

VANISHING AND BASES FOR COHOMOLOGY  
OF PARTIALLY TRIVIAL LOCAL SYSTEMS  
ON HYPERPLANE ARRANGEMENTS

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ABSTRACT. In this paper we prove a vanishing theorem and construct bases for the cohomology of partially trivial local systems on complements of hyperplane arrangements. As a result, we obtain a non-resonance condition for partially trivial local systems.

1. INTRODUCTION

Let  $\mathcal{A}$  be an arrangement of hyperplanes in  $\mathbb{C}^\ell$ , with complement  $M = M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$ . For a complex weight system  $\lambda = (\lambda_H)_{H \in \mathcal{A}}$ , we have a rank one local system  $\mathcal{L}_\lambda$  on  $M$  whose monodromy around the hyperplane  $H$  is  $\exp(-2\pi\sqrt{-1}\lambda_H)$ . For a hyperplane  $H$  defined by a degree one polynomial  $\alpha_H$ , let  $\omega_H = d \log \alpha_H = d\alpha_H/\alpha_H$ . Denote the sheaves of holomorphic  $p$ -forms on  $M$  by  $\Omega_M^p$  with  $\Omega_M^0 = \mathcal{O}_M$  and denote  $\Omega^p(M) = \Gamma(M, \Omega_M^p)$ . Define

$$\omega_\lambda = \sum_{H \in \mathcal{A}} \lambda_H \omega_H, \quad \nabla_\lambda = d + \omega_\lambda \wedge$$

where  $d$  is the ordinary differential. The twisted de Rham theorem

$$H^p(M, \mathcal{L}_\lambda) \simeq H^p(\Omega(M), \nabla_\lambda)$$

is well known ([De]). In general, we know  $H^p(M, \mathcal{L}_\lambda) = 0$  for  $p > \ell$ . Let  $k$  be a weight of  $\mathcal{A}$  such that all  $k_H$ 's are integers. The multiplication by the non-zero holomorphic function  $\prod_{H \in \mathcal{A}} \alpha_H^{-k_H}$  on  $M(\mathcal{A})$  induces an isomorphism of twisted de Rham complexes:  $(\Omega(M), \nabla_\lambda) \simeq (\Omega(M), \nabla_{\lambda+k})$ . Namely, weights  $\lambda$  and  $\lambda + k$  give rise to the same local system (see [OT2, 2.1]). Since the local system  $\mathcal{L}_k$  is equivalent to the constant sheaf  $\mathbb{C}$ , the local system cohomology is the usual cohomology  $H^*(M, \mathbb{C})$ . A weight  $k$  is said to be *trivial*.

The local system cohomology  $H^*(M, \mathcal{L}_\lambda)$ , or the twisted de Rham cohomology, is an important subject in the Aomoto-Gelfand multivariable theory of hypergeometric functions [AK, Gel, OT2] and appears in various areas: the representation theory of Lie algebras and quantum groups [Va]; the Knizhnik-Zamolodchikov differential equation in conformal field theory [EFK]. In order to compute the local

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system cohomology, there are several methods: algebraic geometry, stratified Morse theory, combinatorics and so on.

We recall the vanishing theorem [ESV, STV, Yu1]. We use the notation of [OT, OT2]. Let  $\mathbb{P}^\ell$  be the complex projective space, which is a compactification of  $\mathbb{C}^\ell$  with the infinite hyperplane  $\bar{H}_\infty$ . The projective closure  $\mathcal{A}_\infty$  of  $\mathcal{A}$  is defined by  $\mathcal{A}_\infty = \{\bar{H} \mid H \in \mathcal{A}\} \cup \{\bar{H}_\infty\}$ , where  $\bar{H}$  is the projective closure of  $H$ . Let  $\lambda_\infty = -\sum_{H \in \mathcal{A}} \lambda_H$  be the weight of  $\bar{H}_\infty$ .

The *intersection poset*  $L = L(\mathcal{A})$  of an (affine or projective) arrangement  $\mathcal{A}$  is the set of nonempty intersections of elements of  $\mathcal{A}$  with a *partial order* defined by reverse inclusion. We call an element of  $L$  an *edge* of  $\mathcal{A}$ . The *rank*  $r(X)$  of  $X \in L$  is defined by  $r(X) = \text{codim}(X)$ . The *rank*  $r(\mathcal{A})$  of  $\mathcal{A}$  is the maximal rank of any edges of  $\mathcal{A}$ , which is the maximal number of linearly independent hyperplanes of  $\mathcal{A}$  with nonempty intersection. For  $X \in L(\mathcal{A})$ , define

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}, \quad \lambda_X = \sum_{H \in \mathcal{A}, X \subset H} \lambda_H.$$

Let  $\mathcal{C}$  be a central arrangement with center  $\bigcap_{H \in \mathcal{C}} H \neq \emptyset$ . We call  $\mathcal{C}$  *decomposable* if there exist nonempty subarrangements  $\mathcal{C}_1$  and  $\mathcal{C}_2$  so that  $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$  is a disjoint union and after a linear coordinate change the defining polynomials for  $\mathcal{C}_1$  and  $\mathcal{C}_2$  have no common variables. An edge  $X \in L(\mathcal{A})$  is called *dense* in  $\mathcal{A}$  if the central arrangement  $\mathcal{A}_X$  is not decomposable. Denote the set of all dense edges in  $\mathcal{A}$  by  $D(\mathcal{A})$ . Note that all hyperplanes of  $\mathcal{A}$  are dense edges in  $\mathcal{A}$ .

**Theorem 1.1** ([ESV, STV, Yu1]). *Let  $\mathcal{A}$  be an affine arrangement of hyperplanes. If a weight  $\lambda$  of  $\mathcal{A}$  satisfies the condition:*

$$\text{(Mon)} \quad \lambda_X \notin \mathbb{Z}_{\geq 0} \quad \text{for every } X \in D(\mathcal{A}_\infty),$$

*then we have*

$$H^p(M(\mathcal{A}), \mathcal{L}_\lambda) = 0, \quad \text{for } p \neq r(\mathcal{A}).$$

This theorem says that a weight  $\lambda$  satisfying (Mon) is non-resonant in the sense of [OT2, 6.5]. In general, non-resonant weights give rise to a local system with vanishing cohomology in all dimensions except the top dimension. In the case of non-resonant weights, much is known (see [OT2]).

For example, a basis of  $H^*(M, \mathcal{L}_\lambda)$  is constructed by [FT] as follows. Let  $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$  be an arrangement with a linear order:  $H_i \prec H_j$  if  $i < j$ . An *independent set* is a central subarrangement whose rank is equal to cardinality. A *dependent set* is a central subarrangement whose rank is less than cardinality. A *base* (or *frame*) is an inclusion-maximal independent set. Every base has cardinality  $r = r(\mathcal{A})$ . A *circuit* is an inclusion-minimal dependent set. A *broken circuit* is a subset  $S$  for which there exists  $H \prec \min(S)$  such that  $\{H\} \cup S$  is a circuit. An **NBC set** is a non-empty subset of  $\mathcal{A}$  that has nonempty intersection and contains no broken circuits. An **NBC-base** is a base which is an **NBC set**. The collection of **NBC sets** is a simplicial complex, called the **NBC complex** of  $\mathcal{A}$ . A base  $B$  is called a  $\beta$ **NBC-base** if  $B$  is an **NBC-base** and, for every  $H \in B$ , there exists  $H' \prec H$  in  $\mathcal{A}$  such that  $(B \setminus \{H\}) \cup \{H'\}$  is a base. Denote by  $\beta\mathbf{NBC}(\mathcal{A})$  the set of  $\beta$ **NBC-bases**. Let  $\lambda$  be a weight of  $\mathcal{A}$ . For  $X \in L(\mathcal{A})$  define

$$\omega_\lambda(X) = \sum_{H \in \mathcal{A}_X} \lambda_H \omega_H.$$

For a base  $B = \{H_{i_1}, H_{i_2}, \dots, H_{i_r}\}$  with  $i_1 < i_2 < \dots < i_r$ , define a flag  $\xi(B) = (X_1 \subset X_2 \subset \dots \subset X_r)$  where  $X_p = \bigcap_{k=p}^r H_{i_k}$ , and define a logarithmic form

$$\zeta(B) = \omega_\lambda(X_1) \wedge \dots \wedge \omega_\lambda(X_r).$$

**Theorem 1.2** ([FT]). *Let  $\mathcal{A}$  be an affine arrangement of hyperplanes and let  $\lambda$  be its weight satisfying the condition (Mon). Then the set*

$$\{\zeta(B) \mid B \in \beta\mathbf{NBC}(\mathcal{A})\}$$

*is a basis of  $H^{r(\mathcal{A})}(M(\mathcal{A}), \nabla_\lambda)$ .*

We can find many works and applications on the subject of calculations of the local system cohomology for any weights or resonant weights. The upper and lower bound of dimensions of the local system cohomology were studied in [Co1, CO1, LY]. The characteristic variety is the subvariety of the torus consisting of local systems for which the cohomology does not vanish [Li, CS]. The characteristic varieties are deeply related to the cohomology support loci [Ar] and the resonance varieties [Fa, Yu2]. The Gauss-Manin connection on the local system cohomology is an important subject [Te, CO2] (see [OT2]).

The condition (Mon) demands that  $\lambda_H \notin \mathbb{Z}_{\geq 0}$  for  $H \in \mathcal{A}$ . In this paper, we remove this and then we treat a partially trivial weight  $\lambda$  ( $\lambda_H \in \mathbb{Z}$  for some  $H \in \mathcal{A}$ ). For some partially trivial weight  $\lambda$  which does not satisfy (Mon), we obtain a vanishing theorem and a basis for  $H^*(M(\mathcal{A}), \mathcal{L}_\lambda)$ , by using the above results and a long exact sequence for a triple. As a result, we obtain a condition on the weight  $\lambda$  which is an extension of (Mon) and insures that the local system  $\mathcal{L}_\lambda$  is non-resonant.

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## 2. RESULTS

Let  $\mathcal{P}$  be an arrangement of hyperplanes in the complex projective space  $\mathbb{P}^\ell$ . For  $H_0 \in \mathcal{P}$ , let  $\mathbf{af}_{H_0}(\mathcal{P})$  denote the affine arrangement  $\{H \setminus H_0 : H \in \mathcal{P}\}$  in  $\mathbb{C}^\ell = \mathbb{P}^\ell \setminus H_0$ . The rank of  $\mathbf{af}_{H_0}(\mathcal{P})$ , denoted by  $r_{\mathbf{af}}(\mathcal{P})$ , does not depend on the choice of  $H_0$ . Note that  $r_{\mathbf{af}}(\mathcal{A}_\infty) = r(\mathcal{A})$  for an affine arrangement  $\mathcal{A}$ .

Let  $\mathcal{A}$  be an arrangement and let  $\lambda$  be a weight of  $\mathcal{A}$ . By adding a trivial weight if necessary, we can assume that  $\lambda_H \notin \mathbb{Z}_{\geq 0}$  or  $\lambda_H = 0$ , that is,  $\lambda_H \notin \mathbb{Z}_{> 0}$ , for  $H \in \mathcal{A}$ . Otherwise, it is possible to assume that  $\lambda_H \notin \mathbb{Z}$  or  $\lambda_H = 0$  for  $H \in \mathcal{A}$ . We define a decomposition  $\mathcal{A} = \mathcal{G}(\mathcal{A}, \lambda) \cup \mathcal{N}(\mathcal{A}, \lambda)$  of  $\mathcal{A}$  by

$$\mathcal{G} = \mathcal{G}(\mathcal{A}, \lambda) = \{H \in \mathcal{A} \mid \lambda_H \notin \mathbb{Z}_{\geq 0}\}$$

and

$$\mathcal{N} = \mathcal{N}(\mathcal{A}, \lambda) = \{H \in \mathcal{A} \mid \lambda_H = 0\}.$$

A weight  $\lambda = \lambda(\mathcal{A}) = (\lambda_H)_{H \in \mathcal{A}}$  of  $\mathcal{A}$  induces a weight of a subarrangement  $\mathcal{B}$  of  $\mathcal{A}$ :  $\lambda(\mathcal{B}) = (\lambda_H)_{H \in \mathcal{B}}$ . Thus, the subarrangements  $\mathcal{G}$  and  $\mathcal{N}$  have weights  $\lambda(\mathcal{G})$  and  $\lambda(\mathcal{N})$  induced by  $\lambda(\mathcal{A})$ , respectively. Note that  $\lambda(\mathcal{N})$  is trivial.

The first result in this paper is a generalization of Theorem 1.1 for partially trivial local systems, as follows.

**Theorem 2.1.** *Let  $\mathcal{A}$  be an affine arrangement and let  $\lambda$  be its weight such that  $\lambda_H \notin \mathbb{Z}_{> 0}$  for  $H \in \mathcal{A}$ . If the weight  $\lambda$  satisfies the condition*

$$(\text{Mon}^*) \quad \lambda_X \notin \mathbb{Z}_{\geq 0} \quad \text{for every } X \in D(\mathcal{A}_\infty) \text{ with } (\mathcal{A}_\infty)_X \cap \mathcal{G}(\mathcal{A}_\infty, \lambda) \neq \emptyset,$$

then we have

$$H^k(M(\mathcal{A}), \mathcal{L}_\lambda) = 0, \quad \text{for } k < r,$$

where  $r = r_{\text{af}}(\mathcal{G}(\mathcal{A}_\infty, \lambda))$ .

*Remark 2.2.* If  $\lambda$  is trivial, then  $r = 0$ . If  $\lambda$  is non-trivial, then  $r > 0$  and  $H^0(M(\mathcal{A}), \mathcal{L}_\lambda) = 0$ . When  $\mathcal{A}_\infty = \mathcal{G}(\mathcal{A}_\infty, \lambda)$ , this is Theorem 1.1. When  $r = r(\mathcal{A}) = r_{\text{af}}(\mathcal{G}(\mathcal{A}_\infty, \lambda))$ , we have

$$H^k(M(\mathcal{A}), \mathcal{L}_\lambda) = 0, \quad \text{for } k \neq r.$$

Consequently, a weight  $\lambda$  of  $\mathcal{A}$  satisfying (Mon\*) and  $r(\mathcal{A}) = r_{\text{af}}(\mathcal{G}(\mathcal{A}_\infty, \lambda))$ , is non-resonant in the sense of [OT2, 6.5].

Next we shall construct a basis for partially trivial local systems. Suppose  $\lambda_\infty \notin \mathbb{Z}$ . We modify the definition of  $\omega_\lambda(X)$  from section 1 as follows:

$$\omega_\lambda(X) = \sum_{H \in \mathcal{A}_X, \lambda_H \notin \mathbb{Z}_{\geq 0}} \lambda_H \omega_H + \sum_{H \in \mathcal{A}_X, \lambda_H \in \mathbb{Z}_{\geq 0}} \omega_H.$$

**Theorem 2.3.** *Under the assumption in Theorem 2.1, we moreover assume that  $\lambda_\infty \notin \mathbb{Z}$  and  $r = r(\mathcal{A}) = r_{\text{af}}(\mathcal{G}(\mathcal{A}_\infty, \lambda))$ . Then the set*

$$\{\zeta(B) \mid B \in \beta\text{NBC}(\mathcal{A})\}$$

*is a basis of  $H^r(M(\mathcal{A}), \mathcal{L}_\lambda)$ .*

### 3. PROOFS

A triple  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')_H$  of arrangements with distinguished hyperplane  $H \in \mathcal{A}$  is defined by  $\mathcal{A}' = \mathcal{A} \setminus \{H\}$  and  $\mathcal{A}'' = \mathcal{A}^H$ . A weight  $\lambda$  of  $\mathcal{A}$  induces a weight of a subarrangement  $\mathcal{A}'$  and a weight of  $\mathcal{A}''$  defined by

$$\lambda_{H''} = \sum_{I \in \mathcal{A}, I \supset H'', I \neq H} \lambda_I$$

for  $H'' \in \mathcal{A}''$ .

Let  $H$  be a hyperplane in  $\mathbb{C}^\ell$ . A tubular neighborhood  $T_H$  of  $H$  in  $\mathbb{C}^\ell$  is a trivial bundle over  $H$  with fiber  $F = \mathbb{C}$ . The complement of the zero section is denoted by  $T_H^* = T_H \setminus H$ . Since  $H$  is contractible, for any fiber  $F_0$  of  $T_H^*$ , we have

$$H^1(F_0) = H^1(T_H^*) = \mathbb{C}\{\omega_H\}.$$

**Theorem 3.1** ([Co2, Ka]). *Let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')_H$  be a triple of arrangements with distinguished hyperplane  $H \in \mathcal{A}$ . Let  $\lambda$  be a weight of  $\mathcal{A}$  with  $\lambda_H = 0$ . Then we have a long exact sequence:*

$$\begin{aligned} \dots \rightarrow H^k(M(\mathcal{A}'), \mathcal{L}_\lambda) &\rightarrow H^k(M(\mathcal{A}), \mathcal{L}_\lambda) \rightarrow H^{k-1}(M(\mathcal{A}''), \mathcal{L}_\lambda) \otimes H^1(T_H^*) \\ &\rightarrow H^{k+1}(M(\mathcal{A}'), \mathcal{L}_\lambda) \rightarrow \dots \end{aligned}$$

Let  $(\mathcal{C}, \mathcal{C}', \mathcal{C}'')_H$  be a triple of central arrangements with distinguished hyperplane  $H$ . We call  $H$  a *separator* if the center  $T(\mathcal{C})$  of  $\mathcal{C}$  does not belong to  $L(\mathcal{C}')$  (see [OT, Definition 2.58]). In other words, we can set  $\mathcal{C} = \mathcal{C}' \times \{H\}$ .

**Lemma 3.2.** *Let  $(\mathcal{C}, \mathcal{C}', \mathcal{C}'')_H$  be a triple of central arrangements with distinguished hyperplane  $H$ . If  $H$  is a separator, then  $\mathcal{C}$  is decomposable and we have*

$$\mathcal{C}' \text{ is decomposable} \iff \mathcal{C}'' \text{ is decomposable.}$$

If  $H$  is not a separator, then we have

$$\mathcal{C} \text{ is decomposable} \iff \mathcal{C}' \text{ and } \mathcal{C}'' \text{ are decomposable.}$$

*Proof.* Assume that  $H$  is a separator. Clearly,  $\mathcal{C}$  is decomposable. Since  $\mathcal{C} = \mathcal{C}' \times \{H\}$ , we have  $\mathcal{C}'' = \mathcal{C}^H = \mathcal{C}'$  and thus the first statement.

Assume that  $H$  is not a separator. By [OT2, Lemma 3.2.4.], if  $\mathcal{C}'$  and  $\mathcal{C}''$  are decomposable, then  $\mathcal{C}$  is decomposable. Suppose that  $\mathcal{C}$  is decomposable with  $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$  and  $H \in \mathcal{C}_2$ . Since  $H$  is not a separator, we have  $\mathcal{C}_2 \setminus \{H\} \neq \emptyset$ . Then  $\mathcal{C}' = \mathcal{C}_1 \times (\mathcal{C}_2 \setminus \{H\})$  is decomposable and  $\mathcal{C}'' = \mathcal{C}^H = \mathcal{C}_1 \times \mathcal{C}_2^H$  is decomposable as well.  $\square$

*Proof of Theorem 2.1.* Let  $\lambda$  be a non-trivial weight of an affine arrangement  $\mathcal{A}$ . By changing the infinite hyperplane, if necessary, we can assume that  $\lambda_\infty \notin \mathbb{Z}$ . Since  $\mathcal{G}(\mathcal{A}_\infty, \lambda)$  contains the infinite hyperplane,  $\mathcal{N}(\mathcal{A}_\infty, \lambda)$  and  $\mathcal{N}(\mathcal{A}, \lambda)$  have the same cardinality  $s$ . We shall prove this theorem using induction on  $s$ . When  $s = 0$ , the condition (Mon\*) is equivalent to the condition (Mon), and then it is true by Theorem 1.1 ([ESV, STV]). Note that, when  $r(\mathcal{A}) = 1$ , the result holds. Write  $\mathcal{G} = \mathcal{G}(\mathcal{A}, \lambda)$ ,  $\mathcal{N} = \mathcal{N}(\mathcal{A}, \lambda)$  and  $r = r(\mathcal{G})$ .

Suppose that the statements are true for all weighted arrangements satisfying (Mon\*) and  $|\mathcal{N}| < s$ . Assume that the weight  $\lambda$  of  $\mathcal{A}$  satisfies (Mon\*) and  $s = |\mathcal{N}(\mathcal{A}, \lambda)|$ . Fixing a hyperplane  $H \in \mathcal{N}$ , we take the triple  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')_H$  with distinguished hyperplane  $H$ . Note that  $\lambda_H = 0$ . Write  $\mathcal{G}' = \mathcal{G}(\mathcal{A}', \lambda)$ ,  $\mathcal{N}' = \mathcal{N}(\mathcal{A}', \lambda)$ ,  $r' = r(\mathcal{G}')$  and  $s' = |\mathcal{N}(\mathcal{A}', \lambda)|$  for  $\mathcal{A}'$  and write  $\mathcal{G}'' = \mathcal{G}(\mathcal{A}'', \lambda)$ ,  $\mathcal{N}'' = \mathcal{N}(\mathcal{A}'', \lambda)$ ,  $r'' = r(\mathcal{G}'')$  and  $s'' = |\mathcal{N}(\mathcal{A}'', \lambda)|$  for  $\mathcal{A}''$ .

First, we shall show that induced weights  $\lambda$  of  $\mathcal{A}'$  and  $\mathcal{A}''$  satisfy the condition (Mon\*). By [OT2, Lemma 3.2.6] we have  $D(\mathcal{A}'_\infty) \subset D(\mathcal{A}_\infty)$ . Since  $\mathcal{G}(\mathcal{A}_\infty, \lambda) = \mathcal{G}(\mathcal{A}'_\infty, \lambda)$ , the weight  $\lambda$  of  $\mathcal{A}'$  satisfies (Mon\*). Let  $X'' \in D(\mathcal{A}''_\infty)$  be such that there exists  $H \in \mathcal{A}''_\infty$  containing  $X''$ , whose weight  $\lambda_H$  is not an integer. As  $X'' \in L(\mathcal{A}_\infty)$ , we denote by  $\mathcal{C}$  the central arrangement  $(\mathcal{A}_\infty)_{X''}$  with center  $X''$ . Since  $H \in \mathcal{C}$ , we take the triple  $(\mathcal{C}, \mathcal{C}', \mathcal{C}'')_H$  with distinguished hyperplane  $H$ , whose centers are  $(X, X', X'')$  respectively. Since  $\lambda_H = 0$ , we have

$$\lambda_X = \lambda_{X'} = \lambda_{X''} = \lambda_{\mathcal{C}} = \lambda_{\mathcal{C}'} = \lambda_{\mathcal{C}''}.$$

On the other hand,  $\mathcal{C}''$  is not decomposable because  $X'' \in D(\mathcal{A}''_\infty)$ . By Lemma 3.2, we obtain the following. If  $H$  is a separator, then  $\mathcal{C}'$  is not decomposable and hence  $X' \in D(\mathcal{A}'_\infty)$ . We get  $\lambda_{X''} = \lambda_{X'} \notin \mathbb{Z}_{\geq 0}$ . If  $H$  is not a separator, then  $\mathcal{C}$  is not decomposable and hence  $X \in D(\mathcal{A}_\infty)$ . We get  $\lambda_{X''} = \lambda_X \notin \mathbb{Z}_{\geq 0}$ . Therefore, the weight  $\lambda$  of  $\mathcal{A}''$  satisfies (Mon\*).

Secondly, we study  $\mathcal{G}'$  and  $\mathcal{G}''$ . Since  $H \in \mathcal{N}$ , it is clear that  $\mathcal{G}' = \mathcal{G}$  and then  $r' = r$ . We shall show that  $\mathcal{G}''$  is the restriction  $\mathcal{G}^H$  of  $\mathcal{G}$  to  $H$ . It is clear that  $\mathcal{G}'' \subset \mathcal{G}^H$ . We take any  $X \in \mathcal{G}^H$ . If  $\mathcal{G}_X = \{H_1, \dots, H_p\}$ , then  $X = H_1 \cap \dots \cap H_p \cap H$ . Since  $X$  is a hyperplane in  $\mathcal{G}^H$ , we have  $r(X) = 2$  as the rank in  $\mathcal{G}$ . On the other hand, because  $r(X) \geq r(H_1 \cap \dots \cap H_p) \neq 1$ , we have  $r(H_1 \cap \dots \cap H_p) = 2$ . If  $p = 1$ , then  $X = H_1 \cap H$  and we get  $\lambda_X = \lambda_{H_1} + 0 \notin \mathbb{Z}_{\geq 0}$ . If  $p > 1$ , then  $X$  is dense and hence  $\lambda_X \notin \mathbb{Z}_{\geq 0}$ . Therefore, we have  $\mathcal{G}'' = \mathcal{G}^H$ .

Finally, since  $s' < s$  and  $s'' < s$ , the vanishing theorem holds for  $\mathcal{A}'$  and  $\mathcal{A}''$  by inductive assumptions. Since  $r' = r$  and  $r''$  is  $r$  or  $r - 1$ , we obtain

$$H^k(M(\mathcal{A}'), \mathcal{L}_\lambda) = 0, \quad H^{k-1}(M(\mathcal{A}''), \mathcal{L}_\lambda) = 0, \quad \text{for } k < r.$$

Therefore, the long exact sequence for the triple  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')_H$  in Theorem 3.1 implies that

$$H^k(M(\mathcal{A}), \mathcal{L}_\lambda) = 0, \quad \text{for } k < r,$$

and

$$0 \rightarrow H^r(M(\mathcal{G}), \mathcal{L}_\lambda) \rightarrow H^r(M(\mathcal{A}), \mathcal{L}_\lambda).$$

□

We obtain the following corollary of the proof of Theorem 2.1:

**Corollary 3.3.** *Under the assumption in Theorem 2.1, the inclusion map from  $M(\mathcal{A})$  to  $M(\mathcal{G}(\mathcal{A}_\infty, \lambda))$  induces an injective map*

$$H^r(M(\mathcal{G}(\mathcal{A}_\infty, \lambda)), \mathcal{L}_\lambda) \rightarrow H^r(M(\mathcal{A}), \mathcal{L}_\lambda),$$

where  $M(\mathcal{G}(\mathcal{A}_\infty, \lambda))$  is the complement of the projective arrangement  $\mathcal{G}(\mathcal{A}_\infty, \lambda)$ . In particular, if  $H^r(M(\mathcal{G}(\mathcal{A}_\infty, \lambda)), \mathcal{L}_\lambda) \neq 0$ , then  $H^r(M(\mathcal{A}), \mathcal{L}_\lambda) \neq 0$ .

*Proof of Theorem 2.3.* By [FT], the space generated by  $\{\zeta(B) \mid B \in \beta\mathbf{NBC}(\mathcal{A})\}$  is not dependent on a linear order in  $\mathcal{A}$ . We choose a linear order in  $\mathcal{A}$  as follows : if  $H_i \in \mathcal{G}(\mathcal{A}, \lambda)$  and  $H_j \in \mathcal{N}(\mathcal{A}, \lambda)$ , then  $i < j$ .

We use the same induction on  $s$  in the proof of Theorem 2.1. When  $s = 0$ , the condition (Mon\*) is equivalent to the condition (Mon), and then Theorem 2.3 holds by Theorem 1.2 [FT]. Assume that  $s > 0$ . We take the triple  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')_{H_n}$ , where  $H_n$  is the last hyperplane. By the definition of the linear order, note that  $H_n \in \mathcal{N}(\mathcal{A}, \lambda)$ . Since  $r = r(\mathcal{A}) = r(\mathcal{A}') = r(\mathcal{G})$ , we note that  $r(\mathcal{G}'') = r(\mathcal{A}'') = r - 1$ . From Theorem 3.1 and Theorem 2.1, we have a short exact sequence:

$$0 \rightarrow H^r(M(\mathcal{A}'), \mathcal{L}_\lambda) \rightarrow H^r(M(\mathcal{A}), \mathcal{L}_\lambda) \rightarrow H^{r-1}(M(\mathcal{A}''), \mathcal{L}_\lambda) \otimes H^1(T_{H_n}^*) \rightarrow 0.$$

Using inductive assumptions, we have

$$\{\zeta(B') \mid B' \in \beta\mathbf{NBC}(\mathcal{A}')\} \quad \text{and} \quad \{\zeta(B'') \mid B'' \in \beta\mathbf{NBC}(\mathcal{A}'')\}$$

are bases of  $H^r(M(\mathcal{A}'), \mathcal{L}_\lambda)$  and  $H^{r-1}(M(\mathcal{A}''), \mathcal{L}_\lambda)$ , respectively. Define a map  $\nu : L(\mathcal{A}) \setminus \{\mathbb{C}^\ell\} \rightarrow \mathcal{A}$  by  $\nu X = \min(\mathcal{A}_X)$ . Note that, for  $H'' \in \mathcal{A}''$ , since  $H_n \in \mathcal{A}_{H''}$  and  $H_n$  is the largest hyperplane, we have  $\nu H'' \in \mathcal{A}'$ . For  $S = (X_1, \dots, X_p)$ , write  $\nu S = (\nu X_1, \dots, \nu X_p)$ . Due to [OT2, Theorem 5.3.2], we have a disjoint union

$$\beta\mathbf{NBC}(\mathcal{A}) = \beta\mathbf{NBC}(\mathcal{A}') \cup \{\{\nu B'', H_n\} \mid B'' \in \beta\mathbf{NBC}(\mathcal{A}'')\}.$$

Note that

$$\zeta(\{\nu B'', H_n\}) = \zeta(B'') \wedge \omega_{H_n}$$

(see [OT2, Lemma 6.3.5] for details). By the above short exact sequence, the set  $\{\zeta(B) \mid B \in \beta\mathbf{NBC}(\mathcal{A})\}$  generates  $H^r(M(\mathcal{A}), \mathcal{L}_\lambda)$ .

By the vanishing theorem, we have

$$\dim H^r(M(\mathcal{A}), \mathcal{L}_\lambda) = |\chi(\mathcal{A})|,$$

where  $\chi(\mathcal{A})$  is the Euler characteristic of  $M(\mathcal{A})$ . Moreover, we know that  $|\chi(\mathcal{A})|$  is equal to the cardinality of  $\beta\mathbf{NBC}(\mathcal{A})$ , which is the beta invariant  $\beta(\mathcal{A})$  defined in [OT2, Section 3.3] (see also [STV]). Therefore, we obtain

$$\dim H^r(M(\mathcal{A}), \mathcal{L}_\lambda) = |\beta\mathbf{NBC}(\mathcal{A})| = \beta(\mathcal{A}).$$

Note that  $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$  (see [OT2, Proposition 3.3.3]).

As a result, the set  $\{\zeta(B) \mid B \in \beta\mathbf{NBC}(\mathcal{A})\}$  is a basis of  $H^r(M(\mathcal{A}), \mathcal{L}_\lambda)$ . We have completed the proof. □

4. EXAMPLES

Let  $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$  be an arrangement and  $\lambda$  be its weight. In this section, we shall show examples of weights of the following types:

- (A)  $\lambda$  satisfies (Mon\*) and does not satisfy (Mon), but there exists a trivial weight  $k$  for which  $\lambda + k$  satisfies (Mon).
- (B)  $\lambda$  satisfies (Mon\*), and there is no trivial weight  $k$  for which  $\lambda + k$  satisfies (Mon).

Weights of type A essentially satisfy (Mon). Examples of weights of type B imply that Theorem 2.1 essentially extends the results of Theorem 1.1. Furthermore, there are occasions when we can compute the local system cohomology by the way of the proof of Theorem 2.1.

Define the subset  $\text{ND}(\lambda)$  of  $D(\mathcal{A}_\infty)$  by

$$\text{ND}(\lambda) = \{X \in D(\mathcal{A}_\infty) : \lambda_X \in \mathbb{Z}_{\geq 0}\}.$$

So  $\lambda$  satisfies (Mon) if and only if  $\text{ND}(\lambda) = \emptyset$ . Let  $h^* = h^*(\mathcal{A}) = h^*(\mathcal{A}, \lambda) = \dim H^*(M(\mathcal{A}), \mathcal{L}_\lambda)$ , for short. Write  $X_{i_1 \dots i_s} = \bar{H}_{i_1} \cap \dots \cap \bar{H}_{i_s}$ ,  $\lambda_i = \lambda_{H_i}$  and  $\lambda_{i_1 \dots i_s} = \lambda_{X_{i_1 \dots i_s}}$ . Note that  $\sum_{1 \leq i \leq n} \lambda_i + \lambda_\infty = 0$ . The Selberg 2-arrangement  $\mathcal{S}$  consists of  $H_1 : x = 0$ ,  $H_2 : y = 0$ ,  $H_3 : x = y$ ,  $H_4 : x = 1$  and  $H_5 : y = 1$  in  $\mathbb{C}(x, y)$ . We have

$$D(\mathcal{S}_\infty) = \{\bar{H}_1, \dots, \bar{H}_5, \bar{H}_\infty, X_{123}, X_{345}, X_{14\infty}, X_{25\infty}\}.$$

**Example 4.1.** Let  $\mathcal{A} = \mathcal{S}$  and  $\lambda = (0, 0, 1/4, 1/4, 1/4)$  with  $\lambda_\infty = -3/4$ . Since  $\text{ND}(\lambda) = \{\bar{H}_1, \bar{H}_2\}$ , the weight  $\lambda$  satisfies (Mon\*) and thus we have  $h^1 = h^0 = 0$ . But, for  $k = (-1, -1, 0, 0, 0)$ , the weight  $\lambda + k$  satisfies (Mon). Then  $\lambda$  is of type A.

The weight  $\lambda = (0, 0, 0, 1/3, 1/3)$  is of type A, as well. Note that  $\text{ND}(\lambda) = \{\bar{H}_1, \bar{H}_2, \bar{H}_3, X_{123}\}$ .

**Example 4.2.** Let  $\mathcal{A} = \mathcal{S}$ . The weight  $\lambda = (0, 2/3, 2/3, 1/3, -2)$  satisfies (Mon\*). Then we have  $h^1 = h^0 = 0$ . Let  $\tilde{\lambda} = \lambda + (-1, 0, 0, 0, 0)$ . Since  $\tilde{\lambda}_{14\infty} = 0$ , the integer shift  $\tilde{\lambda}$  does not satisfy (Mon). By concrete computation, there is no trivial weight  $k$  for which  $\lambda + k$  satisfies (Mon). Therefore the weight  $\lambda$  is of type B.

**Example 4.3.** Let  $\mathcal{A} = \mathcal{S}$ . The weight  $\lambda = (0, 0, 1/3, 0, 0)$  is of type A. But, since  $r_{\text{af}}(\mathcal{G}(\mathcal{A}_\infty, \lambda)) = 1$ , Theorem 2.1 says that  $h^0 = 0$ . On the other hand, we have  $\lambda + (-1, -1, 0, -1, -1)$  satisfies (Mon) and thus we know  $h^1 = h^0 = 0$ . Otherwise, we can compute the local system cohomology by using the long exact sequence for a triple. Let  $\mathcal{A}_1 = \{H_1, H_3, H_5\}$ ,  $\mathcal{A}_2 = \mathcal{A}_1 \cup \{H_2\}$  and  $\mathcal{A}_3 = \mathcal{A}_2 \cup \{H_4\} = \mathcal{A}$  be subarrangements of  $\mathcal{A}$ . They have weights induced by  $\lambda$ . The weight  $\lambda(\mathcal{A}_1)$  satisfies (Mon\*), and then  $h^1(\mathcal{A}_1) = h^0(\mathcal{A}_1) = 0$ . For the triple  $(\mathcal{A}_2, \mathcal{A}'_2 = \mathcal{A}_1, \mathcal{A}''_2)_{H_2}$ , we have  $H^*(M(\mathcal{A}''_2), \mathcal{L}_\lambda) = 0$  and the long exact sequence for the triple induces  $h^1(\mathcal{A}_2) = h^0(\mathcal{A}_2) = 0$ . Next, for the triple  $(\mathcal{A}_3 = \mathcal{A}, \mathcal{A}'_3 = \mathcal{A}_2, \mathcal{A}''_3)_{H_4}$ , we have  $h^0(\mathcal{A}''_3) = 0$  and the long exact sequence for this triple induces  $h^1(\mathcal{A}) = h^0(\mathcal{A}) = 0$ .

**Example 4.4.** Let  $\mathcal{A} = \mathcal{S}$  and  $\lambda = (1/3, 0, 0, 0, 0)$  with  $\lambda_\infty = -1/3$ . Since  $\lambda_{14\infty} = 0$ , the weight  $\lambda$  does not satisfy (Mon\*) and there is no integer shift of  $\lambda$  satisfying (Mon). But, we can compute the local system cohomology in the following way. Let  $\mathcal{B}_1 = \{H_1\}$ ,  $\mathcal{B}_2 = \{H_2, H_5\}$ ,  $\mathcal{A}_1 = \{H_1, H_2, H_5\}$ ,  $\mathcal{A}_2 = \mathcal{A}_1 \cup \{H_4\}$  and  $\mathcal{A}_3 = \mathcal{A}_2 \cup \{H_3\} = \mathcal{A}$  be subarrangements of  $\mathcal{A}$ . They have weights induced by  $\lambda$ . Since

essentially  $\mathcal{B}_1 \times \mathcal{B}_2 = \mathcal{A}_1$ , the twisted version of the Künneth formula and  $h^*(\mathcal{B}_1) = 0$ , we have  $h^*(\mathcal{A}_1) = 0$ . By the long exact sequence for the triple  $(\mathcal{A}_2, \mathcal{A}'_2 = \mathcal{A}_1, \mathcal{A}''_2)_{H_4}$ , since  $\lambda(\mathcal{A}''_2)$  is trivial and  $h^1(\mathcal{A}''_2) = 2, h^0(\mathcal{A}''_2) = 1$ , we get  $h^2(\mathcal{A}_2) = 2, h^1(\mathcal{A}_2) = 1, h^0(\mathcal{A}_2) = 0$ . Finally, the long exact sequence for the triple  $(\mathcal{A}_3 = \mathcal{A}, \mathcal{A}'_3 = \mathcal{A}_2, \mathcal{A}''_3)_{H_3}$  induces

$$h^2(\mathcal{A}, \lambda) = 3, h^1(\mathcal{A}, \lambda) = 1, h^0(\mathcal{A}, \lambda) = 0.$$

On the other hand, let  $\mathcal{A}_0 = \mathcal{S} \setminus \{H_5\}$  be the subarrangement of  $\mathcal{A}$ . The induced weight  $\lambda(\mathcal{A}_0)$  satisfies (Mon\*), and thus we have  $h^2(\mathcal{A}_0) = 1, h^1(\mathcal{A}_0) = h^0(\mathcal{A}_0) = 0$ . Note that  $h^0(\mathcal{A}'') = 1, h^1(\mathcal{A}'') = 2$ . In the long exact sequence for the triple  $(\mathcal{A}, \mathcal{A}' = \mathcal{A}_0, \mathcal{A}'')_{H_5}$ , the coboundary map  $H^0(M(\mathcal{A}''), \mathcal{L}_\lambda) \rightarrow H^1(M(\mathcal{A}'), \mathcal{L}_\lambda)$  is an isomorphism. Therefore, in general, coboundary maps in the long exact sequence for a triple are not zero maps. In the case for a trivial weight, all coboundary maps are zero maps ([OT, Theorem 5.87]).

**Example 4.5.** Let  $\mathcal{A} = \mathcal{S} \cup \{H_6\}$  defined by  $H_6 : x + y = 1$ . We have

$$D(\mathcal{A}_\infty) = D(\mathcal{S}_\infty) \cup \{\bar{H}_6, X_{156}, X_{246}\}.$$

It is easy to check the following:

- (1) The weight  $\lambda = (0, 0, 0, 1/4, 1/4, 1/4)$  is of type A.
- (2) The weight  $\lambda = (0, -2/3, 4/3, -5/3, -2/3, 4/3)$  is of type B.
- (3) The weight  $\lambda = (0, 2/3, 1/2, 1/2, -2, 0)$  is of type B.

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