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MEROMORPHIC FUNCTIONS AND FACTORIALITY

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ABSTRACT. Let K be a compact subset of a connected Stein manifold X. We study algebraic properties of the ring of meromorphic functions on X without poles in K.

1. INTRODUCTION AND MAIN RESULTS

Let X be a connected Stein manifold. Denote by $\mathcal{O}(X)$ (resp. $\mathcal{O}_x$) the ring of holomorphic functions on X (resp. the ring of germs of holomorphic functions at a point x in X). We regard $\mathcal{O}(X)$ as a subring of the field $\mathcal{M}(X)$ of meromorphic functions on X. Since X is a Stein manifold, $\mathcal{M}(X)$ is the field of fractions of $\mathcal{O}(X)$; cf. [9, Theorem 7.4.6]. Given a compact subset K of X, we let

$$\mathcal{M}_K(X)$$

denote the subring of $\mathcal{M}(X)$ consisting of all meromorphic functions on X with no poles in K. In this paper we investigate algebraic properties of $\mathcal{M}_K(X)$. Some mild assumptions on K allow us to prove that the ring $\mathcal{M}_K(X)$ is regular (Theorem 1.1) and to give necessary and sufficient conditions for $\mathcal{M}_K(X)$ to be a unique factorization domain (Theorem 1.2). If $\dim X = 1$, then $\mathcal{M}_K(X)$ is always a regular ring and a unique factorization domain (Corollary 1.3). Assuming that X is an algebraic subset of $\mathbb{C}^N$, for some N, the ring $\mathcal{M}_K(X)$ is a unique factorization domain for every compact subset K of X if and only if $H^2(X,\mathbb{Z}) = 0$ (Theorem 1.4). The reader may consult [2, 5, 7, 13] for related results concerning the ring of germs at K of holomorphic functions defined in a neighborhood of K.

We need some preparation in order to state precisely our results. Denote by $\hat{K}$ the holomorphic hull of K in X,

$$\hat{K} = \{x \in X \mid |f(x)| \leq \sup_{y\in K} |f(y)| \text{ for every } f \text{ in } \mathcal{O}(X)\}.$$

Since X is a Stein manifold, $\hat{K}$ is compact. Recall that K is said to be holomorphically convex in X if $K = \hat{K}$. Of course, $K \subseteq \hat{K}$ and $\hat{K}$ is holomorphically convex in X.

We write $H^*(-,\mathbb{Z})$ to denote the Čech cohomology with coefficients in $\mathbb{Z}$. Let

$$G(K)$$

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denote the image of the restriction homomorphism $H^2(X,\mathbb{Z}) \to H^2(K,\mathbb{Z})$ (that is, the homomorphism induced by the inclusion map $K \hookrightarrow X$). The restriction homomorphism $H^2(\hat{K},\mathbb{Z}) \to H^2(K,\mathbb{Z})$ gives rise to a homomorphism from $G(\hat{K})$ onto $G(K)$, written

$$\alpha_K : G(\hat{K}) \to G(K),$$

whose kernel will be denoted by $\hat{G}(K)$,

$$\hat{G}(K) = \mathrm{Ker}\, \alpha_{\mathrm{K}}.$$

The groups $G(K)$ and $\hat{G}(K)$ will play an important role. Note that $\hat{G}(K) = 0$ if K is holomorphically convex in X. Furthermore, $G(K) = 0$ and $\hat{G}(K) = 0$, provided $H^2(X,\mathbb{Z}) = 0$.

Let us set

$$S(K) = \{f \in \mathcal{O}(X) \mid f^{-1}(0) \subseteq X \backslash K\}.$$

Clearly, $S(K)$ is a multiplicatively closed subset of the ring $\mathcal{O}(X)$. The ring $\mathcal{O}_K(X)$ of fractions of $\mathcal{O}(X)$ with denominators in $S(K)$ is a subring of $\mathcal{M}(X)$. We have

$$\mathcal{O}_K(X) \subseteq \mathcal{M}_K(X).$$

Theorem 1.1. *For any compact subset K of X, the ring $\mathcal{O}_K(X)$is regular. If $\hat{G}(K) = 0$, then $\mathcal{O}_K(X) = \mathcal{M}_K(X)$.*

Let us recall that a commutative ring with identity is said to be regular if it is Noetherian and its localization at each maximal ideal is a regular local ring (cf. [10, p. 140]). It is well known that a regular ring A with no zero divisors is a unique factorization domain if and only if $\mathrm{Pic}(A) = 0$, where $\mathrm{Pic}(A)$ is the Picard group of A (the group of isomorphism classes of finitely generated projective A-modules of rank 1); cf. [4] or, more precisely, see the references to [4] given in [3, pp. 306, 307] and [10, p. 142, Theorem 48].

Theorem 1.2. *Assume that $\hat{G}(K) = 0$. Then $\mathrm{Pic}(\mathcal{M}_K(X))$ is canonically isomorphic to $G(K)$. In particular, $\mathcal{M}_K(X)$ is a unique factorization domain if and only if $G(K) = 0$.*

Specializing to $\dim X = 1$, we obtain the following.

Corollary 1.3. *If $\dim X = 1$, then for any compact subset K of X, the ring $\mathcal{M}_K(X)$ is regular and a unique factorization domain.*

Proof. Since $\dim X = 1$, we have $H^2(X,\mathbb{Z}) = 0$. Hence $G(K) = 0$ and $\hat{G}(K) = 0$. The conclusion follows from Theorems 1.1 and 1.2. □

Let us also mention that if $\dim X \geq 1$, then the ring $\mathcal{O}(X)$ is *not* Noetherian and *not* a unique factorization domain.

Theorem 1.4. *Let X be a nonsingular irreducible algebraic subset of $\mathbb{C}^N$ for some N. Then the following conditions are equivalent:*

(a) *for every compact subset K of X, the ring $\mathcal{M}_K(X)$ is a unique factorization domain,*
(b) *for every compact holomorphically convex subset K of X, the ring $\mathcal{M}_K(X)$ is a unique factorization domain,*
(c) $H^2(X,\mathbb{Z}) = 0$.

2. Proofs

We shall frequently use the fact that any compact holomorphically convex subset of a Stein manifold has a base of neighborhoods consisting of open Stein submanifolds; cf. [9, Theorems 5.1.6, 5.2.10].

For any point x in X,

$$m_x = \{f \in \mathcal{O}(X) \mid f(x) = 0\}$$

is a maximal ideal of the ring $\mathcal{O}(X)$ (not every maximal ideal of $\mathcal{O}(X)$ is of this form). Given a compact subset K of X, we shall now regard $\mathcal{O}(X)$ as a subring of $\mathcal{O}_K(X)$. Clearly, if x is a point in K, then $m_x\mathcal{O}_K(X)$ is a maximal ideal of $\mathcal{O}_K(X)$.

Lemma 2.1. *Let m be a maximal ideal of the ring $\mathcal{O}_K(X)$. If K is holomorphically convex in X, then*

$$m = m_x\mathcal{O}_K(X)$$

for some point x in K.

Proof. It suffices to show that there is a point x in K such that $f(x) = 0$ for all f in $m \cap \mathcal{O}(X)$. Suppose that this assertion does not hold. Then one can find functions $f_1, \dots, f_r$ in $m \cap \mathcal{O}(X)$ which have no common zero in a neighborhood U of K in X. We may assume that U is an open Stein neighborhood.

Let $\mathcal{J}$ be the sheaf of ideals on X generated by $f_1, \dots, f_r$,

$$\mathcal{J}_x = (f_1, \dots, f_r)\mathcal{O}_x$$

for all x in X. Clearly, 1 belongs to $\Gamma(U, \mathcal{J})$. Hence there is an f in $\Gamma(X, \mathcal{J})$ such that f is close to 1 on K; cf. [9, Theorem 7.2.7]. In particular, f is in $S(K)$ and, by [9, Theorem 7.2.9], f is also in the ideal $(f_1, \dots, f_r)\mathcal{O}(X)$. It follows that f is in $S(K) \cap m$. This implies $m = \mathcal{O}_K(X)$, a contradiction. □

Denote by $\mathcal{O}_K$ the ring of germs at K of holomorphic functions defined in a neighborhood of K. Note the canonical ring homomorphisms $\mathcal{O}(X) \to \mathcal{O}_K$ and $\mathcal{O}_K(X) \to \mathcal{O}_K$.

Lemma 2.2. *If K is holomorphically convex in X, then the ring $\mathcal{O}_K$ is faithfully flat over $\mathcal{O}_K(X)$.*

Proof. We shall first show that $\mathcal{O}_K$ is flat over $\mathcal{O}(X)$. To this end we shall use a characterization of flatness in terms of solutions of linear equations [10, p. 17, Theorem 1]. Given $f_1, \dots, f_r$ in $\mathcal{O}(X)$ and $g_1, \dots, g_r$ in $\mathcal{O}_K$ with

$$f_1g_1 + \cdots + f_rg_r = 0 \text{ in } \mathcal{O}_K,$$

we have to show the existence of f_{ij} in $\mathcal{O}(X)$ and h_j in $\mathcal{O}_K$, $1 \le i \le r$, $1 \le j \le s$ for some positive integer s, such that

$$f_1f_{1j} + \cdots + f_rf_{rj} = 0 \text{ in } \mathcal{O}(X) \text{ for all } 1 \le j \le s \tag{1}$$

and

$$g_i = f_{i1}h_1 + \cdots + f_{is}h_s \text{ in } \mathcal{O}_K \text{ for all } 1 \le i \le r. \tag{2}$$

We proceed as follows. Let $\mathcal{R}$ be the sheaf of relations among $f_1, \dots, f_r$,

$$\mathcal{R}_x = \{(a_1, \dots, a_r) \in \mathcal{O}_x^r \mid f_1a_1 + \cdots + f_ra_r = 0 \text{ in } \mathcal{O}_x\}$$

for all x in X. By Oka's theorem [9, Theorem 6.4.1], $\mathcal{R}$ is a coherent sheaf, and hence it follows from Cartan's Theorem A [9, Theorem 7.2.8] that there exist global sections $F_1, \dots, F_s$ of $\mathcal{R}$ which generate each stalk $\mathcal{R}_x$ for x in an open neighborhood U of K in X. Since K is holomorphically convex in X, we may assume that U is Stein and the germs $g_1, \dots, g_r$ have holomorphic representatives on U. Hence writing $G = (g_1, \dots, g_r)$, we obtain

$$G = h_1 F_1 + \cdots + h_s F_s$$

for some $h_1, \dots, h_s$ in $\mathcal{O}_K$; cf. [9, Theorem 7.2.9]. Setting $F_j = (f_{1j}, \dots, f_{rj})$ for $1 \leq j \leq s$, we see that (1) and (2) are satisfied. It follows that $\mathcal{O}_K$ is flat over $\mathcal{O}(X)$.

It immediately follows that $\mathcal{O}_K$ is flat over $\mathcal{O}_K(X)$. In order to prove that $\mathcal{O}_K$ is faithfully flat over $\mathcal{O}_K(X)$ it suffices to demonstrate that given a maximal ideal m of $\mathcal{O}_K(X)$, we have $m\mathcal{O}_K \neq \mathcal{O}_K$; cf. [10, p. 25, Theorem 2]. By Lemma 2.1, $m = m_x \mathcal{O}_K(X)$ for some point x in K. Clearly, $m\mathcal{O}_K = m_x\mathcal{O}_K$. Since $m_x\mathcal{O}_K \neq \mathcal{O}_K$, the proof is complete. □

Proposition 2.3. *For any compact subset K of X, the ring $\mathcal{O}_K(X)$ is regular.*

Proof. Step 1. Assume that K is a semianalytic compact holomorphically convex subset of X. It follows that the ring $\mathcal{O}_K$ is Noetherian [7, Théorème (I, 9)]. By Lemma 2.2, $\mathcal{O}_K(X)$ is a Noetherian ring; cf. [4, Chapitre I, p. 50].

In order to complete Step 1 it remains to show that for each maximal ideal m of $\mathcal{O}_K(X)$, the localization $\mathcal{O}_K(X)_m$ of $\mathcal{O}_K(X)$ at m is a regular local ring. By Lemma 2.1, $m = m_x\mathcal{O}_K(X)$ for some point x in K. We have $S(K) \cap m_x = \emptyset$, which implies that the local rings $\mathcal{O}_K(X)_m$ and $\mathcal{O}(X)_{m_x}$ are isomorphic. Hence it suffices to prove that the local ring $\mathcal{O}(X)_{m_x}$ is regular. We have $\mathcal{O}(X)_{m_x} = \mathcal{O}_{\{x\}}(X)$, and hence the ring $\mathcal{O}(X)_{m_x}$ is Noetherian (the set $\{x\}$ being semianalytic and holomorphically convex in X). Since X is a Stein manifold, there exist $f_1, \dots, f_n$ in $\mathcal{O}(X)$, where $n = \dim X$, such that $(f_1, \dots, f_n)\mathcal{O}_x$ is the maximal ideal of $\mathcal{O}_x$; cf. [9, Definition 5.1.3 (γ)]. By Lemma 2.2, $\mathcal{O}_x$ is flat over $\mathcal{O}(X)_{m_x}$ and therefore $(f_1, \dots, f_n)\mathcal{O}(X)_{m_x}$ is the maximal ideal of $\mathcal{O}(X)_{m_x}$. It readily follows that the completions of $\mathcal{O}(X)_{m_x}$ and $\mathcal{O}_x$ with respect to their maximal ideals are isomorphic. Since $\mathcal{O}_x$ is a regular local ring, so is $\mathcal{O}(X)_{m_x}$; cf. [10, p. 175].

Step 2. Let K be an arbitrary compact subset of X. We may assume that X is a closed submanifold of $\mathbb{C}^N$ for some N. Choose a closed ball B in $\mathbb{C}^N$ containing K. Clearly, the compact subset $C = B \cap X$ of X is semianalytic and holomorphically convex in X. According to Step 1, the ring $\mathcal{O}_C(X)$ is regular. Since $S(C) \subseteq S(K)$, the ring $\mathcal{O}_K(X)$ is a localization of $\mathcal{O}_C(X)$, and hence $\mathcal{O}_K(X)$ is also a regular ring. □

In what follows we shall make use several times of the following consequence of Cartan's Theorem B: For any Stein manifold Y,

$$c_1 : \mathrm{Pic}(Y) \to H^2(Y, \mathbb{Z})$$

is an isomorphism, where $\mathrm{Pic}(Y)$ is the group of isomorphism classes of holomorphic line bundles on Y and c_1 is the first Chern class homomorphism; cf. [8, p. 197, Theorem C]. In particular, any holomorphic line bundle on Y is holomorphically trivial if and only if it is topologically trivial.

Proof of Theorem 1.1. In view of Proposition 2.3 and the inclusion $\mathcal{O}_K(X) \subseteq \mathcal{M}_K(X)$ it suffices to show $\mathcal{M}_K(X) \subseteq \mathcal{O}_K(X)$. Let φ be in $\mathcal{M}_K(X) \backslash \{0\}$. To show that φ is in $\mathcal{O}_K(X)$ we proceed as follows. For each point x in X,

$$\mathcal{J}_x = \{h \in \mathcal{O}_x \mid h\varphi \text{ is in } \mathcal{O}_x\}$$

is a nonzero ideal of the ring $\mathcal{O}_x$. It is well known that the union $\mathcal{J}$ of all $\mathcal{J}_x$ is a coherent sheaf of principal ideals on X; cf. [9, proof of Theorem 7.4.6]. There is a family $\{U_i, h_i\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open cover of X and $h_i : U_i \to \mathbb{C}$ is a holomorphic function generating $\mathcal{J}|U_i$ for each i in I. On $U_i \cap U_j$ we have $h_i = g_{ij}h_j$, where $g_{ij} : U_i \cap U_j \to \mathbb{C}\backslash\{0\}$ is a holomorphic function. Note that $\{U_i, h_i\}_{i\in I}$ describes the divisor of poles of φ. Let L be the holomorphic line bundle on X determined by $\{U_i, g_{ij}\}$ and let $s : X \to L$ be the holomorphic section corresponding to $\{h_i\}_{i\in I}$. Note

$$s^{-1}(0) \subseteq X\backslash K. \tag{1}$$

Consider the dual line bundle $L^\vee$ on X. In view of (1), the restriction $L^\vee|K$ is topologically trivial on K and hence $c_1(L^\vee|K) = 0$ in $H^2(K, \mathbb{Z})$, where $c_1(-)$ stands for the first Chern class. Since $c_1(L^\vee|\hat{K})$ is in $G(\hat{K})$, $\alpha_K(c_1(L^\vee|\hat{K})) = c_1(L^\vee|K)$, and $\mathrm{Ker}\alpha_K = \hat{G}(K) = 0$, we get $c_1(L^\vee|\hat{K}) = 0$. The last equality implies that $L^\vee|\hat{K}$ is topologically trivial on $\hat{K}$. Choose an open neighborhood U of $\hat{K}$ in X such that $L^\vee|U$ is topologically trivial. We may assume that U is Stein, and hence $L^\vee|U$ is holomorphically trivial. By [9, Theorem 7.2.7], there exists a holomorphic section $u : X \to L^\vee$ satisfying

$$u^{-1}(0) \subseteq X\backslash\hat{K}. \tag{2}$$

Define $g : X \to \mathbb{C}$ by

$$g(x) = u(x)(s(x)) \text{ for all } x \text{ in } X.$$

Clearly, g is a holomorphic function and moreover g belongs to $\Gamma(X, \mathcal{J})$ (s is locally h_i and u acts on s by multiplying h_i by a holomorphic function). Hence $f = g\varphi$ is in $\mathcal{O}(X)$. Combining (1) and (2), we get $g^{-1}(0) \subseteq X\backslash K$, and hence $\varphi = f/g$ belongs to $\mathcal{O}_K(X)$. □

For the proof of Theorem 1.2 we need another result.

Theorem 2.4. *Let Y be a Stein manifold. If E is a holomorphic vector bundle on Y, then the $\mathcal{O}(Y)$-module $\Gamma(Y, E)$ of global holomorphic sections of E is finitely generated and projective. Furthermore, the correspondence $E \to \Gamma(Y, E)$ is an equivalence between the category of holomorphic vector bundles on Y and the category of finitely generated projective $\mathcal{O}(Y)$-modules.*

Reference for the proof. [6, Satz 6.2 and Satz 6.3]. □

Henceforth, when no confusion is possible, we shall identify modules and their isomorphism classes.

Proof of Theorem 1.2. Let A be a compact subset of X. Denote by

$$\beta_A : \mathrm{Pic}(\mathcal{O}_A(X)) \to \mathrm{Pic}(\mathcal{O}_A)$$

the group homomorphism induced by the canonical ring homomorphism $\mathcal{O}_A(X) \to \mathcal{O}_A$. We assert

(1) β_A is injective if A is holomorphically convex in X.

Indeed, endow $\mathcal{O}_A$ with the topology of uniform convergence on A. If A is holomorphically convex in X, then the image of $\mathcal{O}(X)$ under the canonical homomorphism $\mathcal{O}(X) \to \mathcal{O}_A$ is dense in $\mathcal{O}_A$; cf. [9, Theorem 7.2.7]. Hence (1) is a consequence of [15, Theorem 2.2].

We shall now define a canonical homomorphism

$$\gamma_A : \mathrm{Pic}(\mathcal{O}_A) \to H^2(A, \mathbb{Z}).$$

Denote by $\mathcal{C}(A)$ the ring of all continuous functions from A into $\mathbb{C}$, and let

$$\delta_A : \mathrm{Pic}(\mathcal{O}_A) \to \mathrm{Pic}(\mathcal{C}(A))$$

be the group homomorphism induced by the canonical ring homomorphism $\mathcal{O}_A \to \mathcal{C}(A)$. Denote by

$$\eta_A : \mathrm{Pic}(\mathcal{C}(A)) \to H^2(A, \mathbb{Z})$$

the homomorphism which assigns to each element Q of $\mathrm{Pic}(\mathcal{C}(A))$ the first Chern class of the topological complex line bundle on A determined by Q; cf. [14]. Note that η_A is an isomorphism. By definition, γ_A is the composite of δ_A and η_A, that is $\gamma_A = \eta_A \circ \delta_A$. We claim

(2) γ_A is injective if A is holomorphically convex in X.

In order to prove (2) note that

$$\mathcal{O}_A = \mathrm{ind}\lim \mathcal{O}(U),$$

where U runs through the family of open Stein neighborhoods of A in X. It follows that

$$\mathrm{Pic}(\mathcal{O}_A) = \mathrm{ind}\lim \mathrm{Pic}(\mathcal{O}(U))$$

(see for example [15, Lemma 7.2]). By Theorem 2.4, every element of $\mathrm{Pic}(\mathcal{O}(U))$ is of the form $\Gamma(U, L)$ for some holomorphic line bundle L on U. Suppose $c_1(L|A) = 0$. Then $L|A$ is topologically trivial on A. There is an open neighborhood U' of A in U such that $L|U'$ is topologically trivial on U'. We may choose U' to be Stein, and hence $L|U'$ is holomorphically trivial on U'. It follows that $\Gamma(U, L)$ represents 0 in $\mathrm{Pic}(\mathcal{O}_A)$. Thus (2) is proved.

Define

$$c_A : \mathrm{Pic}(\mathcal{O}_A(X)) \to H^2(A, \mathbb{Z})$$

by setting $c_A = \gamma_A \circ \beta_A$. In view of (1) and (2), we have

(3) c_A is injective if A is holomorphically convex in X.

If B is a compact subset of X containing A, then the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Pic}(\mathcal{O}_B(X)) & \xrightarrow{c_B} & H^2(B, \mathbb{Z}) \\ {\scriptstyle \rho_{B,A}}\downarrow & & {\scriptstyle r_{B,A}}\downarrow \\ \mathrm{Pic}(\mathcal{O}_A(X)) & \xrightarrow{c_A} & H^2(A, \mathbb{Z}), \end{array} \tag{4}$$

where $r_{B,A}$ is the restriction homomorphism and $\rho_{B,A}$ is the homomorphism induced by the inclusion $\mathcal{O}_B(X) \to \mathcal{O}_A(X)$. Since $S(B) \subseteq S(A)$, the ring $\mathcal{O}_A(X)$ is a localization of $\mathcal{O}_B(X)$. By Proposition 2.3, $\mathcal{O}_B(X)$ is a regular ring and hence

(5) $\rho_{B,A}$ is surjective;

cf. [1, p. 144, Proposition 7.17, p. 147, Theorem 7.21].

We shall now show that

$$c_A(\mathrm{Pic}(\mathcal{O}_A(X))) \subseteq G(A). \tag{6}$$

This can be done as follows. We may assume that X is a closed submanifold of $\mathbb{C}^N$ for some N. Let $\{B_i\}_{i\geq 1}$ be a sequence of closed balls in $\mathbb{C}^N$ such that $A \subseteq B_1 \cap X$, $B_i \subseteq B_{i+1}$, and X is the union of the $B_i \cap X$ for $i \geq 1$. Let us set $A_0 = A$ and $A_i = B_i \cap X$ for $i \geq 1$. Clearly, A_i is semianalytic and holomorphically convex in X for $i \geq 1$. There is a triangulation of X for which each A_i with $i \geq 1$ is a compact polyhedron [12, Theorem II.2.1′].

Let P be in $\mathrm{Pic}(\mathcal{O}_A(X))$. We define recursively a sequence $\{P_i\}_{i\geq 0}$, where P_i belongs to $\mathrm{Pic}(\mathcal{O}_{A_i}(X))$. Set $P_0 = P$. Suppose that P_i in $\mathrm{Pic}(\mathcal{O}_{A_i}(X))$ is already defined, and let P_{i+1} be an element of $\mathrm{Pic}(\mathcal{O}_{A_{i+1}}(X))$ which is sent to P_i by the homomorphism

$$\rho_{A_{i+1},\, A_i} : \mathrm{Pic}(\mathcal{O}_{A_{i+1}}(X)) \to \mathrm{Pic}(\mathcal{O}_{A_i}(X))$$

(by (5), this homomorphism is surjective).

The restriction homomorphisms

$$r_{A_{i+1},\, A_i} : H^2(A_{i+1}, \mathbb{Z}) \to H^2(A_i, \mathbb{Z}),$$

for $i \geq 0$, give rise to the projective limit

$$\mathrm{proj}\ \lim H^2(A_i, \mathbb{Z}).$$

In view of (4), $(c_{A_0}(P_0), c_{A_1}(P_1), c_{A_2}(P_2), \ldots)$ belongs to this limit. Since the canonical homomorphism

$$H^2(X, \mathbb{Z}) \to \mathrm{proj}\ \lim H^2(A_i, \mathbb{Z})$$

is surjective [11, Lemma 2], it follows that $c_A(P) = c_{A_0}(P_0)$ is in $G(A)$. This completes the proof of (6).

Our next step is the proof of

$$G(A) \subseteq c_A(\mathrm{Pic}(\mathcal{O}_A(X))). \tag{7}$$

Let u be in $G(A)$. Choose a cohomology class v in $H^2(X, \mathbb{Z})$ which is sent to u by the restriction homomorphism $H^2(X, \mathbb{Z}) \to H^2(A, \mathbb{Z})$. There is a holomorphic line bundle M on X with $c_1(M) = v$. By Theorem 2.4, $S(A)^{-1}\Gamma(X, M)$ is in $\mathrm{Pic}(\mathcal{O}_A(X))$ and by definition of c_A, we have $c_A(S(A)^{-1}\Gamma(X, M)) = u$. Hence (7) is proved.

Let

$$\bar{c}_A : \mathrm{Pic}(\mathcal{O}_A(X)) \to G(A) \tag{8}$$

be the surjective homomorphism determined by c_A ($\bar{c}_A$ is just c_A regarded as a homomorphism onto its image; cf. (6) and (7)).

After this preparation we are ready to complete the proof of the theorem. We shall now show that

$$\bar{c}_K : \mathrm{Pic}(\mathcal{O}_K(X)) \to G(K) \text{ is an isomorphism.} \tag{9}$$

Since $\bar{c}_K$ is a surjective homomorphism, it remains to demonstrate injectivity of $\bar{c}_K$. Diagram (4), with $A = K$ and $B = \hat{K}$, and (8) yield the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Pic}(\mathcal{O}_{\hat{K}}(X)) & \xrightarrow{\bar{c}_{\hat{K}}} & G(\hat{K}) \\ {\scriptstyle \rho_{\hat{K},K}} \downarrow & & {\scriptstyle \alpha_K} \downarrow \\ \mathrm{Pic}(\mathcal{O}_K(X)) & \xrightarrow{\bar{c}_K} & G(K), \end{array}$$

where α_K is the restriction of $r_{\hat{K},K}$ (cf. Section 1). According to (3), $\bar{c}_{\hat{K}}$ is injective. Recall that $\hat{G}(K) = \mathrm{Ker}\alpha_K = 0$, and hence α_K is injective. Since by (5), $\rho_{\hat{K},K}$ is surjective, we obtain that $\bar{c}_K$ is injective. Thus (9) is proved, and the groups $\mathrm{Pic}(\mathcal{O}_K(X))$ and $G(K)$ are canonically isomorphic.

By Theorem 1.1, the ring $\mathcal{O}_K(X) = \mathcal{M}_K(X)$ is regular, and therefore the proof is complete. □

Proof of Theorem 1.4. If (c) holds, then $\hat{G}(K) = 0$ and $G(K) = 0$ for every compact subset K of X, and hence (a) is satisfied in view of Theorem 1.2. It is obvious that (a) implies (b). It remains then to show that (b) implies (c).

Suppose (b) holds. Choose a compact subset C of X such that the inclusion map $C \hookrightarrow X$ is a homotopy equivalence; cf. for example [3, Corollary 9.3.7]. The restriction homomorphism $H^2(X,\mathbb{Z}) \to H^2(C,\mathbb{Z})$ is an isomorphism. It follows that for $K = \hat{C}$ the restriction homomorphism $H^2(X,\mathbb{Z}) \to H^2(K,\mathbb{Z})$ is injective (note $C \subseteq K$). This implies that $H^2(X,\mathbb{Z})$ is isomorphic to $G(K)$. Since K is holomorphically convex in X, we have $\hat{G}(K) = 0$ and hence, by Theorem 1.2 and condition (b), $G(K) = 0$. Therefore $H^2(X,\mathbb{Z}) = 0$ and (c) is satisfied. □

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