

AN EXTREMAL FUNCTION
FOR THE CHANG-MARSHALL INEQUALITY
OVER THE BEURLING FUNCTIONS

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(Communicated by Juha M. Heinonen)

ABSTRACT. S.-Y. A. Chang and D. E. Marshall showed that the functional $\Lambda(f) = (1/2\pi) \int_0^{2\pi} \exp\{|f(e^{i\theta})|^2\} d\theta$ is bounded on the unit ball \mathcal{B} of the space \mathcal{D} of analytic functions in the unit disk with $f(0) = 0$ and Dirichlet integral not exceeding one. Andreev and Matheson conjectured that the identity function $f(z) = z$ is a global maximum on \mathcal{B} for the functional Λ . We prove that Λ attains its maximum at $f(z) = z$ over a subset of \mathcal{B} determined by kernel functions, which provides a positive answer to a conjecture of Cima and Matheson.

1. INTRODUCTION

Let \mathcal{D} be the Dirichlet space of functions f analytic on the unit disk \mathbb{D} , with $f(0) = 0$ and a finite Dirichlet integral

$$\|f\|_{\mathcal{D}}^2 = \frac{1}{\pi} \int \int_{\mathbb{D}} |f'(z)|^2 dx dy.$$

It is well known that \mathcal{D} is a Hilbert space with inner product

$$\langle f, g \rangle_{\mathcal{D}} = \frac{1}{\pi} \int \int_{\mathbb{D}} f'(z) \overline{g'(z)} dx dy.$$

Let $\mathcal{B} = \{f \in \mathcal{D} : \|f\|_{\mathcal{D}} \leq 1\}$ be its closed unit ball.

We shall be concerned with functionals Λ_{Φ} on \mathcal{B} defined by

$$\Lambda_{\Phi}(f) = \frac{1}{\pi} \int_0^{2\pi} \Phi(|f(e^{i\theta})|) d\theta,$$

for $f \in \mathcal{B}$ and $\Phi : (-\infty, \infty) \rightarrow \mathbb{R}$ being a continuous convex nondecreasing function. A function f is a **maximum** for Λ_{Φ} if $f \in \mathcal{B}$ and $\Lambda_{\Phi}(f) \geq \Lambda_{\Phi}(g)$ for all $g \in \mathcal{B}$.

Chang and Marshall [3] proved that if $\Phi_{\alpha}(t) = e^{\alpha t^2}$ for $\alpha > 0$, then $\Lambda_{\Phi_{\alpha}}$ is bounded on \mathcal{B} if and only if $\alpha \leq 1$. In their proof they compared functions in \mathcal{B} to the *Beurling functions*

$$B_{\alpha}(z) = \frac{\log \frac{1}{1-\bar{\alpha}z}}{\sqrt{\log \frac{1}{1-|a|^2}}},$$

Received by the editors August 1, 2003 and, in revised form, March 12, 2004.

2000 *Mathematics Subject Classification*. Primary 30H05; Secondary 30A10, 30D99, 49K99.

Key words and phrases. Dirichlet space, Chang-Marshall inequality, Baernstein star-function, extremal functions.

for $a \in \mathbb{D} \setminus \{0\}$, where the branch of the logarithm is chosen so that $B_a(a)$ is real. The denominator assures that $\|B_a\|_{\mathcal{D}} = 1$. Up to a normalizing factor, the B_a are the kernel functions for \mathcal{D} . We shall denote by \mathcal{B}_0 the set of all Beurling functions and by $\tilde{\mathcal{B}}_0$ its closed convex hull.

A shorter proof of this fact has since been found by Marshall [9]. A significantly more general and stronger inequality has been found by Essén [7]. Andreev and Matheson [1] showed that the identity function $f(z) = z$ is a local maximum for Λ_{Φ_1} on \mathcal{B} and conjectured that it is also a global maximum. Cima and Matheson [4] showed that the identity function is a local maximum on the set \mathcal{B}_0 and that the functional Λ_{Φ_1} attains its maximum on $\tilde{\mathcal{B}}_0$. On the other hand, they showed that Λ_{Φ_1} , when restricted to \mathcal{B} , is not weakly continuous at 0, and thus it is an open question whether there exists a global maximum for Λ_{Φ_1} on \mathcal{B} . Matheson and Pruss [10] studied the regularity of the extremal functions. We refer the reader to their paper for an excellent discussion of this and other related problems and for a list of open problems.

Our principle result is:

Theorem 1.1. *The inequality*

$$(1.1) \quad \Lambda_{\Phi_1}(f) < \Lambda_{\Phi_1}(z)$$

holds true for all $f \in \tilde{\mathcal{B}}_0$.

Our result proves Conjecture 1 of Cima and Matheson in [4].

2. PROOF OF THEOREM 1.1

It is natural to set $B_0(z) = z$ (see [4]). A function $\Phi(x)$ continuous on $-\infty < x < \infty$ is said to be *convex* if $\Phi((x+y)/2) \leq [\Phi(x) + \Phi(y)]/2$, and *strictly convex* if strict inequality holds whenever $x \neq y$. Theorem 1.1 is a consequence of the following result.

Theorem 2.1. *Let $\Phi(x)$ be a convex nondecreasing function on $-\infty < x < \infty$. For all $a_0, a \in \mathbb{D} \setminus \{0\}$ such that $0 \leq |a_0| < |a| < 1$, we have*

$$(2.1) \quad \int_0^{2\pi} \Phi(\log |B_a(re^{i\theta})|) d\theta \leq \int_0^{2\pi} \Phi(\log |B_{a_0}(re^{i\theta})|) d\theta,$$

$0 < r < 1$. If Φ is strictly convex, then the inequality is strict for all r .

Proof. Our proof is based on the deep results of Albert Baernstein [2, Theorem 1] on integral means of univalent functions (see also Chapter 7 of Duren's book [5]). In particular, we need the following proposition [2, Proposition 3].

Proposition 2.2. *For $g, h \in L^1(-\pi, \pi)$, the following statements are equivalent.*

(a) *For each function $\Phi(s)$ convex and nondecreasing on $-\infty < s < \infty$,*

$$\int_{-\pi}^{\pi} \Phi(g(x)) dx \leq \int_{-\pi}^{\pi} \Phi(h(x)) dx.$$

(b) *For each $t \in \mathbb{R}$,*

$$\int_{-\pi}^{\pi} [g(x) - t]^+ dx \leq \int_{-\pi}^{\pi} [h(x) - t]^+ dx.$$

(c) $g^*(\theta) \leq h^*(\theta)$, $0 \leq \theta \leq \pi$.

Here for each $r \in (r_1, r_2)$ and $u(re^{i\theta}) \in L^1(0, 2\pi)$ the *Baernstein star-function* of u is defined as

$$(2.2) \quad u^*(re^{i\theta}) = \sup_{|E|=2\theta} \int_E u(re^{it}) dt,$$

$0 \leq \theta \leq \pi$, where $|E|$ denotes the Lebesgue measure of the set $E \subset [-\pi, \pi]$.

In view of Proposition 2.2, we want first to show that

$$(2.3) \quad \int_{-\pi}^{\pi} \log^+ \left[\frac{|B_a(re^{i\theta})|}{\rho} \right] d\theta \leq \int_{-\pi}^{\pi} \log^+ \left[\frac{|B_{a_0}(re^{i\theta})|}{\rho} \right] d\theta,$$

$0 < r < 1$, for each $\rho > 0$ and all a and a_0 such that $0 \leq |a_0| < |a| < 1$. Notice that

$$\int_{-\pi}^{\pi} \log^+ \left[\frac{|B_{a'}(re^{i\theta})|}{\rho} \right] d\theta = \int_{-\pi}^{\pi} \log^+ \left[\frac{|B_{a''}(re^{i\theta})|}{\rho} \right] d\theta$$

whenever $|a'| = |a''|$. Hence we may assume from now on that $0 \leq a_0 < a < 0$.

We can apply Jensen's theorem to obtain

$$(2.4) \quad \int_{-\pi}^{\pi} \log^+ \left[\frac{|B_a(re^{i\theta})|}{\rho} \right] d\theta = \int_{-\pi}^{\pi} N(r, \rho e^{i\phi}) d\phi,$$

since $B_a(0) = 0$. It is easy to see that B_a is a univalent function in the unit disk \mathbb{D} , $B_a(0) = 0$ and $B'_a(0) = a/A$, where $A = \{\log[1/(1 - |a|^2)]\}^{1/2}$, for each $0 < a < 1$, with a continuous extension to the closed unit disk $\overline{\mathbb{D}}$, and if $\alpha = \rho e^{i\phi} \neq 0$ is in the range D_a of B_a , then

$$(2.5) \quad N(r, \alpha) = \int_0^r \frac{n(t, \alpha)}{t} dt = \log^+ \left[\frac{r}{|\alpha|} \right] = \log^+ \left[\frac{r}{|B_a^{-1}(\alpha)|} \right],$$

$0 < r < 1$. Let $u_a(\zeta) = -\log |B_a^{-1}(\zeta)|$ be the Green's function of D_a with pole at 0. Extend it to a continuous function in the punctured plane by setting $u_a(\zeta) = 0$, $\zeta \notin D_a$. The formula (2.5) takes the form

$$N(r, \zeta) = [u_a(\zeta) + \log r]^+,$$

$0 < r < 1$, for arbitrary ζ , and equation (2.4) becomes

$$(2.6) \quad \int_{-\pi}^{\pi} \log^+ \left[\frac{|B_a(re^{i\theta})|}{\rho} \right] d\theta = \int_{-\pi}^{\pi} [u_a(\rho e^{i\phi}) + \log r]^+ d\phi.$$

Let $u_{a_0}(\zeta) = -\log |B_{a_0}^{-1}(\zeta)|$ for $\zeta \in D_{a_0}$, and let $u_{a_0}(\zeta) = 0$ elsewhere. In view of (2.6) the inequality (2.3) can be recast in the form

$$\int_{-\pi}^{\pi} [u_a(\rho e^{i\phi}) + \log r]^+ d\phi \leq \int_{-\pi}^{\pi} [u_{a_0}(\rho e^{i\phi}) + \log r]^+ d\phi,$$

$0 < r < 1$, $0 < \rho < \infty$. By Proposition 2.2, this is implied by the inequality

$$(2.7) \quad u_a^*(\rho e^{i\phi}) \leq u_{a_0}^*(\rho e^{i\phi}),$$

$0 < \rho < \infty$, $0 \leq \phi \leq \pi$.

The function $u(\zeta)$ is continuous in $0 < |\zeta| < \infty$, it is positive and harmonic in D_a , and identically zero outside D_a . Thus it is subharmonic in $0 < |\zeta| < \infty$. Hence by [2, Theorem A] and the definition (2.2) of the star-function, u_a^* is subharmonic in the open upper half-plane and continuous in the closed upper half-plane, except at the origin.

Since $B_a^{-1}(\zeta) = (1 - e^{-A\zeta})/a$, then, near the origin, u_a has the form

$$(2.8) \quad u_a(\zeta) = -\log|\zeta| - \log\frac{A}{a} + u_{1a}(\zeta),$$

where u_{1a} is harmonic and $u_{1a}(0) = 0$. Thus

$$u_a^*(\rho e^{i\phi}) + 2\phi \log \rho \rightarrow -2\phi \log \frac{A}{a}$$

as $\rho \rightarrow 0$ for $0 \leq \phi \leq \pi$. Similarly, near the origin, u_{a_0} has the form

$$u_{a_0}(\zeta) = -\log|\zeta| - \log\frac{A_0}{a_0} + u_{1a_0}(\zeta),$$

where u_{1a_0} is harmonic and $u_{1a_0}(0) = 0$. Thus

$$u_{a_0}^*(\rho e^{i\phi}) + 2\phi \log \rho \rightarrow -2\phi \log \frac{A_0}{a_0}$$

as $\rho \rightarrow 0$ for $0 \leq \phi \leq \pi$. It follows that

$$[u_a^*(\rho e^{i\phi}) - u_{a_0}^*(\rho e^{i\phi})] \rightarrow -2\phi \log \frac{a_0 A}{a A_0}$$

as $\rho \rightarrow 0$ for $0 \leq \phi \leq \pi$. It is easy to see that $a_0 A / (a A_0) > 1$ for $a_0 < a$ and hence that $-2\pi \log \frac{a_0 A}{a A_0} \leq -2\phi \log \frac{a_0 A}{a A_0} \leq 0$ for $a_0 < a$.

Hence $(u_a^* - u_{a_0}^*)$ is subharmonic in the upper half-plane and continuous in its closure except at the origin, where it has a bounded discontinuity: for $\phi = 0$,

$$\lim_{\rho \rightarrow 0} (u_a^*(\rho) - u_{a_0}^*(\rho)) = 0,$$

and for $\phi = \pi$,

$$\lim_{\rho \rightarrow 0} (u_a^*(-\rho) - u_{a_0}^*(-\rho)) = -2\pi \log \frac{a_0 A}{a A_0}.$$

We want to show that $(u_a^* - u_{a_0}^*) < 0$ in the open upper half-plane. Since $u_a^* - u_{a_0}^*$ is discontinuous at the origin, we cannot apply the maximum principle for subharmonic functions to $u_a^* - u_{a_0}^*$ at this point. The proof of the inequality $(u_a^* - u_{a_0}^*) < 0$ for $\Im \zeta > 0$ will be based on the following four steps (a)–(d).

(a) On the positive real axis, by definition, $u_a^*(\zeta) = v^*(\zeta) = 0$ for $\zeta > 0$.

(b) Next let d_a be the distance from 0 to the complement of D_a . It is obvious that $\Re(1 - ae^{i\theta})^{-1} > 0$. Since the branch of the logarithm was chosen so that $B_a(a)$ is real, then

$$|B_a(e^{i\theta})| = \frac{1}{A} \{ [\log \frac{1}{|1 - ae^{i\theta}|}]^2 + [\arg \frac{1}{1 - ae^{i\theta}}]^2 \}^{1/2}.$$

Since $\max |1 - ae^{i\theta}| = |1 - ae^{i\pi}| = 1 + a$ and $[\arg \frac{1}{1 - ae^{i\pi}}]^2 = 0$, it is easy to see that

$$-\frac{1}{A} \log \frac{1}{1 + a} \leq |B_a(e^{i\theta})| \leq \frac{1}{A} \log \frac{1}{1 - a}$$

for $0 < a < 1$. Thus $d_a = -\frac{1}{A} \log \frac{1}{1 + a}$. We want to show that d_a is a decreasing function of a for $0 < a < 1$. It is clear that $d_a \rightarrow 1$ as $a \rightarrow 0$. Let

$$f(a) = \frac{\log(1 + a)}{A}.$$

Then

$$f'(a) = -\frac{[(1 - a) \log(1 - a) + \log(1 + a)]}{(1 - a^2)A^3}.$$

Let

$$f_1(a) = (1 - a) \log(1 - a) + \log(1 + a).$$

An easy computation shows that $f_1''(a) > 0$ for $0 < a < 1$. Thus f_1' is an increasing function of a , and it follows that $f_1'(a) > 0$ for $0 < a < 1$ since $f_1'(0) = 0$. Therefore f_1 is an increasing function of a for $0 < a < 1$ and $f_1(a) > 0$ since $f_1(0) = 0$. Finally, this implies that $f'(a) < 0$ for $0 < a < 1$, and thus f is a decreasing function of a . Therefore $d_{a_0} > d_a$ for all $a, a_0 < a < 1$.

In the disk $|\zeta| < d_a$, $u_a(\zeta)$ has the form (2.8), where u_{1a} is harmonic in $|\zeta| < d_a$ and $u_{1a}(0) = 0$. Thus

$$u_a^*(\rho e^{i\pi}) = -2\pi \log \frac{1}{\rho} - 2\pi \log \frac{A}{a}$$

and, similarly,

$$u_{a_0}^*(\rho e^{i\pi}) = -2\pi \log \frac{1}{\rho} - 2\pi \log \frac{A_0}{a_0}$$

for $0 < \rho < d_a$. Hence $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ for $-d_a \leq \zeta < 0$.

(c) Since $u_{1a}(\zeta)$ and $u_{1a_0}(\zeta)$ are harmonic in $|\zeta| < d_a$ and $u_{1a}(0) = u_{1a_0}(0) = 0$, then for every $\epsilon > 0$ there is a $\rho_0, \rho_0 = |\zeta_0| < d_a$, such that $|u_{1a}(\zeta)| < \epsilon/2$ and $|u_{1a_0}(\zeta)| < \epsilon/2$ for all $\zeta, |\zeta| \leq \rho_0$. Thus

$$\begin{aligned} u_a^*(\rho e^{i\phi}) &= \sup_{|E|=2\phi} \int_E u_a(\rho e^{it}) dt \\ &= -2\phi \log \rho - 2\phi \log \frac{A}{a} + \sup_{|E|=2\phi} \int_E u_{1a}(\rho e^{it}) dt \\ &\leq -2\phi \log \rho - 2\phi \log \frac{A}{a} + \phi\epsilon \end{aligned}$$

and

$$\begin{aligned} u_{a_0}^*(\rho e^{i\phi}) &= \sup_{|E|=2\phi} \int_E u_{a_0}(\rho e^{it}) dt \\ &= -2\phi \log \rho - 2\phi \log \frac{A_0}{a_0} + \sup_{|E|=2\phi} \int_E u_{1a_0}(\rho e^{it}) dt \\ &\geq -2\phi \log \rho - 2\phi \log \frac{A_0}{a_0} - \phi\epsilon \end{aligned}$$

for $0 < \rho \leq \rho_0$ and $0 < \phi < \pi$. Now choose ϵ such that $\epsilon < \log(Aa_0/aA_0)$. Then

$$u_a^*(\rho e^{i\phi}) - u_{a_0}^*(\rho e^{i\phi}) \leq -2\phi \log \frac{Aa_0}{aA_0} + 2\phi\epsilon < 0$$

for all $0 < \rho \leq \rho_0$ and $0 < \phi < \pi$. Hence $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ for $|\zeta| \leq \rho_0 < d_a$ and $0 < \phi < \pi$.

(d) To establish the inequality on $-\infty < \zeta < -d_a$, we fix $\epsilon > 0$ and consider the function

$$Q(\zeta) = u_a^*(\zeta) - u_{a_0}^*(\zeta) - \epsilon\phi,$$

$\zeta = \rho e^{i\phi}$, which is subharmonic in $\mathcal{A} = \{\zeta : \rho_0 < |\zeta|, 0 \leq \Im\zeta\}$ and continuous in the closure of \mathcal{A} . Let M be the maximum of $Q(\zeta)$ in $\overline{\mathcal{A}}$. Then $M \geq 0$ and, according to the maximum principle for subharmonic functions, the maximum is attained somewhere on the boundary of \mathcal{A} . Suppose $M > 0$. Since $u_a^*(\zeta) \leq u_{a_0}^*(\zeta)$ on the set $\{\zeta : -d_a \leq \zeta \leq \rho_0\} \cup \{\zeta : |\zeta| = \rho_0, \Im\zeta > 0\} \cup \{\zeta : \rho_0 \leq \zeta < \infty\}$, there

is some point $-\zeta_1 = -\rho_1$ for which $-\infty < \zeta_1 < -d_a$ and $Q(\zeta_1) = M$. Let $G_a(\phi)$ denote the symmetric decreasing rearrangement of $u_a(\rho_1 e^{i\phi})$. Then

$$\frac{\partial u_a^*}{\partial \phi}(\rho_1 e^{i\phi}) = 2G_a(\phi)$$

for $0 \leq \phi \leq \pi$ by [2, Proposition 2]. But because $\rho_1 > d_a$, there is some point on the circle $|\zeta| = \rho_1$ that lies outside D_a , so

$$G_a(\pi) = \inf_{0 \leq \phi \leq \pi} u_a(\rho_1 e^{i\phi}) = 0.$$

Applying the same argument to $u_{a_0}^*$ we obtain

$$\frac{\partial u_{a_0}^*}{\partial \phi}(\rho_1 e^{i\phi}) = 2G_{a_0}(\phi)$$

for $0 \leq \phi \leq \pi$. If $d_a < \rho_1 \leq d_{a_0}$, then

$$G_{a_0}(\phi) = \inf_{0 \leq \phi < \pi} \{t : \lambda(t) \leq 2\phi\},$$

where λ is the distribution function of u_{a_0} , $\lambda(t) = |\{\phi : u_{a_0}(\rho_0 e^{i\phi}) > t\}|$, and

$$G_{a_0}(\pi) = \lim_{\phi \rightarrow \pi^-} G_{a_0}(\phi).$$

Hence $G_{a_0}(\pi) \geq 0$ if $d_a < \rho_1 \leq d_{a_0}$. If $d_{a_0} < \rho_1$, there is some point on the circle $|\zeta| = \rho_1$ that lies outside D_{a_0} , so

$$G_{a_0}(\pi) = \inf_{0 \leq \phi \leq \pi} u_{a_0}(\rho_1 e^{i\phi}) = 0.$$

Therefore

$$\frac{\partial Q}{\partial \phi}(\zeta_1) \leq -\epsilon < 0,$$

which contradicts the assumption that $Q(\zeta)$ has a relative maximum at ζ_1 . Hence $M = 0$ and

$$u_a^*(\zeta) \leq u_{a_0}^*(\zeta) + \epsilon\phi \leq u_{a_0}^*(\zeta) + \epsilon\pi$$

for $\zeta \in \overline{\mathcal{A}}$. Letting $\epsilon \rightarrow 0$ we obtain that

$$u_a^*(\rho e^{i\phi}) \leq u_{a_0}^*(\rho e^{i\phi})$$

for $\zeta \in \overline{\mathcal{A}}$.

We are in a position now to prove that $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ in the open upper half-plane. Combining (a)–(d) we obtain (2.7). Furthermore, $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ on the set $\{\zeta : -d_a \leq \zeta \leq \rho_0\} \cup \{\zeta : |\zeta| = \rho_0, \Im \zeta > 0\}$ by (b) and (c). Hence $u_a^* - u_{a_0}^*$ is a subharmonic function on \mathcal{A} that is not identically equal to zero there and, by the maximum principle, this implies that $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ everywhere in \mathcal{A} . Also, $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ for $\{\zeta : 0 < |\zeta| \leq \rho_0 < d_a, 0 < \Im \zeta\}$ by (c). Therefore,

$$u_a^*(\zeta) < u_{a_0}^*(\zeta)$$

in the open upper half-plane.

It follows from Proposition 2.2 that

$$(2.9) \quad \int_0^{2\pi} \Phi(\log |B_a(re^{i\theta})|)d\theta \leq \int_0^{2\pi} \Phi(\log |B_{a_0}(re^{i\theta})|)d\theta$$

for all $0 \leq a_0 < a < 0$ and $0 < r < 1$. The proof of strict inequality in (2.9) is identical to the proof of strict inequality in Theorem 1 in [2, pp. 157-158] and will be omitted. This completes the proof of Theorem 2.1. \square

Proof of Theorem 1.1. The choice $\Phi(x) = e^{e^{2x}}$ in (2.1) allows us to conclude that

$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) < \Lambda_{\Phi_1}(B_{a_0}(re^{i\theta}))$$

for all $0 \leq a_0 < a < 0$ and $0 < r < 1$. Let

$$\|B_a(re^{i\theta})\|_p^p = \frac{1}{2\pi} \int_0^{2\pi} |B_a(re^{i\theta})|^p d\theta.$$

Since

$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) = 1 + \sum_{n=1}^{\infty} \frac{\|B_a(re^{i\theta})\|_{2n}^{2n}}{n!},$$

and, by Lemma 1 of [1], $B_a \in H^p$ for $0 < p < \infty$, we can choose a sequence $r_n \rightarrow 1$ as $n \rightarrow \infty$ for which the inequalities $\Lambda_{\Phi_1}(B_a(r_n e^{i\theta})) < \Lambda_{\Phi_1}(B_{a_0}(r_n e^{i\theta}))$ hold. Hence

$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) \leq \Lambda_{\Phi_1}(B_{a_0}(re^{i\theta}))$$

for all $0 < r \leq 1$ by Hardy's convexity theorem for integral means (see, e.g., [6, Theorem 1.5]).

It now remains to demonstrate that strict inequality holds true in Theorem 1.1. According to Theorem 2 of [4], B_0 is a local maximum on the set of Beurling functions. Thus there is an $a_0, 0 < a_0$, such that

$$\Lambda_{\Phi_1}(B_a(e^{i\theta})) < \Lambda_{\Phi_1}(B_0(e^{i\theta}))$$

for $0 < a \leq a_0$. (James and Matheson [8] have informed the author that, using a numerical method, they have proved the last inequality for $0 < a < 1/2$.)

Finally, combine the last inequality with the fact that Λ_{Φ_1} is log-convex [4, p. 387] to complete the proof of Theorem 1.1. \square

It was pointed out in [1] that B_0 does not maximize the integral means over \mathcal{B} . If we choose $\Phi(x) = e^{px}, 0 < p < \infty$, in Theorem 2.1, we obtain that B_0 maximizes the integral means over \mathcal{B}_0 .

Corollary 2.3. *The inequality*

$$\frac{1}{2\pi} \int_0^{2\pi} |B_a(re^{i\theta})|^p d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |B_{a_0}(re^{i\theta})|^p d\theta$$

holds true for all $0 \leq |a_0| < |a| < 0, 0 < r \leq 1$, and all $0 < p < \infty$.

It will be interesting to see if the approach in Theorem 2.1 can be extended to the univalent functions in \mathcal{D} . The result of this paper provides further evidence in favor of a conjecture made in [1]:

Conjecture 1. Λ_{Φ_1} attains its maximum on \mathcal{B} at B_0 .

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