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THE HOMOLOGICAL DETERMINANT OF QUANTUM GROUPS OF TYPE A

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ABSTRACT. Let R be a Hecke symmetry depending algebraically on a parameter $q \in \mathbb{C}$. We show that the homology of the Koszul complex associated with R is one-dimensional when q is not a root of unity. A generator of this homology group then induces the homological determinant of the quantum group associated with R.

INTRODUCTION

Let V be a vector space over a field k and $GL(V)$ the general linear group. It is well known that elements of $GL(V)$ act on the n-th homogeneous component of the exterior algebra over V by means of the determinant. More precisely, let $x_1, x_2, \ldots, x_d$ be a basis of V. Then $\wedge_d(V)$ is one-dimensional and a non-zero vector is $x_1 \wedge x_2 \wedge \ldots \wedge x_d$. If $g \in GL(V)$ has the matrix A with respect to this basis, then

$$g \cdot (x_1 \wedge x_2 \wedge \ldots \wedge x_d) = \det A \cdot x_1 \wedge x_2 \wedge \ldots \wedge x_d.$$

Now let V be a vector superspace of dimension $(r|s)$, $r+s=d$. The super group $GL(V)$ is defined as follows. Let $x_1, x_2, \ldots, x_d$ be a homogeneous basis of V, where the parity of the first r elements is even and the parity of the rest is odd. Let z_j^i be the endomorphism that maps x_i to x_j and other basis elements to zero. We consider z_j^i as a generator with parity being the sum of those of x_i and x_j. The super semi-group $\mathrm{End}(V)$ is the spectrum of the super commutative algebra

$$M := k\left\langle \{z_j^i\}_{1\le i,j\le d}\right\rangle / (z_j^i z_l^k = (-1)^{(\hat{i}+\hat{j})(\hat{k}+\hat{l})} z_l^k z_j^i)$$

(where $k\langle\{z_j^i\}_{1\le i,j\le d}\rangle$ denotes the free non-commutative algebra and $\hat{i}$ denotes the parity of x_i). Thus, for a super commutative algebra K, an endomorphism of $V_K := V \otimes K$ is a K-point of M, i.e. an algebra homomorphism $M \longrightarrow K$.

The invertibility of a super matrix can be given in terms of the super determinant or Berezinian, which was introduced by Berezin. Let K be a super commutative algebra and Z be a K-point of $\mathrm{End}(V)$. The matrix $Z = (z_j^i)$ has the following form: $Z = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where A, D are square matrices of dimension $m \times m$ and $n \times n$, respectively, whose entries' parities are even, and B, D are matrices of types $m \times n$

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and $n \times m$, whose entries' parities are odd. The super determinant of Z is defined to be

$$\mathrm{Ber} Z = \det T^{-1} \det(A - CD^{-1}B).$$

It is shown that the matrix Z is invertible iff its super determinant is and that the super determinant is multiplicative. Thus, the invertible super matrix forms a group $GL(V)$, which is an algebraic super-subgroup of $\mathrm{End}(V)$. It is however not clear why the definition of Ber is independent of the choice of bases (our basis is a distinguished basis).

In [17] Manin suggested the following construction to define the super determinant. Let V^* denote the vector space dual to V with the dual basis $\xi^1, \xi^2, \ldots, \xi^n$, $\xi^i(x_j) = \delta^i_j$. Manin introduced the following Koszul complex: its (k,l)-term is given by $K^{k,l} := \wedge_k \otimes \mathsf{S}_l{}^*$, where $\wedge_n$ and S_n are the n-th homogeneous components of the exterior and the symmetric tensor algebra over V. The differential $d^{k,l} : K^{k,l} \longrightarrow K^{k+1,l+1}$ is given by

$$d^{k,l}(h \otimes \phi) = \sum_i hx_i \otimes \xi^i \wedge \phi.$$

It is easy to check that $d_{k,l}$ is $GL(V)$-equivariant; hence the homology groups of this complex are representations of $GL(V)$. On the other hand, one can show that this complex is exact everywhere except at the term (m,n), where the homology group is one-dimensional; thus, it defines a one-dimensional representation of $GL(V)$. It turns out that elements of $GL(V)$ act on this representation by means of its super determinant; in other words, the definition of the super determinant is basis free.

The quantum semigroup of type A is the "spectrum" of the bialgebra

$$E := k\langle \{z^i_j\}_{1 \le i,j \le d} \rangle / (R^{ij}_{uv} z^u_k z^v_l = z^i_t z^j_s R^{ts}_{kl})$$

where R is a Hecke symmetry (see §1). The Hecke symmetry resembles the usual flipping operator $a \otimes b \longmapsto b \otimes a$ or $a \otimes b \longmapsto (-1)^{\hat{a}\hat{b}} b \otimes a$ (a, b are homogeneous) in super symmetry.

In [5, 14], a Koszul complex is defined for R. For that, one first has to define the quantum exterior and quantum symmetric tensors by means of certain projectors on $V^{\otimes n}$. It is still an open question whether this complex has the homology group concentrated at a certain term and its dimension is one. Some efforts have been made. Gurevich [5] showed this for even Hecke symmetries (i.e., those that induce a finite-dimensional exterior algebra); Lyubashenko and Sudbery [14] showed this for Hecke sums of an odd and an even Hecke symmetry.

In this paper, assuming that R depends algebraically on q, where q runs in $\mathbb{C}$, we give the affirmative answer to this question for an algebraically dense set of values of q. Our tactic is first to use a new result of Deligne [1] to check the case $q = 1$. Then using a standard argument we show that for a dense set of values of q, the homology group of K has dimension less than that of the corresponding homology groups when $q = 1$. In other words, for an algebraically dense subset of $\mathbb{C}$, the homology group has dimension at most 1. It remains to show the non-vanishing of the homology.

1. Hecke symmetries and the associated quantum groups

We work over an algebraically closed field k of characteristic zero. Let V be a vector space over k of dimension d. Let $R : V \otimes V \longrightarrow V \otimes V$ be an invertible operator. R is called a *Hecke symmetry* if the following conditions are fulfilled:

- $R_1R_2R_1 = R_2R_1R_2$, where $R_1 := R \otimes \mathrm{id}_V$, $R_2 := \mathrm{id}_V \otimes R$,
- $(R+1)(R-q) = 0$ for some $q \in k$.
- The half adjoint to R, $R^\sharp : V^* \otimes V \longrightarrow V \otimes V^*$, $\langle R^\sharp(\xi \otimes v), w\rangle = \langle \xi, R(v \otimes w)\rangle$, is invertible.

Throughout this work we will assume that q *is not a root of unity other than the unity itself*. If $q = 1$, R is called vector symmetry. Vector symmetries were introduced by Lyubashenko [13] and generalized to Hecke symmetries by Gurevich [5].

Let us fix a basis $x_1, x_2, \ldots, x_d$ of V. Then R can be given in terms of a matrix, also denoted by R, $R(x_i \otimes x_j) = x_k \otimes x_l R^{kl}_{ij}$, where we adopt the convention of summing over the indices that appear in both the lower and upper places. The matrix ${R^\sharp}^{kl}_{ij}$ is given by ${R^\sharp}^{kl}_{ij} = R^{ik}_{jl}$. Therefore, the invertibility of $R^\sharp$ can be expressed as follows: there exists a matrix P such that

$$P^{im}_{jn} R^{nk}_{ml} = \delta^i_l \delta^k_j, \quad R^{im}_{jn} P^{nk}_{ml} = \delta^i_l \delta^k_j. \tag{1}$$

Consider the following algebra:

$$E_R := k\left\langle \{z^i_j\}_{1 \le i,j \le d} \right\rangle / (z^i_m z^j_n R^{mn}_{kl} = R^{ij}_{pq} z^p_k z^q_l),$$

which is in fact a coquasitriangular bialgebra [12, 13] with the coproduct given by $\Delta(z^i_j) = z^i_k \otimes z^k_j$ and the counit given by $\varepsilon(z^i_j) = \delta^i_j$. The coquasitriangular structure is given by $r(z^i_j, z^k_l) = R^{ki}_{jl}$. The bialgebra E is called the "function algebra" on the corresponding quantum endomorphism space or the matrix quantum semigroup.

There is a right coaction of E_R on V, given by $\delta(x_i) = x_j \otimes z^j_i$. This coaction induces actions of E_R on $V^{\otimes n}$ for $n \ge 1$. The braiding on $V \otimes V$ induced from the coquasitriangular structure r is precisely the operator R. There is a natural $\mathbb{N}$-grading on E_R, where the n-th homogeneous component consists of homogeneous polynomials of total degree n and is denoted by E_n. Then E_n is a coalgebra and coacts on $V^{\otimes n}$ from the right; hence its dual E^*_n acts on $V^{\otimes n}$ from the left.

The Hecke algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ has generators $T_i, 1 \le i \le n-1$, subject to the relations: $T_iT_j = T_jT_i, |i-j| \ge 2; T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, 1 \le i \le n-2; T_i^2 = (q-1)T_i + q$. There is a k-basis in $\mathcal{H}_n$ indexed by permutations of n elements: $T_w, w \in \mathfrak{S}_n$ ($\mathfrak{S}_n$ is the permutation group), in such a way that $T_{(i,i+1)} = T_i$ and $T_wT_v = T_{wv}$ if the length of wv is equal to the sum of the length of w and the length of v. If q is not a root of unity of degree greater than 1, $\mathcal{H}_n$ is a semisimple algebra. For more details, the reader is referred to [2, 3].

The Hecke symmetry R induces an action of the Hecke algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ on $V^{\otimes n}$, $T_i \longmapsto R_i = \mathrm{id}^{i-1} \otimes R \otimes \mathrm{id}^{n-i-1}$ that commutes with the coaction of E_R. The action of T_w will be denoted by R_w. We have the following "Double centralizer theorem" [6, Thm. 2.1].

1.1. The algebras $\rho_n(\mathcal{H}_n)$ and E^*_n are centralizers of each other in $\mathrm{End}_k(V^{\otimes n})$.

Consequently, simple E^*_n-modules (and hence simple E_n-comodules) can be given as the image of primitive idempotents of $\mathcal{H}_n$, and conjugate idempotents determine isomorphic (co)modules. Since conjugate classes of primitive idempotents of $\mathcal{H}_n$ are indexed by partitions of n, simple subcomodules of $V^{\otimes n}$ are indexed by a subset of partitions of n. Thus E is cosemisimple, and its simple comodules are indexed by a subset of partitions.

Let I_λ denote the simple comodule corresponding to the partition λ. Then I_λ and I_μ can be realized as the images of two primitive idempotents $e_\lambda \in \mathcal{H}_r$ and $e_\mu \in \mathcal{H}_s$. Thus $I_\lambda \otimes I_\mu$ is the image of a (not necessarily primitive) idempotent in $\mathcal{H}_{r+s}$. This idempotent decomposes into an orthogonal sum of primitive idempotents, which yields a decomposition of I_λ and I_μ into a direct sum of simple subcomodules. Taking into account that conjugate idempotents define isomorphic comodules, we have [7]

$$I_\lambda \otimes I_\mu \cong \bigoplus_\gamma I_\gamma{}^{\oplus c^\gamma_{\lambda\mu}} \tag{2}$$

where the $c^\gamma_{\lambda\mu}$ are the Littlewood-Richardson coefficients describing the multiplicity of the Schur function s_γ in the product of two other Schur functions s_λ and s_μ (cf. [15]).

Example (Quantum symmetrizers)**.** The primitive idempotent

$$X_n := \frac{1}{[n]_q} \sum_{w \in \mathfrak{S}_n} R_w$$

determines a simple comodule S_n called the n-th quantum symmetric tensor power, and the primitive idempotent

$$Y_n := \frac{1}{[n]_{1/q}} \sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} R_w$$

determines a simple comodule $\wedge_n$ called the n-th quantum anti-symmetric tensor power. Notice that $\mathsf{S}_n = I_{(n)}$ and $\wedge_n = I_{(1^n)}$.

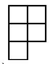
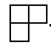
Let us briefly recall here the Littlewood-Richardson algorithm for computing the coefficients $c^\gamma_{\lambda\mu}$ [15]. Let γ and λ be partitions with $\gamma_i \geq \lambda_i$ for all i. We define the skew diagram $[\gamma\backslash\lambda] := \{(i,j) : (i,j) \in [\gamma], \lambda_i < j \leq \gamma_i\}$. The i-th row of the diagram consists of nodes (i,j) with fixed i, and the j-th column consists of nodes (i,j) with fixed j.

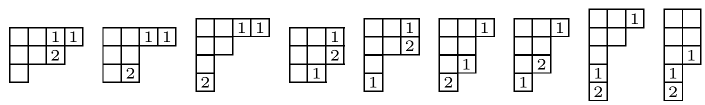
Let μ be a partition. A sequence of positive integers is said to have type μ if each i occurs μ_i times. Such a sequence is said to be "good" if for any term $i > 1$, the number of previous $i-1$ in the sequence is strictly greater than the number of previous i. For example, the good sequences of type $\mu = (2,1)$ are $112, 121$.

The coefficient $c^\gamma_{\lambda\mu}$, where λ is a partition of r, μ is a partition of s and γ is a partition of $r+s$, can be computed as follows:

(i) if $\lambda_i > \gamma_i$ for some i, then $c^\gamma_{\lambda\mu} = 0$;

(ii) if $\lambda_i \leq \gamma_i$ for every i, then $c^\gamma_{\lambda\mu}$ is the number of ways of replacing the nodes (i,j) of $[\gamma \setminus \lambda]$ by integers, such that
- each k occurs μ_k times;
- the numbers are non-decreasing along rows and strictly increasing down columns;
- when reading from right to left in successive rows, we have a good sequence of type μ.

Example. Let $[\lambda] =$  and $[\mu] =$ . There are two good sequences of type $\mu = (2,1) : (112), (121)$. We have the following possibilities for γ for which $c^{\gamma}_{\lambda\mu} \neq 0$:



which means

$$I_{(2,2,1)} \otimes I_{(2,1)} = I_{(4,3,1)} \oplus I_{(4,2^2)} \oplus I_{(4,2,1^2)} \oplus I_{(3^2,2)} \oplus I_{(3^2,1^2)} \oplus I_{(3,2^2,1)}{}^{\oplus 2} \oplus I_{(3,2,1^3)} \oplus I_{(2^3,1^2)}.$$

Note however that not every partition defines a simple comodule, as some of them may give zero-modules. To have more precise information on the simple comodules of E_R, we need the notion of birank of R. Consider the following formal series:

$$P_{\wedge}(t) := \sum_{i=0}^{\infty} \dim \wedge_i t^i.$$

We have the following theorem [7, Thm. 3.5]:

1.2. $P_{\wedge}(t)$ is a rational function having negative roots and positive poles:

$$P_{\wedge}(t) = \frac{\prod_{i=1}^{r}(1 + x_i t)}{\prod_{j=1}^{s}(1 - y_j t)}, \quad x_i, y_j > 0.$$

The pair (r, s) is called the *birank* of the Hecke symmetry R. A partition λ determines a non-zero simple E_R-comodule if and only if $\lambda_{r+1} \leq s$.

The Hecke symmetry R is called *even* of rank r if it has birank $(r, 0)$, i.e., if the series $P_{\wedge}$ is a polynomial of degree r. The Hecke symmetry R is called *odd* of rank s if it has birank $(0, s)$, i.e., if ${P_{\wedge}}^{-1}$ is a polynomial of degree s. There is generally no relationship between the dimension of V and the (bi)rank of R (see examples below).

Examples. The following are examples of Hecke symmetries that are known so far.

- The solutions of the Yang-Baxter equation of series A, due to Drinfel'd and Jimbo [11], provide an example of even Hecke symmetries. The associated quantum groups are called standard deformations of $\mathrm{GL}(n)$.
- Cremmer and Gervais [4] found another series of solutions that are also even Hecke symmetries.
- Hecke sums of odd and even Hecke symmetries [5, 16] are examples of non-even, non-odd Hecke symmetries [14].
- Takeuchi and Tambara found a Hecke symmetry that is neither even nor a Hecke sum of an odd and an even Hecke symmetry [18].
- Even Hecke symmetries of rank 2 were classified by Gurevich [5]. He also shows that on each vector space of dimension ≥ 2, there exists an even Hecke symmetry of rank 2.
- Hecke symmetries of birank $(1, 1)$ were classified by the author [9].

The quantum group of type A is defined to be the "spectrum" of the subsequently defined Hopf algebra. Let $T = (t_i^j)$ be a $d \times d$ matrix of new variables. The Hopf algebra associated to R is a factor algebra of the free non-commutative algebra over entries of Z and T:

$$H_R := T\left\langle \{z^i_j, t^i_j\}_{1\le i,j\le d}\right\rangle \big/ \left(z^i_m z^j_n R^{mn}_{kl} = R^{ij}_{pq} z^p_k z^q_l, t^i_p z^q_j = z^i_p t^p_j = \delta^i_j\right). \tag{3}$$

H_R is a Hopf algebra, the antipode is given by $S(z^i_j) = t^i_j$, and the coquasitriangular structure on E_R can be extended to H_R thanks to the closedness of R: $r(z^i_j, t^k_l) = P^{ki}_{jl}, r(t^i_j, z^i_l) = R^{-1}{}^{ik}_{lj}$ [8, Thm. 2.1.1].

If the Hecke symmetry R is even of rank r, H_R is cosemisimple and its simple comodules can be parameterized by sequences of r non-decreasing integers [8, Thm. 3.2.1]. A similar statement holds for odd Hecke symmetries.

If the Hecke symmetry R is neither even nor odd, the structure of H_R-comodules is more complicated than the structure of E_R-comodules. In particular, the category H_R-comod is not semisimple. We have, however, the following result [8, Thm. 2.3.5].

1.3. The natural map $E_R \longrightarrow H_R$ is injective. Consequently, every simple E_R-comodule is a simple H_R-comodule.

Among H_R-comodules that are not E-comodules, the super determinant plays an important role. The well-known tool for defining the quantum super determinant serves the Koszul complex (of second type) introduced by Manin [17]. This is a (bi-)complex, whose (k, l) term is $\wedge_k \otimes \mathsf{S}_l{}^*$. The differential is induced from the dual basis map. The homology group of this complex is an H_R-comodule; if it is one dimensional over k, it defines a group-like element in H_R called the homological determinant or quantum super determinant or, in some cases, the quantum Berezinian.

2. The Koszul complex

We begin with the description of the Koszul complex. We first recall the dual comodule of a tensor product of two comodules. For two (finite-dimensional) comodules V, W, the dual to $V \otimes W$ is isomorphic to $W^* \otimes V^*$, with the pairing given by $(\varphi\otimes\psi)(v\otimes w) := \varphi(w)\psi(v)$, $\varphi \in W^*, \psi \in V^*, v \in V, w \in W$. The dual to longer tensor products is defined in a similar way.

Fix a basis $x_1, x_2, \dots, x_d$ of V and let $\xi^1, \xi^2, \dots, \xi^d$ be the dual basis in V^*. Let ev_V, be the evaluation map $\mathrm{ev}_V(\varphi \otimes v) = \varphi(v)$ and db_V be the dual basis map defined as follows: $\mathrm{db} : k \longrightarrow V \otimes V^*$, $\mathrm{db}(1) = \sum_i x_i \otimes \xi^i$. These maps clearly do not depend on the choice of basis and are maps of H_R-comodules. The term $K^{k,l}$ of the Koszul complex associated to R is $\wedge_k \otimes \mathsf{S}_l{}^*$, and the differential $d_{k,l}$ is given by:

$$\wedge_k \otimes \mathsf{S}_l{}^* \to V^{\otimes l} \otimes V^{*\otimes l} \overset{\mathrm{id}\otimes\mathrm{db}_V\otimes\mathrm{id}}{\longrightarrow} V^{\otimes k+1} \otimes V^{*\otimes l+1} \overset{Y_{k+1}\otimes X^*_{l+1}}{\longrightarrow} \wedge_{k+1} \otimes \mathsf{S}_{l+1}{}^*,$$

where X_l, Y_k are the q-symmetrizer operators introduced in the previous section. Thus we have in fact a collection of diagonal subcomplexes, each of which contains the terms $K^{k,l}$ with $k - l$ equal to a fixed number. One defines another differential d' as follows:

$$\wedge_k \otimes \mathsf{S}_l{}^* \to V^{\otimes l} \otimes V^{*\otimes l} \overset{\mathrm{id}\otimes\mathrm{ev}_V\tau_{V\otimes V^*}\otimes\mathrm{id}}{\longrightarrow} V^{\otimes k-1} \otimes V^{*\otimes l-1} \overset{Y_{k-1}\otimes X^*_{l-1}}{\longrightarrow} \wedge_{k-1} \otimes \mathsf{S}_{l-1}{}^*,$$

where τ_{V,V^*} denotes the braiding on $V \otimes V^*$ induced from the coquasitriangular structure on H_R, its matrix is given by P, the inverse to the half-adjoint of R.

Since all vector spaces are H_R-comodules and all maps are H_R-comodule maps, we have in fact complexes in H_R-comod.

The differentials d and d' satisfy [5]

$$(qdd' + d'd)\,|_{K^{k,l}} = q^k(\mathrm{rank}_q R + [l-k]_q)\mathrm{id},$$

where $\mathrm{rank}_q R := P_{ij}^{ij}$, and P is given in (1). Hence, if $\mathrm{rank}_q R \neq -[l-k]_q$, the cohomology group at the term (k,l) vanishes.

Theorem 1. *Let R be a Hecke symmetry of birank (r,s). Then*

(i) $\mathrm{rank}_q R = -[s-r]_q$*;*
(ii) *the simple comodule I_λ is injective and projective in the category of H_R-comodules if and only if $\lambda_r \geq s$;*
(iii) *the homology of the Koszul complex at the term (r,s) is non-vanishing.*

Proof. Since R has birank (r,s), the simple E-comodule $I_\lambda \neq 0$ iff $\lambda_{r+1} \leq s$. Using this fact and the Littlewood-Richardson formula, we can easily show that (Hom means Hom^{H_R}):

$$\mathrm{Hom}(I_{((s+1)^r)}, I_{(s^{r+1})} \otimes \wedge_r \otimes \mathsf{S}_s{}^*) \cong \mathrm{Hom}(I_{((s+1)^r)} \otimes \mathsf{S}_s, I_{(s^{r+1})} \otimes \wedge_r) = k,$$
$$\mathrm{Hom}(I_{((s+1)^r)}, I_{(s^{r+1})} \otimes \wedge_{r-1} \otimes \mathsf{S}_{s-1}{}^*) \cong \mathrm{Hom}(I_{((s+1)^r)} \otimes \mathsf{S}_{s-1}, I_{(s^{r+1})} \otimes \wedge_{r-1}) = 0,$$
$$\mathrm{Hom}(I_{((s+1)^r)}, I_{(s^{r+1})} \otimes \wedge_{r+1} \otimes \mathsf{S}_{s+1}{}^*) \cong \mathrm{Hom}(I_{((s+1)^r)} \otimes \mathsf{S}_{s+1}, I_{(s^{r+1})} \otimes \wedge_{r+1}) = 0.$$

As a consequence, $I_{(s^{r+1}))} \otimes \wedge_r \otimes \mathsf{S}_s{}^*$ contains $I_{((s+1)^r)}$ as a subcomodule while the comodules $I_{(s^{r+1})} \otimes \wedge_{r-1} \otimes \mathsf{S}_{s-1}{}^*$, $I_{(s^{r+1})} \otimes \wedge_{r+1} \otimes \mathsf{S}_{s+1}{}^*$ do not.

Assume that $\mathrm{rank}_q R \neq -[s-r]_q$. Then the complex is exact at $K^{r,s}$ and $dd' + d'd = q^r(\mathrm{rank}_q R + [s-r]_q)\mathrm{id} \neq 0$. On the other hand, since $I_{(s+1)^r}$ is a submodule of $I_{(s^{r+1})} \otimes \wedge_{r+1} \otimes \mathsf{S}_{s+1}{}^*$, the restriction of $\mathrm{id}_{I_{((s+1)^r)}} \otimes d^{r,s}$ to it should be zero. Analogously, the restriction of $\mathrm{id}_{I_{((s+1)^r)}} \otimes d^{m,n}$ to $I_{((s+1)^r)}$ is zero. Thus, the restriction of $dd' + d'd$ on $I_{(s+1)^r}$ is zero, a contradiction. Therefore, $\mathrm{rank}_q R = -[s-r]_q$.

According to [10, Thm. 3.2] if $\mathrm{rank}_q R = -[s-r]_q$, then H possesses a non-zero integral (i.e. an H-comodule homomorphism $H \longrightarrow k$, where H coacts on itself by the coproduct and on k by the unit map). Then, according to [10, Prop. 5.1] and to [9, Thm. 3.1], I_λ is injective and projective in H-comod iff $\lambda_r \geq s$. Thus, $I_{((s+1)^r)}$ is projective and injective. Therefore, if $I_{((s+1)^r)}$ is a subquotient of a comodule, it is a direct summand; hence it cannot be a subquotient of $I_{(s^{r+1})} \otimes \wedge_{m-1} \otimes \mathsf{S}_{n-1}{}^*$, and in particular, it cannot be a subcomodule of $I_{(s^{r+1})} \otimes \mathrm{Im}d^{r-1,s-1}$. Consequently,

$$I_{(s^{r+1})} \otimes \mathrm{Im}d^{r-1,s-1} \neq I_{(s^{r+1})} \otimes \mathrm{Ker}d^{m,n}.$$

Thus, the sequence

$$\cdots \to I_{(s^{r+1})} \otimes \wedge_{r-1} \otimes \mathsf{S}_{s-1}{}^* \to I_{(s^{r+1}))} \otimes \wedge_r \otimes \mathsf{S}_s{}^* \to I_{(s^{r+1})} \otimes \wedge_{r+1} \otimes \mathsf{S}_{r+1}{}^* \to \cdots,$$

which is obtained by tensoring $K^{\cdot\cdot}$ with $I_{(s^{r+1})}$, is not exact at the term (r,s), whence neither is $K^{\cdot\cdot}$. □

3. The case $q = 1$

Assume in this section that $q = 1$; thus, R^2 is the identity map and H-comod is a tensor category (i.e., symmetric rigid monoidal). By a theorem of Deligne [1], there exists a faithful and exact tensor (i.e. symmetric monoidal) functor $\mathcal{F}$ from H-comod to the category of vector superspaces. Under this functor, V is mapped to a certain vector superspace $\overline{V}$ and R is mapped to the supersymmetry on $\overline{V}\otimes\overline{V}$, denoted by T.

We can therefore reconstruct a super bialgebra $\overline{E}$ and a Hopf super algebra $\overline{H}$ from $\overline{V}$ and T. We will show that this Hopf superalgebra is isomorphic to the function algebra over the general linear supergroups $GL(r|s)$, where (r,s) is the birank of R, or, in other words, the super dimension of $\overline{V}$ is $(r|s)$. Indeed, $\overline{E}$ is the function algebra on $\mathrm{End}(\overline{V})$ and the images of I_λ under the embedding $\mathcal{F}$ are simple $\overline{E}$-comodules. Since $\mathcal{F}$ is faithful and exact and since $I_\lambda \neq 0 \Leftrightarrow \lambda_{r+1} \leq s$, we conclude that $\overline{E}$ is isomorphic to the function algebra on $M(r|s)$. Hence $\overline{H}$ is isomorphic to the function algebra on $GL(r|s)$, by virtue of 1.3.

Let $\overline{K}^{\cdot\cdot}$ denote the image of the complex $K^{\cdot\cdot}$. Then the homology of $\overline{K}^{\cdot\cdot}$ is concentrated at the term (r,s), and is one-dimensional; it defines the super determinant. As a consequence, the homology of $K^{\cdot\cdot}$ is also concentrated at the term (r,s), for $\mathcal{F}$ is faithful and exact. Let D denote the homology of $K^{\cdot\cdot}$. Then $\overline{D}$, the image of D under $\mathcal{F}$, is one-dimensional and hence invertible; consequently,

$$\mathcal{F}(D^*\otimes D)\cong\mathcal{F}(D^*)\otimes\mathcal{F}(D)\cong\overline{D}^*\otimes\overline{D}\cong k,$$

where the last isomorphism is given by the evaluation morphism, that is, the image of $\mathrm{ev}_D : D^*\otimes D \longrightarrow k$ under $\mathcal{F}$. Since $\mathcal{F}$ is faithful and exact, we conclude that $D^*\otimes D\cong k$, that is D is invertible, hence one-dimensional. Thus, we have proved:

Theorem 2. *Let R be a vector symmetry of birank (r,s). Then the associated Koszul complex is exact everywhere except at the term (r,s) where it has a one-dimensional homology group, which determines a group-like element called the homological determinant.*

4. The case q generic

Using the result of the previous section we show in this section that given a Hecke symmetry of birank (r,s) that depends algebraically on q, then, for a dense set of values q, the associated Koszul complex is exact everywhere except at the term (r,s), where it has a one-dimensional homology group and thus determines a group-like element in H_R, called the homological determinant. In this section k will be assumed to be the field $\mathbb{C}$ of complex numbers.

Thus let $R = R_q$ be a Hecke symmetry depending on a parameter $q\in\mathbb{C}$. We first observe that the dimension of $\wedge_{q,k}$ does not depend on q, so far as q is not a root of unity. Indeed, $\wedge_{q,k}$ is the image of a projection, and its dimension can be given as the trace of a matrix that depends algebraically on q. Since $\mathbb{C}$ without the set of roots of unity is still connected, we conclude that this trace, being always integral, must be a constant. The same happens with $\mathsf{S}_{q,l}$. Thus, the terms of $K^{\cdot\cdot}$ have dimension not depending on q.

On the other hand, observe that the rank of the operator $d_q^{k,l}$, for almost any q (that is, except for a finite number of values of q) is larger than the rank of $d_1^{k,l}$ and for the kernel of $d_q^{k,l}$ we have the reversed inequality. Consequently, the dimension over k of the homology group $H(K_q^{k,l})$ for almost any q is less than or equal to the dimension of $H(K_q^{k,l})$. According to Theorems 1 and 2, we conclude that for an algebraically dense set of values of q, $H(K_q^{k,l}) = 0$, for all $(k,l)\neq(r,s)$ and $H(K_q^{r,s})$ is one-dimensional.

Theorem 3. *Let $R = R_q$ be a Hecke symmetry over $\mathbb{C}$, depending algebraically on q. Then there is an algebraically dense set of values of q for which the homology of the Koszul complex is one-dimensional and concentrated at the term (r, s), where (r, s) is the birank of R.*

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