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BLOWUP FOR $u_t = \Delta u + |\nabla u|^2 u$ FROM $\mathbb{R}^n$ INTO $\mathbb{R}^m$

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ABSTRACT. In this note we consider the global regularity of smooth solutions $u = (u^1, \dots, u^m)$ to the vector-valued Cauchy problem

$$u_t = \Delta u + |\nabla u|^2 u \quad \text{in } \mathbb{R}^n \times [0,\infty), \quad u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n.$$

We show that if $n, m \geq 3$, the gradient-blowup phenomenon occurs in finite time for suitably chosen u_0 vanishing at infinity. We also present a simple example of the L^∞-blowup solutions for $|u_0| \equiv 1 + \epsilon$ for any $\epsilon > 0$, if $m \geq 2$.

1. Introduction

In this note we consider the global (in time) regularity of smooth solutions $u = (u^1, \dots, u^m)$ to the Cauchy problem

$$\begin{cases} u_t = \Delta u + |\nabla u|^2 u & \text{in } \mathbb{R}^n \times [0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where

$$|\nabla u|^2 = \sum_{i=1}^m |\nabla u^i|^2.$$

More precisely, we are concerned with the existence and nonexistence of global smooth solutions to (1.1) in the case when $m \geq 2$. Here, recall that for every $u_0 \in C_b^1(\mathbb{R}^n, \mathbb{R}^m)$, there exists a unique, local in time, smooth solution u for a sufficiently small $T > 0$ depending only on $||\nabla u||_{L^\infty}$, which can be proved via the contraction mapping principle argument (see e.g. [3, Proposition 15.3.1]).

In the scalar-valued case ($m = 1$), all solutions can be extended globally. Indeed, since the transformation

$$v = \int_0^u e^{s^2/2} ds$$

satisfies $v_t = \Delta v$, one can always obtain a unique global smooth solution u to the single equation for every $u_0 \in L^\infty(\mathbb{R}^n, \mathbb{R})$.

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On the other hand, the vector-valued problem is completely different from the single case. For instance, consider the harmonic map heat flow into the sphere $S^{m-1} \subset \mathbb{R}^m$:

$$\begin{cases} u_t = \Delta u + |\nabla u|^2 u & \text{in } \mathbb{R}^n \times [0,\infty), \\ u(x,0) = u_0(x) \in S^{m-1}. \end{cases} \tag{1.2}$$

It is well known that the problem can admit the formation of finite-time singularities. Indeed, Coron and Ghidaglia [2] proved that if $m = n+1$ and $n \geq 3$, there does occur the gradient-blowup phenomenon in finite time for some rotationally symmetric initial data. Furthermore, for the two-dimensional domain it has been shown by Chang, Ding and Ye [1] that the Dirichlet problem from D^2 into S^2 also develops finite-time singularities. Hence, we would like to know what occurs for smooth local solutions to (1.1) with the exception of such a geometric constraint.

Our main result in this note is as follows.

Theorem 1.1. *Let $m \geq 2$. There is a family of initial data u_0 with $|u_0| = 1 + \epsilon$ for any $\epsilon > 0$ such that the L^∞-blowup solutions to (1.1) occur in finite time.*

If $n, m \geq 3$, then there exists a family of initial data with $||u_0||_{L^\infty} \leq 1$ and $\lim_{|x|\to\infty} u_0(x) = 0$ such that the gradient of the solution blows up in finite time.

2. Proof of Theorem 1.1

Suppose that u_0 is given by

$$u_0(x) = \mu_0 \Big(\cos\Big(a_0 + \sum_{j=1}^n a_j x_j\Big), \sin\Big(a_0 + \sum_{j=1}^n a_j x_j\Big), 0, \dots, 0 \Big), \tag{2.1}$$

where μ_0 and $\{a_j\}_{0 \leq j \leq n}$ are constants. By uniqueness, the solution u to (1.1) reduces to two functions $f = f(x,t)$ and $\varphi = \varphi(x,t)$ satisfying

$$\begin{cases} f_t = \Delta f + |\nabla f|^2 f + f(f^2-1)|\nabla\varphi|^2 & \text{in } \mathbb{R}^n \times [0,\infty), \\ \varphi_t = \Delta\varphi + 2\langle \nabla f, \nabla\varphi \rangle & \text{in } \mathbb{R}^n \times [0,\infty), \\ f(x,0) = f_0(x), \quad \varphi(x,0) = \varphi_0(x) & \text{in } \mathbb{R}^n, \end{cases}$$

where $f_0(x) \equiv \mu_0$ and $\varphi_0(x) := a_0 + \sum_{j=1}^n a_j x_j$. Let $\mu = \mu(t)$ be a solution to the ODE:

$$\mu_t = (a_1^2 + \cdots + a_n^2)\mu(\mu^2 - 1), \qquad \mu(0) = \mu_0.$$

Then it is obvious that

$$u(x,t) = \mu(t) \Big(\cos\Big(a_0 + \sum_{j=1}^n a_j x_j\Big), \sin\Big(a_0 + \sum_{j=1}^n a_j x_j\Big), 0, \dots, 0 \Big)$$

is a solution to (1.1). Hence, if $|\mu_0| > 1$ and $a_1^2 + \cdots + a_n^2 \neq 0$, the solution u blows up in finite time.

Next, we shall show the existence of the gradient-blowup solutions. Let k be an integer $\leq \min\{m, n\}$. Suppose that u_0 is a smooth data such that

$$u_0(x) = \frac{f_0(r)}{r}(x_1, \dots, x_k, 0, \dots, 0), \quad r = \sqrt{x_1^2 + \cdots + x_k^2}, \tag{2.2}$$

which must vanish at the origin for the sake of compatibility. By uniqueness, one deduces that

$$u(x,t) = \frac{f(r,t)}{r}(x_1, \dots, x_k, 0, \dots, 0). \tag{2.3}$$

Then it follows from straightforward calculations that f satisfies

$$\begin{cases} f_t = f_{rr} + \dfrac{k-1}{r} f_r - \dfrac{k-1}{r^2} f + \left(f_r^2 + \dfrac{k-1}{r^2} f^2 \right) f =: \tau(f), \\ f(r,0) = f_0(r), \quad f(0,t) = 0 \end{cases} \tag{2.4}$$

for $r, t \geq 0$.

Here, let us establish the following comparison principle.

Lemma 2.1. *Let*

$$f, \tilde{f} \in C^{2,1}((0,\infty) \times (0,T)) \cap C_b([0,\infty) \times [0,T]).$$

Suppose that f and $\tilde{f}$ satisfy

$$\begin{cases} e^{f^2/2}(f_t - \tau(f)) \geq e^{\tilde{f}^2/2}(\tilde{f}_t - \tau(\tilde{f})), \\ f(r,0) \geq \tilde{f}(r,0), \quad f(0,t) \geq \tilde{f}(0,t) \end{cases} \tag{2.5}$$

for $(r,t) \in [0,\infty) \times [0,T]$. Then $f \geq \tilde{f}$ on $[0,\infty) \times [0,T]$.

Proof. Define

$$G(y) = \int_0^y e^{s^2/2} ds$$

and its inverse function $\mathfrak{g} = G^{-1}$. If we set

$$h = G(f) - G(\tilde{f}) = \int_{\tilde{f}}^{f} e^{s^2/2} ds,$$

then (2.5) corresponds to

$$h_t - h_{rr} - \frac{k-1}{r} h_r + \frac{k-1}{r^2} \psi h \geq 0, \quad h(r,0) \geq 0, \; h(0,t) \geq 0,$$

where

$$\psi(r,t) = 1 - \int_0^1 \Big(2\mathfrak{g}\big(\theta G(f) + (1-\theta) G(\tilde{f})\big)^2 + \mathfrak{g}\big(\theta G(f) + (1-\theta) G(\tilde{f})\big)^4 \Big) d\theta.$$

Then the classical maximum principle argument yields that $h \geq 0$, from which it follows that $f \geq \tilde{f}$. $\square$

Let $\lambda = \lambda(t)$ solve the ODE: $\lambda_t = -1/\lambda$, $\lambda(0) = \lambda_0 > 0$, i.e., $\lambda(t) = \sqrt{\lambda_0^2 - 2t}$. Let us prove that the following ansatz:

$$\tilde{f}(r,t) = \frac{2\mu_0 \lambda r}{\lambda^2 + r^2} \qquad (0 < \mu_0 \le 1) \tag{2.6}$$

is a blowup subsolution to (2.4), provided that λ_0 is small enough and for some μ_0. Notice that $\tilde{f}_r(0,t) \to \infty$ as $t \to T = \lambda_0^2/2$ and also that $||\tilde{f}(\cdot,t)||_{L^\infty} = \mu_0$. We have

$$\begin{aligned}
\tau(\tilde{f}) &= 2\mu_0\lambda\bigg(\frac{2r(-3\lambda^2+r^2)}{(\lambda^2+r^2)^3} + \frac{(k-1)(\lambda^2-r^2)}{r(\lambda^2+r^2)^2} - \frac{k-1}{r(\lambda^2+r^2)} \\
&\qquad + \frac{4\mu_0^2\lambda^2 r}{\lambda^2+r^2}\bigg(\frac{(\lambda^2-r^2)^2}{(\lambda^2+r^2)^4} + \frac{k-1}{(\lambda^2+r^2)^2}\bigg)\bigg) \\
&= \frac{2\mu_0\lambda}{r(\lambda^2+r^2)^5}\Big((\lambda^2+r^2)^2\big(2r^2(-3\lambda^2+r^2) + (k-1)(\lambda^4-r^4) \\
&\qquad - (k-1)(\lambda^2+r^2)^2\big) + 4\mu_0^2\lambda^2 r\big((\lambda^2-r^2)^2 + (k-1)(\lambda^2+r^2)^2\big)\Big) \\
&= \frac{2\mu_0\lambda r}{(\lambda^2+r^2)^5}\Big(-2r^2(\lambda^2+r^2)^2\big((k+2)\lambda^2 + (k-2)r^2\big) \\
&\qquad + 4\mu_0^2\lambda^2\big(k\lambda^4 + 2(k-2)\lambda^2 r^2 + kr^4\big)\Big) \\
&= \frac{2\mu_0\lambda r}{(\lambda^2+r^2)^5}\Big(-2r^2\big((k+2)\lambda^6 + (3k+2)\lambda^4 r^2 + (3k-2)\lambda^2 r^4 + (k-2)r^6\big) \\
&\qquad + 4\mu_0^2\lambda^2\big(k\lambda^4 + 2(k-2)\lambda^2 r^2 + kr^4\big)\Big) \\
&= \frac{2\mu_0\lambda r}{(\lambda^2+r^2)^5}\Big(4k\mu_0^2\lambda^6 + 2\big(4(k-2)\mu_0^2 - (k+2)\lambda^2\big)\lambda^4 r^2 \\
&\qquad + \big(4k\mu_0^2 - 2(3k+2)\lambda^2\big)\lambda^2 r^4 - 2(3k-2)\lambda^2 r^6 - 2(k-2)r^8\Big).
\end{aligned}$$

Since

$$\tilde{f}_t = \frac{2\mu_0 r(\lambda^2-r^2)}{\lambda(\lambda^2+r^2)^2} = \frac{2\mu_0 r(\lambda^8 + 2\lambda^6 r^2 - 2\lambda^2 r^6 - r^8)}{\lambda(\lambda^2+r^2)^5},$$

we have

$$\begin{aligned}
&\tilde{f}_t - \tau(\tilde{f}) \\
&= -\frac{2\mu_0 r}{\lambda(\lambda^2+r^2)^5}\Big((-1+4k\mu_0^2)\lambda^8 + 2\big(-1+4(k-2)\mu_0^2 - (k+2)\lambda^2\big)\lambda^6 r^2 \\
&\qquad + 2\big(2k\mu_0^2 - (3k+2)\lambda^2\big)\lambda^4 r^4 + 2\big(1-(3k-2)\lambda^2\big)\lambda^2 r^6 + \big(1-2(k-2)\lambda^2\big)r^8\Big).
\end{aligned}$$

Taking $\lambda_0 \le 1/\sqrt{3k-2}$, and if one chooses $\mu_0 \in (0,1]$ such that

$$\mu_0^2 \ge \max\bigg\{\frac{1}{4k},\ \frac{(k+2)\lambda_0^2+1}{4(k-2)},\ \frac{(3k+2)\lambda_0^2}{2k}\bigg\},$$

then $\tilde{f}$ is a blowup subsolution to (2.4). On the other hand, since the constant one is a global supersolution, if $\tilde{f}_0 \le f_0 \le 1$, then by Lemma 2.1 the solution with $f = f_0$ satisfies $\tilde{f} \le f \le 1$. Therefore, the derivative f_r must blow up at $r = 0$ in finite time. Hence, this completes the proof of Theorem 1.1.

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