal{C}_v = \emptyset,$$

$$(3.4) \quad \text{if } x \in \mathcal{N}_v \cap \gamma, \text{ then } \gamma \text{ intersects } \mathcal{N}_v \text{ at } x \text{ orthogonally.}$$

Proof. We order the nodal domains $A_1, A_2, \dots, A_n, \dots$ according to Proposition 3.2. Without loss of generality, we can assume that $x_1 \in A_1$ and $y_1 \in A_i$ for some $i > 1$. By Proposition 3.2, we can find i_l , with $l = 1, \dots, n$, such that $i_1 = 1, i_n = i$, and, for any $l = 2, \dots, n, i_{l-1} < i_l$ and there exists a regular portion of $\mathcal{N}_v, \Sigma_{i_l}$, such that $\Sigma_{i_l} \subset \partial A_{i_{l-1}} \cap \partial A_{i_l}$.

Let σ_l be a line segment crossing Σ_{i_l} orthogonally and let it be small enough such that $\sigma_l \subset A_{i_{l-1}} \cup \Sigma_{i_l} \cup A_{i_l}$. Let $y_l^- \in A_{i_{l-1}}, y_l^+ \in A_{i_l}$ be the endpoints of σ_l . Let β_1 be a regular path within A_1 which joins x_1 to y_2^- and has a C^1 -smooth junction with σ_2 at y_2^- . For every $l = 2, \dots, n - 1$, let β_l be a regular path within A_{i_l} which joins y_l^+ to y_{l+1}^- and has C^1 -smooth junctions with the segments σ_l and σ_{l+1} , at the points y_l^+, y_{l+1}^- , respectively. Let β_n be a regular path within A_{i_n} which joins y_n^+ to y_1 and has a C^1 -smooth junction with σ_n at y_n^+ . We form the curve γ by attaching consecutively the curves $\beta_1, \sigma_1, \beta_2, \sigma_2, \dots$ up to β_n . \square

We have what is needed to build up our *hidden path*. From now on we consider $G = \mathbb{R}^N \setminus D$ and $v = \Re u$. Note that $\mathcal{N}_u \subset \mathcal{N}_v$.

Proposition 3.6. *Let $x_1 \in \partial G$ be such that x_1 belongs to the interior of one of the cells forming ∂G and $\frac{\partial v}{\partial \nu}(x_1) \neq 0, \nu$ being the unit normal to ∂G at x_1 , pointing to the interior of G . Let $y_1 \in \mathcal{N}_u \setminus \mathcal{C}_v$ be fixed.*

Then there exists a regular curve $\gamma : [0, +\infty) \mapsto \mathbb{R}^N$, such that the following conditions are satisfied:

- (1) $\gamma(0) = x_1$;
- (2) $\gamma(t) \in G$ for every $t > 0$;
- (3) there exists t_1 such that $\gamma(t_1) = y_1$;

- (4) $\lim_{t \rightarrow +\infty} \|\gamma(t)\| = +\infty$;
 (5) if, for some t , $\gamma(t) \in \mathcal{N}_u$, then $\gamma(t) \notin \mathcal{C}_v$ and γ intersects \mathcal{N}_v at $x = \gamma(t)$ orthogonally.

Proof. Let A_1 be the nodal domain of v such that $x_1 \in \partial A_1$ and let η_1 be a line segment in A_1 having x_1 as an endpoint and which is orthogonal to ∂G there. Let $x'_1 \in A_1$ be the other endpoint of η_1 . Let η_2 be a line segment crossing \mathcal{N}_v orthogonally at the point y_1 . Let it be small enough so that v is strictly monotone on η_2 . Let y'_1, y''_1 be the endpoints of η_2 . By Proposition 3.5, we can find a regular curve γ_1 joining x'_1 to y'_1 and satisfying conditions (3.3), (3.4). We can also choose γ_1 in such a way that it has C^1 -smooth junctions with the segments η_1, η_2 at its endpoints. Let $R > 0$ be large enough so that $\mathcal{N}_u \subset B_R(0)$ and let us fix $z_1, |z_1| > R$. Again by Proposition 3.5, we can find a regular curve γ_2 joining y''_1 to z_1 and satisfying conditions (3.3), (3.4) and also such that it has a C^1 -smooth junction with η_2 at the point y''_1 . Next let us fix a regular path γ_3 in $\mathbb{R}^N \setminus B_R(0)$ joining z_1 to ∞ having a C^1 -smooth junction with γ_2 at z_1 . The resulting path γ is obtained by attaching the paths $\eta_1, \gamma_1, \eta_2, \gamma_2, \gamma_3$. \square

Lemma 3.7. *Let the assumptions of Proposition 3.6 be satisfied and let γ be the path constructed there. If $y' = \gamma(t') \in \mathcal{N}_u$ is a flat point, then there exists $t'' > t'$ such that $y'' = \gamma(t'') \in \mathcal{N}_u$ is a flat point.*

Proof. Let Π' be the plane through y' and let $r > 0$ be such that $S' = \Pi' \cap B_r(y') \subset \mathcal{N}_u$.

Let \tilde{S}' be the connected component of $\Pi' \setminus D$ containing y' . We have that, by analytic continuation, u is identically zero on \tilde{S}' . Therefore, we can immediately notice that, by Lemma 3.1, \tilde{S}' is bounded.

Let $\epsilon > 0$ be small enough so that $v(\gamma(t))$ is strictly monotone for $t' - \epsilon \leq t \leq t' + \epsilon$, and let us set $y^- = \gamma(t' - \epsilon)$, $y^+ = \gamma(t' + \epsilon)$.

Let G^+ be the connected component of $G \setminus \tilde{S}'$ containing y^+ and let G^- be the connected component of $G \setminus \tilde{S}'$ containing y^- . Let us remark that it may happen that $G^+ = G^-$.

Let us denote with R the reflection in Π' . We call E^+ the connected component of $G^+ \cap R(G^-)$ containing y^+ and E^- the connected component of $G^- \cap R(G^+)$ containing y^- . We observe that $E^- = R(E^+)$ and we set $E = E^+ \cup E^- \cup \tilde{S}'$.

We have that E is a connected open set and, by construction, the boundary of E is composed by cells, more precisely by subsets of the cells of ∂G and of $R(\partial G)$. Furthermore, in E we have that $u = -Ru$ where $Ru(x) = u(R(x))$. In fact, $u + Ru$ is a solution of the Helmholtz equation in E with zero Cauchy data on \tilde{S}' .

In other words, u is odd symmetric in E , with respect to the plane Π' . Hence, we infer that $u = 0$ on $E \cap \Pi'$ and, moreover, u is continuous up to the interior of each cell forming ∂E and $u = 0$ there. Furthermore, since u is continuous in G , we have that $u = 0$ in all of $\partial E \cap G$. That is, $\partial E \cap G \subset \mathcal{N}_u$.

Let us exclude now the case that E is unbounded. In fact, ∂E is bounded and, if E were unbounded, then E would contain $\mathbb{R}^N \setminus B_\rho(0)$, for some sufficiently large $\rho > 0$. Then $u = 0$ on $\Pi' \setminus B_\rho(0)$ and this contradicts Lemma 3.1.

Thus E is a bounded open set containing y' . Since γ is not bounded, there exists $t'' > t'$ such that $\gamma(t'') \in \partial E \cap G$. We have that $y'' = \gamma(t'') \in \mathcal{N}_u$ and, by the properties of γ , it is not a critical point of v . Let C be a cell of ∂E such that $y'' \in C$ and let Π'' be the hyperplane containing C . Let $r > 0$ be such that $B_r(y'') \subset G$.

We have that $u = 0$ on $C \cap B_r(y'')$ and hence, by analytic continuation, $u = 0$ on $\Pi'' \cap B_r(y'')$; therefore $\Pi'' \cap B_r(y'') \subset \mathcal{N}_u$. \square

Proof of Theorem 2.4. Let us assume, by contradiction, that $y_1 \in \mathcal{N}_u$ is a flat point. Let Π_1 be the plane through y_1 and $r > 0$ such that $S_1 = \Pi_1 \cap B_r(y_1) \subset \mathcal{N}_u$. By the uniqueness for the Cauchy problem, S_1 contains at least one point $y'_1 \notin \mathcal{C}_v$. Thus, without loss of generality, we can assume that there exists a flat point $y_1 \in \mathcal{N}_u \setminus \mathcal{C}_v$.

We arbitrarily fix a point x_1 belonging to the interior of one of the cells of ∂G . Again by the uniqueness for the Cauchy problem, we can assume, without loss of generality, that $\frac{\partial v}{\partial \nu} \neq 0$, ν being the interior unit normal to ∂G at the point x_1 .

We choose γ according to Proposition 3.6. Then, applying iteratively Lemma 3.7, we can find a strictly increasing sequence $\{t_n\}_{n \in \mathbb{N}}$ such that, for any n , $y_n = \gamma(t_n)$ is a flat point of u and, by construction of γ , y_n is not a critical point of v . Since \mathcal{N}_u is bounded and $\lim_{t \rightarrow +\infty} \|\gamma(t)\| = +\infty$, there exists a finite T such that $\lim_{n \rightarrow +\infty} t_n = T$. We have that $\tilde{y} = \gamma(T)$ belongs to \mathcal{N}_u and, again by the properties of γ , \tilde{y} is not a critical point of v and γ is orthogonal to \mathcal{N}_v there. Therefore, there exists $\delta > 0$ such that $v(\gamma(t)) \neq 0$ for every $T - \delta < t < T$ and this contradicts the fact that $\gamma(t_n) \in \mathcal{N}_u$ for any n . \square

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