PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 7, Pages 2077–2081
S 0002-9939(05)07722-1

AN INTERPOLATION THEOREM FOR HILBERT SPACES WITH NEVANLINNA-PICK KERNEL

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(Communicated by Juha M. Heinonen)

Abstract. We prove an interpolation theorem for Hilbert spaces of analytic functions that have the Nevanlinna-Pick property. This result applies to Dirichlet and Dirichlet-type spaces, and in particular a short proof of the theorem by Marshall-Sundberg on interpolating sequences is obtained.

1. Introduction

We consider Hilbert spaces of analytic functions in the unit disk with inner product

$$\langle f, g\rangle = \sum_n \frac{1}{c_n} a_n \bar{b}_n,$$

where $f(z) = \sum_n a_n z^n$, $g(z) = \sum_n b_n z^n$. The reproducing kernels $k_w(z)$ satisfying

$$\langle f, k_w\rangle = f(w)$$

are given by $k_w(z) = k(\bar{w}z)$, where

$$k(z) = \sum_n c_n z^n.$$

We will assume that k satisfies

$$1 - \frac{1}{k(z)} = \sum_n d_n z^n \tag{1}$$

with $d_n \geq 0$ and $k(r) > 0$ for $r > 0$. This holds e.g. for Dirichlet and Dirichlet-type spaces ([6, p. 22]).

A Hilbert space H has the Nevanlinna-Pick property when the matrix

$$(1 - w_i \bar{w}_j)\langle k_{z_i}, k_{z_j}\rangle$$

being positive semi-definite is necessary and sufficient for the existence of ϕ in the multiplier algebra M_H of H satisfying $\phi(z_i) = w_i$, $\|\phi\|_{M_H} \leq 1$. In [6, Lemma 10] it is proved that (1) implies the Nevanlinna-Pick property.

We define $\{z_n\}$ to be interpolating for H when

$$\{\frac{f(z_n)}{\|k_{z_n}\|_H} : f \in H\} = l^2$$

Received by the editors September 30, 2003 and, in revised form, March 12, 2004.
2000 *Mathematics Subject Classification.* Primary 30H05, 46E22.

and to be interpolating for M_H when

$$\{\phi(z_n) : \phi \in M_H\} = l^\infty.$$

Also, $\{\frac{k_{z_n}}{\|k_{z_n}\|_H}\}$ is a Riesz sequence if

$$\|\sum_n a_n \frac{k_{z_n}}{\|k_{z_n}\|_H}\| \ \sim \ (\sum_n |a_n|^2)^{\frac{1}{2}}.$$

For spaces with the Nevanlinna-Pick property, these three conditions are satisfied by the same sequences ([6, Corollary 7], [2, Theorem 9.19]). Our main result is a characterisation of these sequences under an extra assumption. To state this, we say $\{z_n\}$ is separated when

$$|\langle \frac{k_{z_n}}{\|k_{z_n}\|}, \frac{k_{z_m}}{\|k_{z_m}\|}\rangle| < \sigma$$

for $n \neq m$, $\sigma < 1$. We also make the further technical assumption that for any sequence $\{z_n\}$, boundedness on l^2 of the Gram matrix

$$\{\langle \frac{k_{z_n}}{\|k_{z_n}\|}, \frac{k_{z_m}}{\|k_{z_m}\|}\rangle\}$$

implies boundedness of the matrix

$$\{|\langle \frac{k_{z_n}}{\|k_{z_n}\|}, \frac{k_{z_m}}{\|k_{z_m}\|}\rangle|\}.$$

Under these assumptions we have the following:

Theorem 1.1. *Suppose H is a Hilbert space of analytic functions satisfying* (1) *and the condition on the Gram matrix. A sequence $\{z_n\}$ is interpolating for H if and only if it is separated and the Gram matrix is bounded.*

The assumptions in Theorem 1.1 can be stated in a way that looks more concrete. For this, define μ to be a Carleson measure for H if

$$\int |f|^2 d\mu \lesssim \|f\|^2.$$

Boundedness of the Gram matrix is equivalent to $\sum_n \|k_{z_n}\|^{-2}\delta_{z_n}$ being a Carleson measure ([2, Proposition 9.5]).

As a convenient reference for most results that we use, we mention [2], especially chapters 7 and 9.

2. PROOF OF THEOREM 1.1

It is clear that an interpolating sequence must have a bounded Gram matrix. The separation condition is proved as in [6, p. 41].

For the converse we will prove that $\{\frac{k_{z_n}}{\|k_{z_n}\|_H}\}$ is a Riesz sequence. According to Bari's theorem ([7, p. 132]), this holds if and only if both matrices

$$\Gamma = \langle \frac{k_{z_n}}{\|k_{z_n}\|}, \ \frac{k_{z_m}}{\|k_{z_m}\|}\rangle, \ \Gamma' = \langle f_n, f_m\rangle$$

are bounded operators on l^2. Here $\{f_n\}$ are biorthogonal functionals defined as minimal norm solutions of

$$\langle f_n, \frac{k_{z_m}}{\|k_{z_m}\|}\rangle = \delta_{n,m}.$$

It is clear that $\{f_n\}$ lie in the span of $\{k_{z_n}\}$.

For $A \subset \{z_i\}$, denote by H_A the Hilbert space $\{f \in H : f_{|A} = 0\}$ and by k_α^A the reproducing kernel for H_A at α. It satisfies

$$k_\alpha^A(\alpha) = \|k_\alpha^A\|_{H_A}^2$$

and maximises $|f(\alpha)|$ among f in H_A with $\|f\| = \|k_\alpha^A\|$. So we have

$$f_n = \frac{k_{z_n}^{A_n}}{\|k_{z_n}^{A_n}\|^2} \cdot \|k_{z_n}\|$$

where $A_n = \{z_i\} \setminus z_n$.

To prove $\Gamma^{'}$ is bounded on l^2 we write

$$\langle f_n, f_m \rangle = \frac{\|k_{z_n}\| \|k_{z_m}\|}{\|k_{z_n}^{A_n}\|^2 \|k_{z_m}^{A_m}\|^2} \langle k_{z_n}^{A_n}, k_{z_m}^{A_m} \rangle. \tag{2}$$

Reproducing kernels for $H_{A\cup a}$ are determined by those for H_A by the relation

$$k_\alpha^{A\cup a}(z) = k_\alpha^A(z) - \frac{k_a^A(z) k_\alpha^A(a)}{k_a^A(a)}.$$

Denoting $A^{'} = \{z_i\} \setminus \{z_n \cup z_m\}$, we have

$$k_{z_n}^{A_n}(z) = k_{z_n}^{A^{'}}(z) - \frac{k_{z_m}^{A^{'}}(z) k_{z_n}^{A^{'}}(z_m)}{k_{z_m}^{A^{'}}(z_m)} \tag{3}$$

and

$$k_{z_m}^{A_m}(z) = k_{z_m}^{A^{'}}(z) - \frac{k_{z_n}^{A^{'}}(z) k_{z_m}^{A^{'}}(z_n)}{k_{z_n}^{A^{'}}(z_n)}. \tag{4}$$

We get from (4)

$$\begin{aligned} \langle k_{z_n}^{A_n}, k_{z_m}^{A_m} \rangle &= \langle k_{z_n}^{A_n}, k_{z_m}^{A^{'}} \rangle - \frac{\overline{k_{z_m}^{A^{'}}(z_n)}}{k_{z_n}^{A^{'}}(z_n)} \langle k_{z_n}^{A_n}, k_{z_n}^{A^{'}} \rangle \\ &= k_{z_n}^{A_n}(z_m) - \frac{\overline{k_{z_m}^{A^{'}}(z_n)}}{k_{z_n}^{A^{'}}(z_n)} k_{z_n}^{A_n}(z_n). \end{aligned}$$

From (3) it follows that $k_{z_n}^{A_n}(z_n) \leq k_{z_n}^{A^{'}}(z_n)$, so

$$|\langle k_{z_n}^{A_n}, k_{z_m}^{A_m} \rangle| \leq |k_{z_m}^{A^{'}}(z_n)|.$$

Referring to (2) we then have

$$|\langle f_n, f_m \rangle| \leq \frac{\|k_{z_n}\| \|k_{z_m}\|}{\|k_{z_n}^{A_n}\|^2 \|k_{z_m}^{A_m}\|^2} |k_{z_m}^{A^{'}}(z_n)|. \tag{5}$$

For $A \subset \{z_i\}$, $z_j \notin A$, the extremal problems

$$c_H = \inf\{\|f\|_H : f(z_j) = 1, f_{|A} = 0\}$$

and

$$c_{M_H} = \inf\{\|\phi\|_{M_H} : \phi(z_j) = 1, \phi_{|A} = 0\}$$

are related by

$$c_{M_H} = c_H \|k_{z_j}\|,$$

and the unique extremals satisfy

$$f_{z_j} = \phi_{z_j} \frac{k_{z_j}}{\|k_{z_j}\|^2}.$$

This follows from the Nevanlinna-Pick property [6, Proposition 6]. In [6, Proposition 23] and [9] it is proved that

$$c_{M_H} \leq \prod_{z_i \in A} (1 - \frac{|\langle k_{z_i}, k_{z_j}\rangle|^2}{\|k_{z_i}\|^2 \|k_{z_j}\|^2}).$$

By the Carleson condition applied to $\frac{k_{z_j}}{\|k_{z_j}\|}$ and separation we get $c_{M_H} \leq C$, with C independent of A and z_j. From this it follows that

$$\|k_{z_n}^{A_n}\| \geq C^{-1}\|k_{z_n}\|, \ \|k_{z_m}^{A_m}\| \geq C^{-1}\|k_{z_m}\|,$$

and

$$\frac{k_{z_m}^{A'}}{\|k_{z_m}^{A'}\|^2} = \phi \frac{k_{z_m}}{\|k_{z_m}\|^2}$$

where $\|\phi\|_{M_H} \leq C$. Thus

$$|k_{z_m}^{A'}(z_n)| \leq C|k_{z_m}(z_n)|$$

and in (5) we get the estimate

$$|\langle f_n, f_m\rangle| \leq C\frac{|k_{z_m}(z_n)|}{\|k_{z_n}\|\|k_{z_m}\|}.$$

Since Γ is bounded we get from the assumption on the Gram matrix that Γ' is also bounded, which finishes the proof.

We remark that this proof works for any Hilbert space with a complete Nevanlinna-Pick kernel. This follows from the characterisation of such kernels in [1, Theorem 1.2].

We now prove that Theorem 1.1 applies to the Dirichlet space. The reproducing kernels are

$$k_w(z) = \frac{1}{z\bar{w}} \log(\frac{1}{1 - z\bar{w}}).$$

Assuming that the $\{a_n\}$ are positive, we need to bound

$$\sum_{n,m} a_n a_m \frac{|k_{z_m}(z_n)|}{\|k_{z_n}\|\|k_{z_m}\|}.$$

One checks that

$$|\frac{1}{z\bar{w}} \log(\frac{1}{1 - z\bar{w}})| \lesssim C + \mathrm{Re}\ \frac{1}{z\bar{w}} \log(\frac{1}{1 - z\bar{w}})$$

so that

$$\sum_{n,m} a_n a_m \frac{|k_{z_m}(z_n)|}{\|k_{z_n}\|\|k_{z_m}\|} \lesssim (\sum_n \frac{a_n}{\|k_{z_n}\|})^2 + \mathrm{Re}\ \sum_{n,m} a_n a_m \frac{k_{z_m}(z_n)}{\|k_{z_n}\|\|k_{z_m}\|}.$$

The second term is by assumption bounded by $\|a_n\|_{l_2}^2$. For the first term it follows from the Carleson measure condition that

$$(\sum_n \frac{a_n}{\|k_{z_n}\|})^2 \leq \|a_n\|_{l_2}^2 \cdot \sum_n \|k_{z_n}\|^{-2} \lesssim \|a_n\|_{l_2}^2.$$

Similarly, Theorem 1.1 applies to the Dirichlet-type spaces D_α that lie between the Hardy and Dirichlet spaces. For these, the reproducing kernels are

$$k_w(z) = \frac{1}{(1-z\bar{w})^{1-\alpha}},\ 0 < \alpha < 1.$$

It is clear that

$$\operatorname{Arg} \frac{1}{(1-z\bar{w})^{1-\alpha}} \in [-\frac{\pi}{2}(1-\alpha), \frac{\pi}{2}(1-\alpha)],$$

which implies

$$\frac{1}{|1-z\bar{w}|^{1-\alpha}} \leq C_\alpha \cdot \operatorname{Re} \frac{1}{(1-z\bar{w})^{1-\alpha}}.$$

From this the conclusion follows easily.

Thus one gets a short proof of the theorem from [6] and [3] on interpolating sequences. We mention also [4], which gives a constructive proof that does not use the Nevanlinna-Pick property and applies to spaces defined with L^p-norms. For characterisations of Carleson measures for the Dirichlet space, we refer to [8].

The result on interpolating sequences for D_α is proved by different means in [5] and [10]. We would like to point out that previous proofs for the Dirichlet space and the spaces D_α have been very different, whereas with the method here we obtain a unified treatment.

Theorem 1.1 is related to [2, Question 9.57], which asks if for a Hilbert space with a complete Nevanlinna-Pick kernel, a sequence being separated and having bounded Gram matrix must be an interpolating sequence. Theorem 1.1 gives an affirmative answer to this under the hypothesis on the Gram matrix. In this connection we point out that this extra hypothesis does not hold for the Hardy space H^2.

Acknowledgment

I would like to thank K. Seip for useful discussions.

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