AN EXTREMAL FUNCTION FOR THE CHANG-MARSHALL INEQUALITY OVER THE BEURLING FUNCTIONS

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Abstract. S.-Y. A. Chang and D. E. Marshall showed that the functional \( \Lambda(f) = \frac{1}{2\pi} \int_0^{2\pi} \exp(|f(e^{i\theta})|^2) d\theta \) is bounded on the unit ball \( B \) of the space \( D \) of analytic functions in the unit disk with \( f(0) = 0 \) and Dirichlet integral not exceeding one. Andreev and Matheson conjectured that the identity function \( f(z) = z \) is a global maximum on \( B \) for the functional \( \Lambda \). We prove that \( \Lambda \) attains its maximum at \( f(z) = z \) over a subset of \( B \) determined by kernel functions, which provides a positive answer to a conjecture of Cima and Matheson.

1. Introduction

Let \( D \) be the Dirichlet space of functions \( f \) analytic on the unit disk \( D \), with \( f(0) = 0 \) and a finite Dirichlet integral

\[
\|f\|_D^2 = \frac{1}{\pi} \int \int_D |f'(z)|^2 \, dx \, dy.
\]

It is well known that \( D \) is a Hilbert space with inner product

\[
\langle f, g \rangle_D = \frac{1}{\pi} \int \int_D f'(z) \overline{g'(z)} \, dx \, dy.
\]

Let \( B = \{ f \in D : \|f\|_D \leq 1 \} \) be its closed unit ball.

We shall be concerned with functionals \( \Lambda_{\Phi} \) on \( B \) defined by

\[
\Lambda_{\Phi}(f) = \frac{1}{\pi} \int_0^{2\pi} \Phi(|f(e^{i\theta})|) \, d\theta,
\]

for \( f \in B \) and \( \Phi : (-\infty, \infty) \to \mathbb{R} \) being a continuous convex nondecreasing function. A function \( f \) is a maximum for \( \Lambda_{\Phi} \) if \( f \in B \) and \( \Lambda_{\Phi}(f) \geq \Lambda_{\Phi}(g) \) for all \( g \in B \).

Chang and Marshall \( \Phi \) proved that if \( \Phi_{\alpha}(t) = e^{\alpha t^2} \) for \( \alpha > 0 \), then \( \Lambda_{\Phi_{\alpha}} \) is bounded on \( B \) if and only if \( \alpha \leq 1 \). In their proof they compared functions in \( B \) to the Beurling functions

\[
B_{\alpha}(z) = \frac{\log \frac{1}{1 - |z|^2}}{\sqrt{\log \frac{1}{1 - |\alpha|^2}}},
\]

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for \( a \in \mathbb{D} \setminus \{0\} \), where the branch of the logarithm is chosen so that \( B_a(a) \) is real. The denominator assures that \( \|B_a\|_D = 1 \). Up to a normalizing factor, the \( B_a \) are the kernel functions for \( D \). We shall denote by \( B_0 \) the set of all Beurling functions and by \( \overline{B}_0 \) its closed convex hull.

A shorter proof of this fact has since been found by Marshall [9]. A significantly more general and stronger inequality has been found by Essén [7]. Andreev and Matheson [1] showed that the identity function \( f(z) = z \) is a local maximum for \( \Lambda \Phi \) on \( B \) and conjectured that it is also a global maximum. Cima and Matheson [4] showed that the identity function is a local maximum on the set \( B_0 \) and that the functional \( \Lambda \Phi \) attains its maximum on \( \overline{B}_0 \). On the other hand, they showed that \( \Lambda \Phi \), when restricted to \( B \), is not weakly continuous at 0, and thus it is an open question whether there exists a global maximum for \( \Lambda \Phi \) on \( B_0 \). Matheson and Pruss [10] studied the regularity of the extremal functions. We refer the reader to their paper for an excellent discussion of this and other related problems and for a list of open problems.

Our principle result is:

**Theorem 1.1.** The inequality

\[
\Lambda \Phi_1(f) < \Lambda \Phi_1(z)
\]

holds true for all \( f \in \overline{B}_0 \).

Our result proves Conjecture 1 of Cima and Matheson in [4].

2. **Proof of Theorem 1.1**

It is natural to set \( B_0(z) = z \) (see [4]). A function \( \Phi(x) \) continuous on \( -\infty < x < \infty \) is said to be convex if \( \Phi((x + y)/2) \leq \frac{\Phi(x) + \Phi(y)}{2} \), and strictly convex if strict inequality holds whenever \( x \neq y \). Theorem 1.1 is a consequence of the following result.

**Theorem 2.1.** Let \( \Phi(x) \) be a convex nondecreasing function on \( -\infty < x < \infty \). For all \( a_0, a \in \mathbb{D} \setminus \{0\} \) such that \( 0 \leq |a_0| < |a| < 1 \), we have

\[
\int_0^{2\pi} \Phi(\log |B_a(re^{i\theta})|)d\theta \leq \int_0^{2\pi} \Phi(\log |B_{a_0}(re^{i\theta})|)d\theta,
\]

\( 0 < r < 1 \). If \( \Phi \) is strictly convex, then the inequality is strict for all \( r \).

**Proof.** Our proof is based on the deep results of Albert Baernstein [2, Theorem 1] on integral means of univalent functions (see also Chapter 7 of Duren’s book [5]). In particular, we need the following proposition [2, Proposition 3].

**Proposition 2.2.** For \( g, h \in L^1(-\pi, \pi) \), the following statements are equivalent.

(a) For each function \( \Phi(s) \) convex and nondecreasing on \( -\infty < s < \infty \),

\[
\int_{-\pi}^{\pi} \Phi(g(x))dx \leq \int_{-\pi}^{\pi} \Phi(h(x))dx.
\]

(b) For each \( t \in \mathbb{R} \),

\[
\int_{-\pi}^{\pi} [g(x) - t]^+dx \leq \int_{-\pi}^{\pi} [h(x) - t]^+dx.
\]

(c) \( g^*(\theta) \leq h^*(\theta), \ 0 \leq \theta \leq \pi. \)
Here for each $r \in (r_1, r_2)$ and $u(re^{i\theta}) \in L^1(0, 2\pi)$ the *Baernstein star-function* of $u$ is defined as

$$u^*(re^{i\theta}) = \sup_{|E|=2\pi} \int_E u(re^{i\theta}) \, dt,$$

$0 \leq \theta \leq \pi$, where $|E|$ denotes the Lebesgue measure of the set $E \subset [-\pi, \pi]$.

In view of Proposition 2.2, we want first to show that

$$\int_{-\pi}^{\pi} \log^+ \left( \frac{|B_a(re^{i\theta})|}{\rho} \right) d\theta \leq \int_{-\pi}^{\pi} \log^+ \left( \frac{|B_a(0)|}{\rho} \right) d\theta,$$

$0 < r < 1$, for each $\rho > 0$ and all $a$ and $a_0$ such that $0 \leq |a_0| < |a| < 1$. Notice that

$$\int_{-\pi}^{\pi} \log^+ \left( \frac{|B_a(re^{i\theta})|}{\rho} \right) d\theta = \int_{-\pi}^{\pi} \log^+ \left( \frac{|B_{a_0}(re^{i\theta})|}{\rho} \right) d\theta,$$

whenever $|a'| = |a''|$. Hence we may assume from now on that $0 \leq a_0 < a < 0$.

We can apply Jensen’s theorem to obtain

$$\int_{-\pi}^{\pi} \log^+ \left( \frac{|B_a(re^{i\theta})|}{\rho} \right) d\theta = \int_{-\pi}^{\pi} N(r, re^{i\theta}) d\phi,$$

since $B_a(0) = 0$. It is easy to see that $B_a$ is a univalent function in the unit disk $D$, $B_a(0) = 0$ and $B_a'(0) = a/A$, where $A = \{\log[1/(1-|a|^2)]\}^{1/2}$, for each $0 < a < 1$, with a continuous extension to the closed unit disk $\overline{D}$, and if $\alpha = re^{i\phi} \neq 0$ is in the range $D_a$ of $B_a$, then

$$N(r, \alpha) = \int_0^r \frac{n(t, \alpha)}{t} dt = \log^+ \left( \frac{r}{|\alpha|} \right) = \log^+ \left( \frac{r}{|B_a^{-1}(\alpha)|} \right),$$

$0 < r < 1$. Let $u_a(\zeta) = -\log |B_a^{-1}(\zeta)|$ be the Green’s function of $D_a$ with pole at 0. Extend it to a continuous function in the punctured plane by setting $u_a(\zeta) = 0$, $\zeta \notin D_a$. The formula (2.3) takes the form

$$N(r, \zeta) = [u_a(\zeta) + \log r]^+, \quad 0 < r < 1, \quad \text{for arbitrary } \zeta, \quad \text{and equation (2.3) becomes}$$

$$\int_{-\pi}^{\pi} \log^+ \left( \frac{|B_a(re^{i\theta})|}{\rho} \right) d\theta = \int_{-\pi}^{\pi} [u_a(re^{i\theta}) + \log r]^+ d\phi.$$

Let $u_{a_0}(\zeta) = -\log |B_{a_0}^{-1}(\zeta)|$ for $\zeta \in D_{a_0}$, and let $u_{a_0}(\zeta) = 0$ elsewhere. In view of (2.6), the inequality (2.3) can be recast in the form

$$\int_{-\pi}^{\pi} [u_a(re^{i\phi}) + \log r]^+ d\phi \leq \int_{-\pi}^{\pi} [u_{a_0}(re^{i\phi}) + \log r]^+ d\phi,$$

$0 < r < 1, 0 < \rho < \infty$. By Proposition 2.2, this is implied by the inequality

$$u_a^*(\rho e^{i\phi}) \leq u_{a_0}^*(\rho e^{i\phi}),$$

$0 < \rho < \infty, 0 \leq \phi \leq \pi$.

The function $u(\zeta)$ is continuous in $0 < |\zeta| < \infty$, it is positive and harmonic in $D_a$, and identically zero outside $D_a$. Thus it is subharmonic in $0 < |\zeta| < \infty$. Hence by [2, Theorem A] and the definition (2.2) of the star-function, $u_a^*$ is subharmonic in the open upper half-plane and continuous in the closed upper half-plane, except at the origin.
Since $B_a^{-1}(\zeta) = (1 - e^{-A\zeta})/a$, then, near the origin, $u_a$ has the form

$$u_a(\zeta) = -\log |\zeta| - \log \frac{A}{a} + u_{1a}(\zeta),$$

where $u_{1a}$ is harmonic and $u_{1a}(0) = 0$. Thus

$$u_a^*(pe^{i\phi}) + 2\phi \log \rho \to -2\phi \log \frac{A}{a}$$

as $\rho \to 0$ for $0 \leq \phi \leq \pi$. Similarly, near the origin, $u_{a_0}$ has the form

$$u_{a_0}(\zeta) = -\log |\zeta| - \log \frac{A_0}{a_0} + u_{1a_0}(\zeta),$$

where $u_{1a_0}$ is harmonic and $u_{1a_0}(0) = 0$. Thus

$$u_{a_0}^*(pe^{i\phi}) + 2\phi \log \rho \to -2\phi \log \frac{A_0}{a_0}$$

as $\rho \to 0$ for $0 \leq \phi \leq \pi$. It follows that

$$|u_a^*(pe^{i\phi}) - u_{a_0}^*(pe^{i\phi})| \to -2\phi \log \frac{a_0 A}{AA_0}$$

as $\rho \to 0$ for $0 \leq \phi \leq \pi$. It is easy to see that $a_0 A/(a A_0) > 1$ for $a_0 < a$ and hence that $-2\pi \log \frac{a_0 A}{AA_0} \leq -2\phi \log \frac{a_0 A}{AA_0} \leq 0$ for $a_0 < a$.

Hence $(u_a^* - u_{a_0}^*)$ is subharmonic in the upper half-plane and continuous in its closure except at the origin, where it has a bounded discontinuity: for $\phi = 0$,

$$\lim_{\rho \to 0} (u_a^* (\rho) - u_{a_0}^* (\rho)) = 0,$$

and for $\phi = \pi$,

$$\lim_{\rho \to 0} (u_a^*(-\rho) - u_{a_0}^*(-\rho)) = -2\pi \log \frac{a_0 A}{AA_0}.$$

We want to show that $(u_a^* - u_{a_0}^*) < 0$ in the open upper half-plane. Since $u_a^* - u_{a_0}^*$ is discontinuous at the origin, we cannot apply the maximum principle for subharmonic functions to $u_a^* - u_{a_0}^*$ at this point. The proof of the inequality $(u_a^* - u_{a_0}^*) < 0$ for $3\zeta > 0$ will be based on the following four steps (a)–(d).

(a) On the positive real axis, by definition, $u_a^*(\zeta) = v^*(\zeta) = 0$ for $\zeta > 0$.

(b) Next let $d_a$ be the distance from 0 to the complement of $D_a$. It is obvious that $\Re(1 - ae^{i\theta})^{-1} > 0$. Since the branch of the logarithm was chosen so that $B_a^*(a)$ is real, then

$$|B_a^*(e^{i\theta})| = \frac{1}{A} \left\{ \log \frac{1}{|1 - a e^{i\theta}|} \right\}^2 + \left| \arg \frac{1}{1 - a e^{i\theta}} \right|^2)^{1/2}.$$

Since $\max |1 - ae^{i\theta}| = |1 - ae^{i\pi}| = 1 + a$ and $|\arg \frac{1}{1 - ae^{i\theta}}|^2 = 0$, it is easy to see that

$$-\frac{1}{A} \log \frac{1}{1 + a} \leq |B_a^*(e^{i\theta})| \leq \frac{1}{A} \log \frac{1}{1 - a}$$

for $0 < a < 1$. Thus $d_a = -\frac{1}{A} \log \frac{1}{1 + a}$. We want to show that $d_a$ is a decreasing function of $a$ for $0 < a < 1$. It is clear that $d_a \to 1$ as $a \to 0$. Let

$$f(a) = \frac{\log (1 + a)}{A}.$$

Then

$$f'(a) = -\frac{[(1 - a) \log (1 - a) + \log (1 + a)]}{(1 - a^2) A^3}.$$
Let

\[ f_1(a) = (1 - a) \log(1 - a) + \log(1 + a). \]

An easy computation shows that \( f_1'(a) > 0 \) for \( 0 < a < 1 \). Thus \( f_1 \) is an increasing function of \( a \), and it follows that \( f_1(a) > 0 \) for \( 0 < a < 1 \) since \( f_1'(0) = 0 \). Therefore \( f_1 \) is an increasing function of \( a \) for \( 0 < a < 1 \) and \( f_1(a) > 0 \) since \( f_1(0) = 0 \). Finally, this implies that \( f'(a) < 0 \) for \( 0 < a < 1 \), and thus \( f \) is a decreasing function of \( a \). Therefore \( d_{a_0} > d_a \) for all \( a, a_0 < a < 1 \).

In the disk \(|\zeta| < d_a\), \( u_\phi(\zeta) \) has the form (2.8), where \( u_{1a} \) is harmonic in \(|\zeta| < d_a\) and \( u_{1a}(0) = 0 \). Thus

\[ u_\phi^*(\rho e^{i\pi}) = -2\pi \log \frac{1}{\rho} - 2\pi \log \frac{A}{a} \]

and, similarly,

\[ u_{a_0}^*(\rho e^{i\pi}) = -2\pi \log \frac{1}{\rho} - 2\pi \log \frac{A_0}{a_0} \]

for \( 0 < \rho < d_a \). Hence \( u_\phi^*(\zeta) < u_{a_0}^*(\zeta) \) for \(-d_a < \zeta < 0\).

(c) Since \( u_{1a}(\zeta) \) and \( u_{1a_0}(\zeta) \) are harmonic in \(|\zeta| < d_a\) and \( u_{1a}(0) = u_{1a_0}(0) = 0 \), then for every \( \epsilon > 0 \) there is a \( \rho_0, \rho_0 = |\zeta| < d_a \), such that \( |u_{1a}(\zeta)| < \epsilon/2 \) and \( |u_{1a_0}(\zeta)| < \epsilon/2 \) for all \( \zeta, |\zeta| \leq \rho_0 \). Thus

\[
\begin{align*}
\sup_{|E|=2\phi} \int_E u_{1a}(\rho e^{i\theta}) dt &= -2\phi \log \rho - 2\phi \log \frac{A}{a} + \sup_{|E|=2\phi} \int_E u_{1a}(\rho e^{i\theta}) dt \\
&\leq -2\phi \log \rho - 2\phi \log \frac{A}{a} + \phi \epsilon
\end{align*}
\]

and

\[
\begin{align*}
\sup_{|E|=2\phi} \int_E u_{a_0}(\rho e^{i\theta}) dt &= -2\phi \log \rho - 2\phi \log \frac{A_0}{a_0} + \sup_{|E|=2\phi} \int_E u_{a_0}(\rho e^{i\theta}) dt \\
&\geq -2\phi \log \rho - 2\phi \log \frac{A_0}{a_0} - \phi \epsilon
\end{align*}
\]

for \( 0 < \rho \leq \rho_0 \) and \( 0 < \phi < \pi \). Now choose \( \epsilon \) such that \( \epsilon < \log(Aa_0/aA_0) \). Then

\[
u_\phi^*(\rho e^{i\phi}) - u_{a_0}^*(\rho e^{i\phi}) \leq -2\phi \log \frac{Aa_0}{aA_0} + 2\phi \epsilon < 0
\]

for all \( 0 < \rho \leq \rho_0 \) and \( 0 < \phi < \pi \). Hence \( u_\phi^*(\zeta) < u_{a_0}^*(\zeta) \) for \(|\zeta| \leq \rho_0 < d_a \) and \( 0 < \phi < \pi \).

(d) To establish the inequality on \(-\infty < \zeta < -d_a \), we fix \( \epsilon > 0 \) and consider the function

\[ Q(\zeta) = u_{1a}^*(\zeta) - u_{a_0}^*(\zeta) - \epsilon \phi, \]

\( \zeta = \rho e^{i\phi} \), which is subharmonic in \( A = \{\zeta : \rho_0 < |\zeta|, 0 < 3\zeta\} \) and continuous in the closure of \( A \). Let \( M \) be the maximum of \( Q(\zeta) \) in \( \overline{A} \). Then \( M > 0 \) and, according to the maximum principle for subharmonic functions, the maximum is attained somewhere on the boundary of \( A \). Suppose \( M > 0 \). Since \( u_\phi^*(\zeta) \leq u_{a_0}^*(\zeta) \) on the set \( \{\zeta : -d_a \leq \zeta \leq \rho_0\} \cup \{\zeta : |\zeta| = \rho_0, 3\zeta > 0\} \cup \{\zeta : \rho_0 \leq \zeta < \infty\} \), there
is some point $-\zeta_1 = -\rho_1$ for which $-\infty < \zeta_1 < -d_a$ and $Q(\zeta_1) = M$. Let $G_\alpha(\phi)$ denote the symmetric decreasing rearrangement of $u_a(\rho_1 e^{i\phi})$. Then
\[ \frac{\partial u^*_a(\rho_1 e^{i\phi})}{\partial \phi} = 2G_\alpha(\phi) \]
for $0 \leq \phi \leq \pi$ by [2, Proposition 2]. But because $\rho_1 > d_a$, there is some point on the circle $|\zeta| = \rho_1$ that lies outside $D_a$, so
\[ G_\alpha(\pi) = \inf_{0 \leq \phi \leq \pi} u_a(\rho_1 e^{i\phi}) = 0. \]
Applying the same argument to $u_{a_0}$ we obtain
\[ \frac{\partial u^*_{a_0}(\rho_1 e^{i\phi})}{\partial \phi} = 2G_{a_0}(\phi) \]
for $0 \leq \phi \leq \pi$. If $d_a < \rho_1 \leq d_{a_0}$, then
\[ G_{a_0}(\phi) = \inf_{0 \leq \phi < \pi} \{ t : \lambda(t) \leq 2\phi \}, \]
where $\lambda$ is the distribution function of $u_{a_0}$, $\lambda(t) = |\{ \phi : u_{a_0}(\rho_0 e^{i\phi}) > t \}|$, and
\[ G_{a_0}(\pi) = \lim_{\phi \to \pi^-} G_{a_0}(\phi). \]
Hence $G_{a_0}(\pi) \geq 0$ if $d_a < \rho_1 \leq d_{a_0}$. If $d_{a_0} < \rho_1$, there is some point on the circle $|\zeta| = \rho_1$ that lies outside $D_{a_0}$, so
\[ G_{a_0}(\pi) = \inf_{0 \leq \phi \leq \pi} u_{a_0}(\rho_1 e^{i\phi}) = 0. \]
Therefore
\[ \frac{\partial Q}{\partial \phi}(\zeta_1) \leq -\epsilon < 0, \]
which contradicts the assumption that $Q(\zeta)$ has a relative maximum at $\zeta_1$. Hence $M = 0$ and
\[ u^*_a(\zeta) \leq u^*_{a_0}(\zeta) + \epsilon \phi \leq u^*_{a_0}(\zeta) + \epsilon \pi \]
for $\zeta \in \mathcal{A}$. Letting $\epsilon \to 0$ we obtain that
\[ u^*_a(\rho e^{i\phi}) \leq u^*_{a_0}(\rho e^{i\phi}) \]
for $\zeta \in \mathcal{A}$.

We are in a position now to prove that $u^*_a(\zeta) < u^*_{a_0}(\zeta)$ in the open upper half-plane. Combining (a)–(d) we obtain (2.7). Furthermore, $u^*_a(\zeta) < u^*_{a_0}(\zeta)$ on the set $\{ \zeta : -d_a \leq \zeta \leq \rho_0 \} \cup \{ \zeta : |\zeta| = \rho_0, \Im \zeta > 0 \}$ by (b) and (c). Hence $u^*_a - u^*_{a_0}$ is a subharmonic function on $\mathcal{A}$ that is not identically equal to zero there and, by the maximum principle, this implies that $u^*_a(\zeta) < u^*_{a_0}(\zeta)$ everywhere in $\mathcal{A}$. Also, $u^*_a(\zeta) < u^*_{a_0}(\zeta)$ for $\{ \zeta : 0 < |\zeta| \leq \rho_0 < d_a, 0 < \Im \zeta \}$ by (c). Therefore,
\[ u^*_a(\zeta) < u^*_{a_0}(\zeta) \]
in the open upper half-plane.

It follows from Proposition 2.2 that
\[ \int_0^{2\pi} \Phi(\log |B_a(re^{i\theta})|)d\theta \leq \int_0^{2\pi} \Phi(\log |B_{a_0}(re^{i\theta})|)d\theta \]
for all $0 \leq a_0 < a < 0$ and $0 < r < 1$. The proof of strict inequality in (2.9) is identical to the proof of strict inequality in Theorem 1 in [2, pp. 157-158] and will be omitted. This completes the proof of Theorem 2.1. \qed
Proof of Theorem 1.1. The choice $\Phi(x) = e^{2x}$ in (2.1) allows us to conclude that
$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) < \Lambda_{\Phi_1}(B_{a_0}(re^{i\theta}))$$
for all $0 \leq a_0 < a < 0$ and $0 < r < 1$. Let
$$\|B_a(re^{i\theta})\|_p = \frac{1}{2\pi} \int_0^{2\pi} |B_a(re^{i\theta})|^p \, d\theta.$$ 
Since
$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) = 1 + \sum_{n=1}^{\infty} \frac{\|B_a(re^{i\theta})\|_{2n}}{n!},$$
and, by Lemma 1 of [1], $B_a \in H^p$ for $0 < p < \infty$, we can choose a sequence $r_n \to 1$ as $n \to \infty$ for which the inequalities $\Lambda_{\Phi_1}(B_a(r_n e^{i\theta})) < \Lambda_{\Phi_1}(B_{a_0}(r_n e^{i\theta}))$ hold. Hence
$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) \leq \Lambda_{\Phi_1}(B_{a_0}(re^{i\theta}))$$
for all $0 < r \leq 1$ by Hardy’s convexity theorem for integral means (see, e.g., [6, Theorem 1.5]).

It now remains to demonstrate that strict inequality holds true in Theorem 1.1. According to Theorem 2 of [4], $B_0$ is a local maximum on the set of Beurling functions. Thus there is an $a_0$, $0 < a_0$, such that
$$\Lambda_{\Phi_1}(B_a(e^{i\theta})) < \Lambda_{\Phi_1}(B_0(e^{i\theta}))$$
for $0 < a \leq a_0$. (James and Matheson [8] have informed the author that, using a numerical method, they have proved the last inequality for $0 < a < 1/2$.)

Finally, combine the last inequality with the fact that $\Lambda_{\Phi_1}$ is log-convex [4, p. 387] to complete the proof of Theorem 1.1.

It was pointed out in [1] that $B_0$ does not maximize the integral means over $B$. If we choose $\Phi(x) = e^{px}$, $0 < p < \infty$, in Theorem 2.1, we obtain that $B_0$ maximizes the integral means over $B_0$.

Corollary 2.3. The inequality
$$\frac{1}{2\pi} \int_0^{2\pi} |B_a(re^{i\theta})|^p \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |B_{a_0}(re^{i\theta})|^p \, d\theta$$
holds true for all $0 \leq |a_0| < |a| < 0$, $0 < r \leq 1$, and all $0 < p < \infty$.

It will be interesting to see if the approach in Theorem 2.1 can be extended to the univalent functions in $D$. The result of this paper provides further evidence in favor of a conjecture made in [1]:

Conjecture 1. $\Lambda_{\Phi_1}$ attains its maximum on $B$ at $B_0$.

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