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PERTURBING A PRODUCT OF STABLE FLOWS

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ABSTRACT. Suppose that f and f' are axiom A flows with attractors A and A'. Then the attractor $A \times A'$ for the product flow $g_t = f_t \times f'_t$ on the product manifold is no longer hyperbolic (although there is a hyperbolic action of $\mathbb{R}^2$).

It is easy to see that the attractor cannot explode but we show here that it cannot implode: for any flow (h_t) sufficiently close to (g_t) any attractor whose basin is not too thin is ε-dense in $A \times A'$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Properties like *structural stability* or Ω-stability [15, page 796] are important for applications because a model devised to make sense of some physical or other system can only be an approximation and so, if it is accurately to reflect that system, the model must be somewhat insensitive to perturbations; see [1, page 374], [16, pages 97–98] or [17, page 94].

Accordingly we study an attractor for a flow $f_t : M \to M$ $(t \in \mathbb{R})$. The flow is Ω-stable if and only if it satisfies Smale's axiom A and the no cycle property; see [12, 13, 10, 6] or [9, §18]. (Examples are the Anosov flows and, in particular, the geodesic flow on (the unit tangent bundle of) a compact manifold of negative sectional curvature; see [2, 4] or [15, page 800].)

The product $g_t : M \times M' \to M \times M'$ $(t \in \mathbb{R})$ of two flows $f_t : M \to M$, $f' : M' \to M'$ $(t \in \mathbb{R})$ is defined by $g_t(x, x') := (f_t(x), f'_t(x'))$. (Thus the projection of a g-orbit onto the first and second factor flows at unit speed along an f- and an f'-orbit, respectively.) A perturbation of g represents a weak coupling of the two flows f and f'. It was noted already by Smale [15, p. 804] that the product of two flows that have periodic orbits has an invariant torus and so cannot be Ω-stable. (This is because the flow on the torus is a rational or irrational flow (depending on the ratio of the periods of the two orbits), and such a flow cannot be structurally stable.) Since an axiom A attractor has a dense subset of periodic orbits, the product of two such attractors *cannot* be structurally stable, even though each of them is. Thus a weak coupling between the two attractors may destroy their structure even though each is stable on its own.

The product of attractors for f and f' is an attractor for g. It still attracts for a perturbation h of g, so the product attractor cannot explode under perturbation. But can it implode? Roughly speaking, our answer is No. We shall exploit the

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normal hyperbolicity of the g-invariant surfaces that occur as the product of periodic orbits to show, under a mixing hypothesis, that an attractor for a perturbation of g is not much smaller than the attractor for g in $M \times M'$ (Theorem 1).

A non-empty closed invariant set Λ will be called an *attractor* for a flow (h_t) if for some $\kappa > 0$ we have $\bigcap_{t>0} h_t(U_\kappa(\Lambda)) = \Lambda$, where $U_\kappa(B) = \{x \in M : d(x,b) < \kappa$ for some $b \in B\}$. If $C \subset U_\varepsilon(B)$, we say that B is ε-dense in C.

We may assume that the metric on M is "adapted" to the flow (f_t) (see [4, page 181]), and then, for a certain small $\zeta > 0$, we have $f_1(U_\zeta(A)) \subset U_\zeta(A)$ so $A = \bigcap_{t>0} f_t U_\zeta(A)$ and similarly for M', A' and (f'_t). The Riemannian metric on $M \times M'$ is obtained by taking at each point the tangent spaces to M, M' to be perpendicular. Then $g_1(U_\zeta(A \times A')) \subset U_\zeta(A \times A')$ and any flow h sufficiently close to g has $h_1(U_\zeta(A \times A')) \subset U_\zeta(A \times A')$ also. Thus $\bigcap_{t>0} h_t(U_\zeta(A \times A')) \subset U_\zeta(A \times A')$. A C^1 flow (h_t) is said to be δ-close to (g_t) in the C^1 metric if, for $0 \leq t \leq 1$, the diffeomorphisms h_t and g_t are δ-close in the C^1 metric.

Theorem 1. *Let A and A' be topologically transitive attractors (but not fixed points) of the axiom A flows $f_t : M \to M$ and $f'_t : M' \to M'$ respectively, and let $g_t : M \times M' \to M \times M'$ denote the product flow. Choose ζ so that $g_1(U_\zeta(A \times A')) \subset U_\zeta(A \times A')$. Suppose, in addition, that $f'_t|A'$ is topologically mixing. Then, for each $\varepsilon > 0$ and $\gamma > 0$, there is $\delta > 0$ such that, if the C^1 flow $h_t : M \times M' \to M \times M'$ is δ close to g in the C^1 metric and Λ is an attractor for h whose basin contains a ball of radius γ lying in $U_\zeta(A \times A')$, then $A \times A' \subset U_\varepsilon(\Lambda)$.*

Note that we do not insist that Λ is indecomposable. $\bigcap_{t>0} h_t U_\zeta(A \times A')$ is an example.

There is earlier related work of Hurley [8] and Moreva [11]. They found a generic stability property for attractors: a perturbation of any flow g in a certain residual set has some attractor close to an attractor of g. This is stronger than our results in that no assumption like axiom A is made. On the other hand, it is weaker in that a product flow might well not be in the residual set, and significantly also in that, if the perturbation has several attractors, they only claim closeness for one of these attractors.

Example 1. Suppose that M and M' are each just a single f-orbit and a single f'-orbit of period 1 and so not mixing. Then, taking coordinates $(r, r') \in [0,1) \times [0,1)$ in $M \times M'$, we have $g_t(r, r') = (r + t, r' + t)$ with each coordinate reduced modulo 1 and the flow g is given by the vector field $(1,1)$. For $k \in \mathbb{N}$ the vector field $(1, 1 + \nu \sin(2\pi k(r - r')))$ determines a C^∞ flow $(h_t)_{t\in\mathbb{R}}$ that converges C^1 to g as $\nu \to 0$. There are k attracting periodic orbits $\Lambda_j := \{(j/k + t, t) : 0 \leq t < 1\}$ for $0 \leq j < k$. Each Λ_j has a basin containing discs of diameter $1/(k\sqrt{2})$ but not larger. And $M \times M' \subset U_\varepsilon(\Lambda_j)$ for $\varepsilon > 1/(2\sqrt{2})$, but not for smaller ε. This shows that, in the absence of mixing, attractors of an arbitrarily small perturbation of g could be much smaller than $A \times A'$.

Remark 1. In the hypotheses of Theorem 1 let us now suppose that A' (but not A) *is* a fixed point. Then $f'_t|A'$ is trivially mixing. In this case $A \times A'$ is an axiom A attractor for g_t and topologically conjugate to $f_t|A$. (The vector field for $g_t|A \times A'$ is nowhere zero, $T_{A\times A'} M \times M'$ has a hyperbolic splitting and, as with $f_t|A$, the g-periodic orbits are dense in $A \times A'$.) By the stability of such C^1 flows (see [13] or [14, §10.8]), h has exactly one attractor in $U_\zeta(A \times A')$ and this is topologically equivalent to $f_t|A$ using a homeomorphism close to the inclusion.

Thus the attractor depends continuously in the Hausdorff metric on h in the case where A' is a fixed point.

We collect the results we need from the book [7] of Hirsch, Pugh and Shub.

Proposition 1. *Assume the hypotheses of Theorem* 1. *Let V denote the non-separable* 2*-manifold which is the disjoint union of all the $\mathbb{R}^2$-orbits $\{(f_t(y), f'_{t'}(y')) : t, t' \in \mathbb{R}\} \subset A \times A'$. Let $i : V \to M \times M'$ denote the inclusion, which is injective; and let η denote a C^∞ vector subbundle of $T_{i(V)}(M \times M')$ complementary to $Ti(TV)$ and C^0-close to $E^s \oplus E^{s'} \oplus E^u \oplus E^{u'}$, where E^s, E^u and $E^{s'}, E^{u'}$ are the contracting and expanding subbundles for $T_A M$ and $T_{A'} M'$. Fix $\varepsilon > 0$ and write $i^*\eta(\varepsilon)$ for the pull-back of the ε-disc bundle $\eta(\varepsilon)$ in η. Write i^*h_t for the map that makes the diagram*

$$\begin{tikzcd}
i^*\eta(\varepsilon) \arrow[r, "i^*h_t"] \arrow[d, "\exp\circ i_*"'] & i^*\eta \arrow[d, "\exp\circ i_*"'] \\
M \times M' \arrow[r, "h_t"] & M \times M'
\end{tikzcd}$$

commute. Then, for any C^1 flow h on $M \times M'$ which is C^1 close to g, there is a unique section $G_h : V \to i^*\eta$ such that

$$\forall t \in \mathbb{R}, \forall q \in V \ (i^*h_t)(G_h(q)) \in i^*\eta(\varepsilon).$$

Also, G_h is C^1 and tends C^1 to 0 as h tends C^1 to g. Moreover, W^s_h, defined as $\bigcap_{n\le 0}(i^*h_1)^n i^*\eta(\varepsilon)$, is a C^1 submanifold and converges C^1 to W^s_g as h tends C^1 to g. And W^s_h is invariantly fibred by C^1 submanifolds W^{ss}_h. Points of each W^{ss}_h are characterized by sharp forward asymptoticity under h_t and W^{ss}_h converges C^1 to W^{ss}_g as h tends C^1 to g. The flow $j = (j_t)$ on V, defined by $j_t := G_h^{-1} \circ h_t \circ G_h$, tends C^1 to g as h tends C^1 to g.

Proof. Because g is normally hyperbolic at $i : V \to M \times M'$, these properties of G_h, W^s_h, W^{ss}_h are given by Theorem 6.8 of [7]; see also Theorem 6.1 and Example $2'$ on page 68 there. The C^1 convergence of j follows from the C^1 convergence of G_h and G_h^{-1}. $\square$

Remark 2. We shall apply Proposition 1 only for compact $\mathbb{R}^2$-orbits that are the product of periodic orbits for f and f', and so we could rely on Theorem 4.1 of [7]; however, that approach does not make it so clear that the distance of the perturbation h from g is independent of the periods of the orbits.

2. The proof in the product space

Proof of Theorem 1. We may assume that $\gamma < \varepsilon$. First take $x \in A$ of some least period T for which

$$U_{\gamma/4}(\{f_t(x) : 0 \le t \le T\}) \supset A \tag{1}$$

and

$$U_{\gamma/4}(W^s_f(\{f_t(x) : 0 \le t \le T\}, \zeta)) \supset U_\zeta(A). \tag{2}$$

Now choose $q_0 \in \mathbb{N}$ such that $T/q_0 < \gamma/4$. Put $\beta = 1/(4q_0)$. Next choose $x'_1 \in A'$ and $T_1 > T$ such that

$$U_{\gamma/8}(\{f'_t(x'_1) : 0 \le t \le T_1\}) \supset A' \tag{3}$$

and

$$U_{\gamma/8}(W^s_{f'}(\{f'_t(x'_1) : 0 \le t \le T_1\}, \zeta)) \supset U_\zeta(A').$$

Then choose an open set U'_1 containing x'_1 so that

$$x'_2 \in U'_1 \Rightarrow d(f'_t(x'_2), f'_t(x'_1)) < \gamma/8 \text{ for } 0 \le t \le T_1 \tag{4}$$

and

$$x'_2 \in U'_1 \Rightarrow U_{\gamma/4}(W^s_{f'}(\{f'_t(x'_2) : 0 \le t \le T_1\}, \zeta)) \supset U_\zeta(A'). \tag{5}$$

The measure m' of maximal entropy for $(f'_t|A')_{t\in\mathbb{R}}$ has $m'(U'_1) > 0$. Now, according to [3], under the mixing hypothesis, m' is the weak limit, as $t \to \infty$, of the measure equidistributed on those periodic orbits of f' that have least period in $((t-\beta)T, (t+\beta)T)$. Thus, there is $\tau \in \mathbb{R}$ for which $\tau - 2\beta \in \mathbb{N}$ and there is a periodic point $x' \in U'_1$ of least period $T' \in ((\tau-\beta)T, (\tau+\beta)T)$. Then $T' > T_1$ and the f'-orbit of x' is $\gamma/4$-dense in A'. Also the fractional part of T'/T is in $(1/(4q_0), 3/(4q_0))$. In particular,

$$|\alpha - T'/T| < \beta \Rightarrow \alpha \notin \{p/q : 0 < q \le q_0, p \in \mathbb{Z}\}.$$

Consider the torus

$$V_1 := \{(f_t(x), f'_{t'}(x')) : 0 \le t < T, 0 \le t' < T'\} \subset A \times A'.$$

Then

$$U_{\gamma/2}(V_1) \supset A \times A'$$

by (1), (3), (4). The flow $(g_t|V_1)_{t\in\mathbb{R}}$ is given, in terms of (t, t')-coordinates on V_1, by the vector field $(1, 1)$.

Let $B(z_1, \gamma)$ be any ball of radius γ in $U_\zeta(A \times A')$. Then $B(z_1, \gamma/2) \cap W^s_g(V_1) \neq \emptyset$ by (2), (5). If a perturbation h of g has an attractor Λ whose basin contains $B(z_1, \gamma)$, we pick $z \in B(z_1, \gamma) \cap W^s_h(G_h(V_1))$. (Technically, $G_h(V_1)$ lies in the vector bundle η and we should have written $\exp \circ G_h(V_1)$.) Then $\omega_h(z) \subset \Lambda \cap G_h(V_1)$. Next take $y \in V_1$ so that $\omega_h(G_h(y)) = \omega_h(z) \subset G_h(V_1)$ and note that $G_h(\omega_j(y)) = \omega_h(G_h(y))$. Choose δ such that G_h is $\gamma/4$ C^0 close to the identity. Using Proposition 1, we choose δ so that also, in (t, t')-coordinates on V_1, the vector field (J_1, J_2) giving the flow $(j_t|V_1)$ satisfies $|1 - J_1/J_2| < T/(8q_0T')$. The j-orbit of the point y in V_1 meets the circle $t' = 0$, $0 \le t < T$ at successive points $(y_n, 0), n \in \mathbb{Z}$ where $T/(8q_0) < y_n - y_{n-1} < 7T/(8q_0)$ (or, for certain n, $T/(8q_0) < y_n - y_{n-1} + T < 7T/(8q_0)$). Between $(y_n, 0)$ and $(y_{n+1}, 0)$, the j-orbit of y meets the circle $t' = a$ at $(y_n(a), a)$ where $0 < y_n(a) - y_{n-1}(a) < T/q_0$. In particular, $U_{\gamma/4}(\omega_j(y)) \supset V_1$ using $T/q_0 = \gamma/4$. Now

$$\begin{aligned} U_\varepsilon(\Lambda) \supset U_\gamma(\Lambda) \supset U_\gamma(\omega_h(z)) &= U_\gamma(G_h(\omega_j(y))) \\ &\supset U_{3\gamma/4}(\omega_j(y)) \supset U_{\gamma/2}(V_1) \supset A \times A' \end{aligned}$$

as required. □

Question 1. Does the time average of a function along most h-orbits keep close to the time average along Lebesgue almost every g-orbit?

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