

A GENERAL FUNCTIONAL EQUATION AND ITS STABILITY

JOHN A. BAKER

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ABSTRACT. Suppose that V and B are vector spaces over \mathbb{Q} , \mathbb{R} or \mathbb{C} and $\alpha_0, \beta_0, \dots, \alpha_m, \beta_m$ are scalar such that $\alpha_j \beta_k - \alpha_k \beta_j \neq 0$ whenever $0 \leq j < k \leq m$. We prove that if $f_k : V \rightarrow B$ for $0 \leq k \leq m$ and

$$(*) \quad \sum_{k=0}^m f_k(\alpha_k x + \beta_k y) = 0 \quad \text{for all } x, y \in V,$$

then each f_k is a “generalized” polynomial map of “degree” at most $m - 1$.

In case $V = \mathbb{R}^n$ and $B = \mathbb{C}$ we show that if some f_k is bounded on a set of positive inner Lebesgue measure, then it is a genuine polynomial function.

Our main aim is to establish the stability of $(*)$ (in the sense of Ulam) in case B is a Banach space.

We also solve a distributional analogue of $(*)$ and prove a mean value theorem concerning harmonic functions in two real variables.

BLANKET ASSUMPTIONS

Throughout this paper $F = \mathbb{Q}$, \mathbb{R} or \mathbb{C} , V and B are vector spaces over F , $m \in \mathbb{N}$, and $\alpha_0, \beta_0, \dots, \alpha_m, \beta_m$ are given members of F . Our aim is to study the functional equation

$$(*) \quad \sum_{k=0}^m f_k(\alpha_k x + \beta_k y) = 0 \quad \text{for all } x, y \in V,$$

where the functions $f_k : V \rightarrow B$ may be thought of as the unknowns. The stability of $(*)$ (in the sense of Ulam) is of principal interest.

BACKGROUND RESULTS

A function $a : V \rightarrow B$ is said to be *additive* provided $a(x + y) = a(x) + a(y)$ for all $x, y \in V$; in this case it is easily seen that $a(rx) = ra(x)$ for all $x \in V$ and all $r \in \mathbb{Q}$.

If $k \in \mathbb{N}$ and $a : V^k \rightarrow B$, then we say that a is *k-additive* provided it is additive in each variable; we say that a is *symmetric* provided $a(x_1, x_2, \dots, x_k) = a(y_1, y_2, \dots, y_k)$ whenever $x_1, x_2, \dots, x_k \in V$ and (y_1, y_2, \dots, y_k) is a permutation of (x_1, x_2, \dots, x_k) .

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If $k \in \mathbb{N}$ and $a : V^k \rightarrow B$ is symmetric and k -additive, let $a^*(x) = a(x, x, \dots, x)$ for $x \in V$ and note that $a^*(rx) = r^k a^*(x)$ whenever $x \in V$ and $r \in \mathbb{Q}$. Such a function a^* will be called a *monomial* function of *degree* k (assuming $a^* \not\equiv 0$).

A function $p : V \rightarrow B$ is called a *generalized polynomial* (GP) function of *degree* $m \in \mathbb{N}$ provided there exist $a_0 \in B$ and symmetric k -additive functions $a_k : V^k \rightarrow B$ (for $1 \leq k \leq m$) such that

$$p(x) = a_0 + \sum_{k=1}^m a_k^*(x) \quad \text{for all } x \in V,$$

and $a_m^* \not\equiv 0$. In this case

$$p(rx) = a_0 + \sum_{k=1}^m r^k a_k^*(x) \quad \text{for } x \in V \text{ and } r \in \mathbb{Q}.$$

Let B^V denote the vector space (over F) consisting of all maps from V into B . For $h \in V$ define the linear “difference” operator Δ_h on B^V by

$$\Delta_h f(x) = f(x+h) - f(x) \quad \text{for } f \in B^V \text{ and } x \in V.$$

Notice that these difference operators commute ($\Delta_{h_1} \Delta_{h_2} = \Delta_{h_2} \Delta_{h_1}$ for all $h_1, h_2 \in V$) and if $h \in V$ and $n \in \mathbb{N}$, then Δ_h^n —the n -th iterate of Δ_h —satisfies

$$\Delta_h^n f(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(x+kh) \quad \text{for } f \in B^V \text{ and } x, h \in V.$$

The following two theorems were proved by Mazur and Orlicz [12] and [13], and in greater generality by Djoković [5].

Theorem A. *If $n \in \mathbb{N}$ and $f : V \rightarrow B$, then the following are equivalent.*

- (i) $\Delta_h^n f(x) = 0$ for all $x, h \in V$.
- (ii) $\Delta_{h_n} \cdots \Delta_{h_1} f(x) = 0$ for all $x, h_1, \dots, h_n \in V$.
- (iii) f is a GP function of degree at most $n - 1$.

The functional equation (i) was (perhaps first) studied by Fréchet [7]. The (Ulam) stability of (i) and (ii) was established by Hyers [9] and by Albert and Baker [1]. From Theorem 3 and Lemma 1(b) thereof we have

Theorem B. *Suppose that $2 \leq n \in \mathbb{N}$, $\delta \geq 0$, B is a Banach space, $f : V \rightarrow B$ and*

$$\|\Delta_{h_1} \cdots \Delta_{h_n} f(x)\| \leq \delta \quad \text{for all } x, h_1, \dots, h_n \in V.$$

Then there exists a GP function $p : V \rightarrow B$ of degree at most $n - 1$ such that $\|f(x) - p(x)\| \leq 2\delta$ for all $x \in V$.

In [15], Whitney proved that if $V = \mathbb{R}$ (or even $(0, +\infty)$), $f : V \rightarrow \mathbb{R}$, $\delta > 0$,

$$|\Delta_h^n f(x)| \leq \delta \quad \text{for all } x, h \in V$$

and f is bounded on some interval, then there is a real polynomial function p , of degree at most $n - 1$, and a constant $k_n > 0$ such that $|f(x) - p(x)| \leq k_n \delta$ for all $x \in V$.

MAIN RESULTS

Now let's address the stability of (*). Suppose that $m \in \mathbb{N}$, $\alpha_k, \beta_k \in F$ for $0 \leq k \leq m$, B is a Banach space, $f_0, f_1, \dots, f_m : V \rightarrow B$, $\delta \geq 0$ and

$$(\dagger) \quad \left\| \sum_{k=0}^m f_k(\alpha_k x + \beta_k y) \right\| \leq \delta \quad \text{for all } x, y \in V.$$

We assume that $\alpha_j \beta_k - \alpha_k \beta_j \neq 0$ whenever $0 \leq j < k \leq m$, i.e. $(\alpha_0, \beta_0), (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)$ are pairwise linearly independent in F^2 . To see why this is a natural assumption, suppose on the contrary (for example) that $(\alpha_m, \beta_m) = \gamma(\alpha_{m-1}, \beta_{m-1})$ for some $\gamma \in F$. Then $f_{m-1}(\alpha_{m-1}x + \beta_{m-1}y) + f_m(\alpha_m x + \beta_m y) = g(\alpha_{m-1}x + \beta_{m-1}y)$ for all $x, y \in V$ where $g(z) = f_{m-1}(z) + f_m(\gamma z)$ for $z \in V$ and thus (\dagger) can be written as an inequality involving m , rather than $m + 1$, functions.

Suppose $m = 1$ so that (\dagger) becomes

$$\|f_0(\alpha_0 x + \beta_0 y) + f_1(\alpha_1 x + \beta_1 y)\| \leq \delta, \quad x, y \in V.$$

Given $u, v \in V$, since $\alpha_0 \beta_1 - \alpha_1 \beta_0 \neq 0$ there exist unique $x, y \in V$ such that $u = \alpha_0 x + \beta_0 y$ and $v = \alpha_1 x + \beta_1 y$. Hence

$$\|f_0(u) + f_1(v)\| \leq \delta \quad \text{for all } u, v \in V.$$

If $c_0 = -f_1(0)$ and $c_1 = -f_0(0)$ we have $\|c_0 + c_1\| \leq \delta$, $\|f_0(u) - c_0\| \leq \delta$ and $\|f_1(v) - c_1\| \leq \delta$ for all $u, v \in V$. Henceforth we therefore assume that $m \geq 2$.

Theorem 1. *Suppose that B is a real (or complex) Banach space, $f_k : V \rightarrow B$ for $0 \leq k \leq m$, $\delta > 0$ and (\dagger) holds. Then for each $k = 0, 1, \dots, m$,*

$$\left\| \Delta_{h_m} \cdots \Delta_{h_1} f_k(x) \right\| \leq 2^m \delta \quad \text{for all } x, h_1, \dots, h_m \in V,$$

and there exists a GP function $p_k : V \rightarrow \mathbb{R}$ and $c_k \in B$ such that

$$\|(f_k(x) - c_k) - p_k(x)\| \leq 2^{m+1} \delta \quad \text{for all } x \in V.$$

Moreover

$$\sum_{k=0}^m p_k(\alpha_k x + \beta_k y) = 0 \quad \text{for all } x, y \in V.$$

Proof. For $0 \leq j \leq k \leq m$ let $d_{jk} = \alpha_j \beta_k - \alpha_k \beta_j$ so that $d_{jk} \neq 0$ if $j < k$ and $d_{kk} = 0$.

For $0 \leq k \leq m$ and $x, y, h_1 \in V$,

$$\alpha_k(x + \beta_m h_1) + \beta_k(y - \alpha_m h_1) = \alpha_k x + \beta_k y + d_{km} h_1$$

so that

$$\left\| \sum_{k=0}^m f_k(\alpha_k x + \beta_k y + d_{km} h_1) \right\| = \left\| \sum_{k=0}^m f_k(\alpha_k(x + \beta_m h_1) + \beta_k(y - \alpha_m h_1)) \right\| \leq \delta.$$

Using this inequality and (\dagger) we find that

$$\begin{aligned} & \left\| \sum_{k=0}^m \{f_k(\alpha_k x + \beta_k y + d_{km} h_1) - f_k(\alpha_k x + \beta_k y)\} \right\| \\ & \leq \left\| \sum_{k=0}^m f_k(\alpha_k x + \beta_k y + d_{km} h_1) \right\| + \left\| \sum_{k=0}^m f_k(\alpha_k x + \beta_k y) \right\| \leq 2\delta \end{aligned}$$

and thus, since $d_{mm} = 0$,

$$(\dagger)_1 \quad \left\| \sum_{k=0}^{m-1} \Delta_{d_{km}h_1} f_k(\alpha_k x + \beta_k y) \right\| \leq 2\delta \quad \text{for all } x, y, h_1 \in V.$$

Repeating the argument that led from (\dagger) to $(\dagger)_1$ we find that

$$(\dagger)_2 \quad \left\| \sum_{k=0}^{m-2} \Delta_{d_{k,m-1}h_2} \Delta_{d_{km}h_1} f_k(\alpha_k x + \beta_k y) \right\| \leq 4\delta$$

for all $x, y, h_1, h_2 \in V$.

Applying this reasoning $m - 2$ more times, we are inclined to admit that

$$\left\| \Delta_{d_{01}h_m} \cdots \Delta_{d_{0m}h_1} f_0(\alpha_0 x + \beta_0 y) \right\| \leq 2^m \delta \quad \text{for all } x, y, h_1, \dots, h_m \in V.$$

Since $d_{0k} \neq 0$ for $1 \leq k \leq m$ and $|\alpha_0| + |\beta_0| > 0$, the last inequality simply asserts that

$$\left\| \Delta_{h_m} \cdots \Delta_{h_1} f_0(z) \right\| \leq 2^m \delta \quad \text{for all } z, h_1, \dots, h_m \in V.$$

By the ‘‘symmetry’’ of our assumptions on f_0, \dots, f_m , for each $k = 0, 1, \dots, m$,

$$\left\| \Delta_{h_m} \cdots \Delta_{h_1} f_k(x) \right\| \leq 2^m \delta \quad \text{for all } x, h_1, \dots, h_m \in V,$$

and hence, according to Theorem B, there exists a GP function $q_k : V \rightarrow B$, of degree at most $m - 1$, such that

$$(1) \quad \|f_k(x) - q_k(x)\| \leq 2^{m+1} \delta \quad \text{for all } x \in V.$$

For $0 \leq k \leq m$ and $1 \leq j \leq m - 1$ there are symmetric, j -additive maps $a_{kj} : V^j \rightarrow B$ such that

$$(2) \quad q_k(x) = q_k(0) + \sum_{j=1}^{m-1} a_{kj}^*(x) \quad \text{for all } x \in V.$$

By (\dagger) and (1), for all $x, y \in V$,

$$(3) \quad \begin{aligned} \left\| \sum_{k=0}^m q_k(\alpha_k x + \beta_k y) \right\| &\leq \left\| \sum_{k=0}^m \{q_k(\alpha_k x + \beta_k y) - f_k(\alpha_k x + \beta_k y)\} \right\| \\ &+ \left\| \sum_{k=0}^m f_k(\alpha_k x + \beta_k y) \right\| \\ &\leq (m + 1)2^{m+1} \delta + \delta =: \delta_1. \end{aligned}$$

Now (3) says, in light of (2), that, for all $x, y \in V$,

$$(3)' \quad \left\| \left(\sum_{k=0}^m q_k(0) \right) + \left(\sum_{j=1}^{m-1} \sum_{k=0}^m a_{kj}^*(\alpha_k x + \beta_k y) \right) \right\| \leq \delta_1.$$

In $(3)'$ replace x by rx and y by ry ($r \in \mathbb{Q}$) to conclude that, for all $x, y \in V$ and all $r \in \mathbb{Q}$,

$$(4) \quad \left\| \left(\sum_{k=0}^m q_k(0) \right) + \sum_{j=1}^{m-1} r^j \left(\sum_{k=0}^m a_{kj}^*(\alpha_k x + \beta_k y) \right) \right\| \leq \delta_1.$$

‘‘By continuity’’, (4) holds for all *real* r , and all $x, y \in V$.

Now suppose that $\Lambda : B \rightarrow \mathbb{R}$ is a continuous linear functional. Then by (4),

$$|\Lambda(\sum_{k=0}^m q_k(0)) + \sum_{j=1}^{m-1} r^j \Lambda\{\sum_{k=0}^m a_{kj}^*(\alpha_k x + \beta_k y)\}| \leq \|\Lambda\| \delta_1$$

for all $x, y \in V$ and all $r \in \mathbb{R}$.

Since a real polynomial function is bounded if and only if it is constant, from the last inequality we surmise that, for $1 \leq j \leq m - 1$,

$$\Lambda\{\sum_{k=0}^m a_{kj}^*(\alpha_k x + \beta_k y)\} = 0 \quad \text{for all } x, y \in V.$$

Since this is so for every continuous linear functional $\Lambda : B \rightarrow \mathbb{R}$, by the Hahn-Banach theorem,

$$(5) \quad \sum_{k=0}^m a_{kj}^*(\alpha_k x + \beta_k y) = 0 \quad \text{for } x, y \in V \text{ and } 1 \leq j \leq m - 1.$$

Let $p_k(x) = q_k(x) - q_k(0) = \sum_{j=1}^{m-1} a_{kj}^*(x)$, $x \in V$, for $0 \leq k \leq m$. Then each p_k is a GP function of degree at most $m - 1$ and from (5) we find that

$$\sum_{k=0}^m p_k(\alpha_k x + \beta_k y) = 0 \quad \text{for all } x, y \in V.$$

Finally, letting $c_k = q_k(0)$ for $0 \leq k \leq m$, we find from (1) that if $0 \leq k \leq m$, then

$$\|(f_k(x) - c_k) - p_k(x)\| \leq \|f_k(x) - q_k(x)\| \leq 2^{m+1} \delta \quad \text{for all } x \in V.$$

□

Note that the possibility that $\delta = 0$ has not been excluded. Thus the solutions of (*) are GP functions of degree at most $m - 1$.

REGULARITY PROPERTIES OF (*) AND (†)

The following result is a special case of Theorem 6.1 (iii) of Kemperman [10].

Theorem C. *Suppose that (†) holds where $F = \mathbb{R}$, $V = \mathbb{R}^n$ and $B = \mathbb{C}$. If some f_k is bounded on a subset of \mathbb{R}^n having positive inner Lebesgue measure, then p_k is a genuine polynomial function.*

A MEAN VALUE THEOREM

The next theorem comes from Walsh [14]; also see [3].

Theorem D. *If $2 \leq m \in \mathbb{N}$, $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and f is continuous, then the following are equivalent.*

- (i) *For every regular m -gon in \mathbb{R}^2 , the value of f at its center is the arithmetic mean of its value at the vertices.*
- (ii) *The function f is a harmonic polynomial of degree at most $m - 1$.*

If we let $\beta_k = \exp(2\pi ki/m)$ for $1 \leq k \leq m$ and identify \mathbb{R}^2 with \mathbb{C} , then (i) asserts that

$$\sum_{k=1}^m f(z + \beta_k w) = mf(z) \quad \text{for all } z, w \in \mathbb{C}.$$

This leads us to

Theorem 2. *Suppose that $2 \leq m \in \mathbb{N}$, $f : \mathbb{C} \rightarrow \mathbb{C}$, $\delta \geq 0$, $\beta_k = \exp(2\pi ki/m)$ for $1 \leq k \leq m$ and*

$$(\#) \quad \left| mf(z) - \sum_{k=1}^m f(z + \beta_k w) \right| \leq \delta \quad \text{for all } z, w \in \mathbb{C}.$$

Then there exists a $c \in \mathbb{C}$ and a GP function $p : \mathbb{C} \rightarrow \mathbb{C}$, of degree at most $m - 1$, such that

$$\begin{aligned} |f(z) - c - p(z)| &\leq 2^{m+1}\delta \quad \text{for all } z \in \mathbb{C} \text{ and} \\ mp(z) &= \sum_{k=1}^m p(z + \beta_k w) \quad \text{for all } z, w \in \mathbb{C}. \end{aligned}$$

Moreover, if f is bounded on some subset of \mathbb{C} ($= \mathbb{R}^2$) having positive inner Lebesgue measure, then p is a harmonic polynomial of degree at most $m - 1$.

Proof. By putting $f_0 = mf$ and $f_k = -f$ for $1 \leq k \leq m$ and letting $\beta_0 = 0$ and $\alpha_k = 1$ for $0 \leq k \leq m$, we deduce from Theorem 1 that there exist $c_0, c \in \mathbb{C}$ and GP functions $p_0, p : \mathbb{C} \rightarrow \mathbb{C}$, each of degree at most $m - 1$, such that

$$\begin{aligned} |f_0(z) - c_0 - p_0(z)| &\leq 2^{m+1}\delta \quad \text{and} \\ |f(z) - c - p(z)| &\leq 2^{m+1}\delta \quad \text{for all } z \in \mathbb{C}, \end{aligned}$$

and moreover

$$mp_0(z) = \sum_{k=1}^m p(z + \beta_k w) \quad \text{for all } z, w \in \mathbb{C}.$$

But when $w = 0$ this last equation tells us that $p_0 = p$.

By Theorem 1, $|\Delta_{h_m} \cdots \Delta_{h_1} f(z)| \leq 2^m \delta$ for all $z, h_1, \dots, h_m \in \mathbb{C}$. Hence, for all $z, h \in \mathbb{C}$,

$$\left| \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} f(z + kh) \right| = |\Delta_h^m f(z)| \leq 2^m \delta$$

and therefore, by Theorem C, if f is bounded on some subset of $\mathbb{C} = \mathbb{R}^2$ of positive inner Lebesgue measure, then p is an ordinary polynomial function on \mathbb{R}^2 . In this case, Theorem D implies that p is harmonic. \square

A DISTRIBUTIONAL ANALOGUE OF (*)

Suppose that I and J are open intervals in \mathbb{R} , $\alpha_0, \beta_0, \dots, \alpha_m, \beta_m \in \mathbb{R}$ with $\alpha_j \beta_k - \alpha_k \beta_j \neq 0$ for $0 \leq j < k \leq m$, $I_k = \{\alpha_k x + \beta_k y : x \in I, y \in J\}$ for $0 \leq k \leq m$, $f_k : I_k \rightarrow \mathbb{C}$ for $0 \leq k \leq m$, $g_j : I \rightarrow \mathbb{C}$ and $h_j : J \rightarrow \mathbb{C}$ for $1 \leq j \leq n$ and

$$(6) \quad \sum_{k=0}^m f_k(\alpha_k x + \beta_k y) = \sum_{j=1}^n g_j(x) h_j(y) \quad \text{for } (x, y) \in I \times J.$$

In [2] it was shown that if the f_k 's are locally Lebesgue integrable, then they are restrictions to I_k of exponential polynomials. It follows that the locally integrable solutions of (*) are polynomial functions.

Let $T_k(x, y) = \alpha_k x + \beta_k y$ for $(x, y) \in I \times J$. Then (*) holds (for $x \in I$ and $y \in J$) if and only if

$$\sum_{k=0}^m f_k \circ T_k = 0.$$

Now each T_k is a *submersion* of $I \times J$ onto I_k —see Chapter VI of Hormander [8]—and if $u \in \mathcal{D}'(I_k)$, there is a member T_k^*u of $\mathcal{D}'(I \times J)$, called the *pullback* of u by T_k , which serves as a suitable generalization of “composition”. In [2] we used the more suggestive notation $u \circ T$ in place of the more popular T^*u . Thus a natural question to ask, concerning (*), in the realm of Schwartz distribution theory is, for which $u_k \in \mathcal{D}'(I_k)$, $0 \leq k \leq m$ is it true that

$$(**) \quad \sum_{k=0}^m u_k \circ T_k = 0.$$

It follows from Theorem 3 of [2], wherein a distributional analogue of (6) was considered, that if (**) holds, then each u_k is a regular distribution corresponding to a polynomial function $p_k : \mathbb{R} \rightarrow \mathbb{C}$ of degree at most $m - 1$.

RELATED CONVEXITY RESULTS

If I is an interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$, then f is said to be *midpoint convex* or *Jensen convex* provided $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$ for all $x, y \in I$; equivalently,

$$f(x) - 2f(x+h) + f(x+2h) = \Delta_h^2 f(x) \geq 0$$

whenever $h > 0$, $x \in I$ and $x + 2h \in I$.

A function $f : I \rightarrow \mathbb{R}$ is said to be (*Jensen*) *convex of order $n - 1$* provided $\Delta_h^n f(x) \geq 0$ whenever $h > 0$, $x \in I$ and $x + nh \in I$. Ciesielski proved in [4] that if such a function is bounded on a subset of I having positive Lebesgue inner measure, then it is continuous on I . This result was generalized in Kemperman [11].

A survey of stability results concerning functional equations can be found in [6]. Numerous other papers on the subject have been published in the journal *Aequationes Mathematicae* in the last two decades.

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO,
CANADA N2L 3G1

E-mail address: `jabaker@math.uwaterloo.ca`