THE LINEAR SPACE OF GENERALIZED BROWNIAN MOTIONS WITH APPLICATIONS

JEONG HYUN LEE

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Abstract. In this paper, we define, motivated by recent works of Chang and Skoug, stochastic integrals for a generalized Brownian motion (gBm) X and then use it to study the representation problem on the linear space $H(X)$ spanned by X. We next establish a translation theorem for $L^p$-functionals of $X$, $p \geq 1$, and then use this translation to establish an integration by parts formula for $L^p$-functionals of $X$.

1. Introduction

Let $(C_0[0,T], B(C_0[0,T]), m_w)$ be the Wiener measure space. Let $a \in C_0[0,T]$ be of bounded variation on $[0,T]$ and let $b \in C_0[0,T]$ be strictly increasing and of bounded variation on $[0,T]$. Then by Theorem 14.2 ([12], p. 187) there exists a Gaussian measure $\mu$ on $(C_0[0,T], B(C_0[0,T]))$ such that the coordinate process defined as $X(t,x) = x(t)$ is a continuous additive process on $(C_0[0,T], B(C_0[0,T]), \mu)$ and $[0,T]$ on which the probability distribution of $X(t_2, \cdot) - X(t_1, \cdot)$, $t_1 < t_2$ is normally distributed with mean $a(t_2) - a(t_1)$ and variance $b(t_2) - b(t_1)$.

Such a process $X = (X_t, t \in [0,T])$ will be referred to as the generalized Brownian motion (gBm) determined by the mean function $a(t)$ and the variance function $b(t)$. We will write the space $(C_0[0,T], B(C_0[0,T]), \mu)$ as $(C_{a,b}[0,T], B(C_{a,b}[0,T]), \mu)$. We note that the Wiener process $W(t,x) = x(t)$ on $(C_0[0,T], B(C_0[0,T]), m_w)$ and $[0,T]$ is free of drift and is stationary in time, while the process $X = (X_t, t \in [0,T])$ on $(C_{a,b}[0,T], B(C_{a,b}[0,T]), \mu)$ is subject to the drift $a(t)$ and is nonstationary in time, and can be used to describe the Black-Scholes model with time-dependent drift and diffusion coefficient in the financial market, as well as the model of interest rates and bond prices. For more details about the Black-Scholes model with time-dependent parameters, see e.g. ([9], p. 80, p. 87). For the model of bond prices and interest rates, see e.g. ([9], p. 125, p. 127).

Let $X = (X_t, t \in [0,T])$ be a gBm determined by $a(t)$ and $b(t)$. Then we note that $X$ is an $L^2$-process and $\sigma(X) = B(C_{a,b}[0,T])$. There are two Hilbert spaces associated to the gBm $X$. The one is the nonlinear space of $X$, $L^2(X) = L^2(C_{a,b}[0,T], \sigma(X), \mu)$ (consisting of all $\sigma(X)$-measurable random variables with...
finite second moment). Elements of $L^2(X)$ are called (nonlinear) $L^2$-functionals of $X$. The second Hilbert space is the linear space $H(X)$ of $X$, which is the closed subspace of $L^2(X)$ spanned by $X$. Elements of $H(X)$ are called linear $L^2$-functionals of $X$.

Let $\bar{X} = (\bar{X}_t \equiv X_t - a(t), t \in [0,T])$. Let $L^2(\bar{X})$ (or $H(\bar{X})$, resp.) denote the nonlinear (linear, resp.) space of $\bar{X}$.

The authors in [4], [5] used a gBm to define a generalized analytic Feynman integral and a generalized analytic Fourier-Feynman transform and then studied the first variations and the integration by parts formula involving these integrals and transforms. Motivated by these recent works ([4], [5]), in this paper we introduce a stochastic integral for the gBm $X$ for which the space of the integrands is wider than the one for the stochastic integrals defined in [4], and then use it to study the representation problem on the linear space $H(X)$. We next establish a translation theorem for $L^p$-functionals, $p \geq 1$, which is proved by using a different method from the one given in [3], [5] and then use this translation theorem to establish an integration by parts formula for $L^p$-functionals of $X$ with $p \geq 1$.

The organization of this paper is as follows. In Section 2, we introduce the function Hilbert space $L_{a,b}^{1,2}[0,T]$, which is our choice of function Hilbert spaces to represent the linear space $H(X)$. In Section 3, we establish a translation theorem for $L^p$-functionals, $p \geq 1$. In Section 4, we use the translation theorem obtained in Section 3 to establish an integration by parts formula for the functionals in $L^p(X)$, $p \geq 1$.

2. Stochastic integration for gBm

In this section we introduce the function Hilbert space $L_{a,b}^{1,2}[0,T]$, which will be our choice of function Hilbert spaces for the linear representation problem on the gBm $X$.

We will assume throughout this paper that $a(t)$ is an absolutely continuous function on $[0,T]$ with $a(0) = 0$ and $a'(t) \in L^2[0,T]$, and $b(t)$ is a differentiable function with $b(0) = 0$ and $b_1 \leq b'(t) \leq b_2$ ($b_1, b_2 > 0$) for all $t \in [0,T]$. This assumption is slightly milder than the one given in [3].

Let $L^2_b[0,T]$ be the Hilbert space of functions on $[0,T]$ given by

$$L^2_b[0,T] = \{ f \in L^2[0,T] \mid \int_0^T f^2(s)db(s) < \infty \}$$

equipped with an inner product

$$\langle f, g \rangle_b = \int_0^T f(s)g(s)db(s).$$

Let $L_{a,b}^{1,2}[0,T]$ be the Hilbert space of functions on $[0,T]$ defined by

$$L_{a,b}^{1,2}[0,T] = \{ f \in L^2_b[0,T] \mid \int_0^T |f(s)|d|a|(s) < \infty \}$$
equipped with an inner product

$$\langle f, g \rangle_{a,b} = \int_0^T f(s)g(s)db(s) + \int_0^T f(s)da(s) \int_0^T g(s)da(s)$$

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Theorem 2.3. Let
\[ L^1_{a,b}[0, T] \]
be a separable Hilbert space.

Let \( S \) be the set of all step functions on \([0, T]\), \( f = \sum_{j=0}^{n-1} c_j 1_{(t_j, t_{j+1}]} \), where \( 0 = t_0 < t_1 < t_2 < \cdots < t_n = T \) and \( c_j \in \mathbb{R} \). We define \( \theta : S \to H(X) \) by

\[ \theta(f) = \sum_{j=0}^{n-1} c_j (X_{t_{j+1}} - X_{t_j}) = \sum_{j=0}^{n-1} c_j [(\tilde{X}_{t_{j+1}} - \tilde{X}_{t_j}) + (a(t_{j+1}) - a(t_j))]. \]

Clearly \( S \) is a linear dense subspace of \( L^1_{a,b}[0, T] \) and for all \( f, g \in S \), we have the following identity:

\[ E[\theta(f)] = \int_0^T f da \quad \text{and} \quad E[\theta(f)\theta(g)] = \langle f, g \rangle_{a,b}, \]

where \( \int_0^T f da \) denotes the Stieltjes integral of \( f \) with respect to the absolutely continuous function \( a \). Thus the correspondence \( f \mapsto \theta(f) \) is an inner product preserving mapping from \( S \) to \( H(X) \). Hence \( \theta \) is uniquely extended to \( L^1_{a,b}[0, T] \).

For \( f \in L^1_{a,b}[0, T] \), we call \( \theta(f) \) the stochastic integral of \( f \) against \( X \) and write it as

\[ \theta(f) = \int_0^T f dX_t \quad \text{or} \quad \int_0^T f d\tilde{X}_t + \int_0^T f da. \]

Definition 2.1. The stochastic integral \( \int_0^T f dX_t \) for \( f \in L^1_{a,b}[0, T] \) is defined by

\[ \int_0^T f dX_t = \int_0^T f d\tilde{X}_t + \int_0^T f da. \]

Proposition 2.2. Let \( X \) be a gBm determined by \( a(t) \) and \( b(t) \). Then

\[ L^1_{a,b}[0, T] \cong H(X). \]

Proof. Let \( \theta : S \to H(X) \) be the mapping defined as above. Then \( \theta(1_{(a,t)}) = \int_0^T 1_{(a,t)}(u) dX(u) = X_t \) for any \( t \in [0, T] \). But the set \( \{X_t \mid t \in [0, T]\} \) generates \( H(X) \). Hence the mapping \( \theta \) is uniquely extended to a unitary isomorphism from \( L^1_{a,b}[0, T] \) onto \( H(X) \). This completes the proof. \( \square \)

Theorem 2.3. Let \( X \) be a gBm determined by \( a(t) \) and \( b(t) \). For \( f, g \in L^1_{a,b}[0, T] \) and \( \alpha, \beta \in \mathbb{R} \), the stochastic integral \( \theta \) satisfies the following:

1. \( \theta(\alpha f + \beta g) = \alpha \theta(f) + \beta \theta(g) \).
2. \( E[\theta(f)] = \int_0^T f da \).
3. \( \langle \theta(f), \theta(g) \rangle_{L^2(X)} = \langle f, g \rangle_{a,b} \).
4. \( \| \theta(f) \|_{L^2(X)}^2 = \| f \|_{a,b}^2 \).
5. \( \theta(f) \) is normally distributed with \( N(\int_0^T f da, \| f \|_{a,b}^2) \).
6. \( \{ \theta(f), f \in L^1_{a,b}[0, T] \} \) is a Gaussian system of random variables.
Proof. The proof of (1)–(4) has been done in Proposition 2.2.

(5) For any $f \in L^2_{\alpha,\beta}\{0,T\}$, there exists a sequence $f_n \in \mathcal{S}$, $n \geq 1$ such that $f_n \to f$. Then $\theta(f_n) \to \theta(f)$ in $L^2(X)$ and clearly the $\theta(f_n)$’s are Gaussian variables. Hence the $L^2$-limit $\theta(f)$ of a sequence $\theta(f_n)$ is normally distributed with $N(\int_0^T f \, da, \|f\|_2^2)$.

(6) The proof follows from the fact that for any finite linear combination of $\theta(f)$’s, $
 \sum_{i=1}^n \alpha_i \theta(f_i)$ is normally distributed with $N(\sum_{i=1}^n \alpha_i \int_0^T f_i \, da, \sum_{i,j=1}^n \alpha_i \alpha_j \langle f_i, f_j \rangle_b). \square$

3. TRANSLATION THEOREM FOR $L^p$-FUNCTIONALS

Let $D$ be any set and $C$ be a real-valued function on $D \times D$. Then $C$ is called a covariance function on $D \times D$ if

1. $C(s, t) = C(t, s)$,
2. $\sum_{t,s \in I} a_t a_s C(t,s) \geq 0$ for all finite subsets $I$ of $D$ and $\{a_t, s \in I\} \subset \mathbb{R}$.

Let $C$ be a real-valued covariance function on $D \times D$. Then according to the following theorem due to Aronszajn [1], there exists a unique Hilbert space $K(C)$ of functions on $D$ satisfying the conditions in the following theorem.

**Theorem 3.1.** Let $D$ be any separable metric space and $C$ be a real-valued covariance function on $D \times D$. Then there exists a unique Hilbert space $K(C)$ of real-valued functions on $D$, with the inner product denoted by $\langle \cdot, \cdot \rangle_{K(C)}$ such that

1. $C(t, \cdot) \in K(C)$, for each $t \in D$,
2. $\langle f(\cdot), C(t, \cdot) \rangle_{K(C)} = f(t)$, for each $t \in D$ and $f \in K(C)$.

The space $K(C)$ in Theorem 3.1 is called the reproducing kernel Hilbert space (RKHS) of $C$. Indeed, $K(C)$ is defined as the closure of the linear span of the functions $\{C(t, \cdot) \mid t \in D\}$ with respect to the inner product $\langle C(t, \cdot), C(s, \cdot) \rangle_{K(C)} = C(t, s)$. It is well known that if $C$ is continuous on $D \times D$, then $K(C)$ is a separable Hilbert space.

**Proposition 3.2.** Let $D = [0,T]$ and $C(t, s) = \min\{b(t), b(s)\}$, $s, t \in [0,T]$. Let $C_b'[0,T]$ be the Hilbert space given by

$$C_b'[0,T] = \{ \gamma \mid \gamma(t) = \int_0^t \frac{d\gamma}{db} \, db, \frac{d\gamma}{db} \in L^2_b[0,T] \}$$

equipped with the inner product defined by

$$\langle \gamma_1, \gamma_2 \rangle_{C_b'[0,T]} = \left( \frac{d\gamma_1}{db} , \frac{d\gamma_2}{db} \right)_b,$$

where $\frac{d\gamma}{db} = \frac{d\gamma}{d\mu}$ with $\mu$ is the Borel-Stieltjes measure induced by $b$ and $\nu$ is the signed measure induced by $\gamma(t) = \int_0^t g \, db$, $g \in L^2_b[0,T]$. Then $K(C) = C_b'[0,T]$.

**Proof.** We first note that for all $s, t \in [0,T]$, $C(t, s) = \min\{b(t), b(s)\} = \int_0^T 1_{[0,t \wedge s]} \, db$ and $\frac{dC(t, \cdot)}{db} = 1_{[0,t]}(\cdot)$. Since $1_{[0,t]} \in L^2_b[0,T]$ for all $t \in [0,T]$, we have

$$C(t, \cdot) \in C_b'[0,T], \quad \text{for each } t \in [0,T].$$
For any \( f \in C'_b[0, T] \) and for each \( t \in [0, T] \), we have
\[
\langle C(t, \cdot), f \rangle_{C'_b[0, T]} = \langle 1_{[0, t]}, \frac{df}{db} \rangle_b
\]
\[
= \int_0^t \frac{df}{db} db = f(t).
\]
Hence by Theorem 3.1, \( K(C) = C'_b[0, T] \).

**Theorem 3.5.** Let \( D = [0, T] \) and \( C(t, s) = \min\{b(t), b(s)\} \), \( s, t \in [0, T] \). Then we have \( K(C) \cong L^2_b[0, T] \).

**Proof.** We define an operator \( U \) from \( L^2_b[0, T] \) to \( C'_b[0, T] \) by
\[
U f(t) = \int_0^t f(s) db(s), \ f \in L^2_b[0, T].
\]
Then we see that \( \langle f, g \rangle_b = \langle \int_0^t f(s) db(s), \int_0^t g(s) db(s) \rangle_{C'_b[0, T]} \). Thus \( U \) is a unitary operator from \( L^2_b[0, T] \) onto \( C'_b[0, T] \). Hence \( K(C) = C'_b[0, T] \cong L^2_b[0, T] \).

For \( p \geq 1 \), let \( L^p(X) \equiv L^p(C_{a, b}[0, T], B(C_{a, b}[0, T]), \mu) \) be the space of all random variables \( F : C_{a, b}[0, T] \to \mathbb{R} \) such that
\[
\|F\|_p = E(|F|^p)^{\frac{1}{p}} < \infty.
\]

**Lemma 3.4.** For \( \phi \in L^p(X) \), \( p \geq 1 \), there exists a sequence \( \{\xi_n\}_{n=1}^\infty \) in \( H(X) \) such that \( \phi \) is \( \sigma(\xi_n, n \geq 1) \)-measurable.

**Proof.** Let \( \{e_n\}_{n=1}^\infty \) be a complete orthonormal basis for \( L^1_{a, b}[0, T] \) and let \( \xi_n = \int_0^T e_n(t) dX_t \). Since \( 1_{[0, t]} \in L^2_{a, b}[0, T] \) for each \( t \in [0, T] \), we have
\[
X_t = \int_0^T 1_{[0, t]}(s) dX_s = \sum_{n=1}^\infty \langle 1_{[0, t]}, e_n \rangle_{a, b} \int_0^T e_n(s) dX_s = \sum_{n=1}^\infty \langle 1_{[0, t]}, e_n \rangle_{a, b} \xi_n.
\]
Hence \( X_t \) is \( \sigma(\xi_n, n \geq 1) \)-measurable for each \( t \in [0, T] \). Therefore, \( \phi \) is \( \sigma(\xi_n, n \geq 1) \)-measurable since \( \phi \) is \( \sigma(X_t, t \in [0, T]) \)-measurable.

**Theorem 3.5.** The set \( A = \{e^\xi | \xi \in H(X)\} \) is total in \( L^p(X) \), where \( e^\xi = \sum_{k=0}^n \frac{\xi^k}{k!} \).

**Proof.** We first note that each \( e^\xi \) is an element of \( L^p(X) \) for each \( p \geq 1 \). Let \( V \) be the \( L^p \)-closure of the linear span of \( A \). Suppose \( V 
\neq L^p(X) \). Then there exists \( \phi \in L^p(X) \) \( \left( \frac{1}{p} + \frac{1}{q} = 1 \right) \) such that \( \phi \neq 0 \) and \( E[\phi \cdot e^\xi] = 0 \) for all \( \xi \in H(X) \).

Let \( \{\xi_n\} \) be as in Lemma 3.4. For any \( E \in \sigma(\xi_1, \xi_2, \cdots, \xi_n) \equiv B_n, \ \xi_i \in H(X), \ i = 1, 2, \cdots, n \), we have
\[
\int_E \phi \cdot e^\xi d\mu = \int_E E[\phi \cdot e^\xi | B_n] d\mu,
\]
for all \( \xi \in H(X) \). Hence we have
\[
\int_{C_{a, b}[0, T]} \phi \cdot e^{t_1 \xi_1 + \cdots + t_n \xi_n} d\mu = \int_{C_{a, b}[0, T]} e^{t_1 \xi_1 + \cdots + t_n \xi_n} E[\phi | B_n] d\mu = 0.
\]
By using the Stone-Weierstrass Theorem, one can show that the set \( \{e^{t_1 \xi_1 + \cdots + t_n \xi_n} | t_i \in \mathbb{R}, \xi_i \in H(X), i = 1, 2, \cdots, n \} \) is total in \( L^p(\xi_1, \cdots, \xi_n) \). This follows because
\[ E[\phi | B_n] = 0 \quad \text{a.e. for all } n. \] Since \( \phi \) is integrable, we have by a well-known martingale convergence theorem and Lemma 4.3
\[ \phi = \lim_{n \to \infty} E[\phi | B_n] = 0. \]

This contradiction completes the proof. \( \square \)

**Theorem 3.6 (Translation Theorem).** Let \( F : C_{a,b} [0,T] \to \mathbb{R} \) be a random variable such that \( F \in L^p(X) \), and \( \gamma(t) = \int_0^t g(s) db(s) \) for \( g \in L^2_b[0,T] \). Then
\[
E[F(\cdot + \gamma)] = E[F(\cdot) \int_0^T g dX(\cdot) - \frac{1}{2} \int_0^T g^2 db] = E[F(\cdot) \int_0^T g d\tilde{X}(\cdot) - \frac{1}{2} \int_0^T g^2 db].
\]

**Proof.** For \( f \in L^{1,2}_{a,b}[0,T] \), let \( F \in L^p(C_{a,b}[0,T]) \) be given by \( F(x) = e^{\int_0^T f(t) dX_t(x)} \). Then we have
\[
F(x + \gamma) = e^{\int_0^T f(t) dX_t(x + \gamma(t))} = e^{\int_0^T f(t) dX_t(x) + \int_0^T f(t) g(t) db(t)}.
\]
Hence it follows that
\[
E[F(\cdot + \gamma)] = e^{\int_0^T f(t) db(t) + \int_0^T f(t) g(t) db(t)}.
\]
On the other hand, we have
\[
E[F(\cdot) e^{\int_0^T g dX - \frac{1}{2} \int_0^T g^2 db}] = E[e^{\int_0^T (f+g) dX(\cdot) - \frac{1}{2} \int_0^T g^2 db}] = e^{\int_0^T f(t) db(t) + \int_0^T f(t) g(t) db(t)}.
\]
Hence the theorem is true for all functionals of the form \( e^{\int_0^T f dX} \), \( f \in L^{1,2}_{a,b}[0,T] \). By Theorem 3.5 the linear span of \( \{e^{\int_0^T f dX} | f \in L^{1,2}_{a,b}[0,T]\} \) is dense in \( L^p(X) \). Hence the theorem is proved. \( \square \)

### 4. Integration by parts formula for \( L^p \)-functionals

In this section, we define the directional derivative of \( L^p \)-functionals and then use the translation theorem obtained in Section 3 to establish an integration by parts formula for \( L^p \)-functionals.

**Definition 4.1.** Let \( p \geq 1 \). The directional derivative of a random variable \( F \in L^p(X) \) in the direction \( \gamma \), \( \gamma(t) = \int_0^t g(s) db(s) \) where \( g \in L^2_b[0,T] \), is defined as
\[
D_\gamma F(\cdot) = \lim_{t \to 0} \frac{1}{t} \{F(\cdot + t\gamma) - F(\cdot)\},
\]
where the limit is taken in the \( L^p(X) \)-sense. Let
\[
D^p(X) = \{F \in L^p(X)| D_\gamma F \text{ exists in } L^p(X) \text{ for all } \gamma \in C_{a,b}[0,T]\}.
\]

**Lemma 4.2.** Let \( \epsilon(h)(x) = e^{\int_0^T h dX_t(x) - \frac{1}{2} \int_0^T h^2 db}, h \in L^2_b[0,T], \) be a functional on \( C_b[0,T] \). Then \( \epsilon(h) \in L^p(X) \) for all \( p \geq 1 \). Moreover, for any \( p \geq 1 \),
\[
\lim_{t \to 0} \frac{1}{t} \{\epsilon(th) - 1\} = \int_0^T h d\tilde{X}
\]
where the limit is taken in the \( L^p(X) \)-sense.
Proof. Let \( h \in L^p_0[0, T] \). We first note that \( e^{\int_0^T h d\tilde{X}} \in L^p(X) \) for all \( p \geq 1 \). So we see that \( \|e(h)\|_p < \infty \). Let \( A > 0 \) be fixed. Define a function \( f \) on \( C_{a,b}[0,T] \times [-A, A] \) by

\[
\frac{d^2}{dt^2} f(x, t) = e^{\int_0^t h d\tilde{X}(s)} \cdot e^{\int_0^t h^2 db(s)} f(x, t).
\]

Then for almost all \( x \in C_{a,b}[0,T] \), \( f(x, \cdot) \) is differentiable on \((-A, A)\)

\[
\frac{\partial f}{\partial t}(x, t) = (\int_0^t h d\tilde{X}(s) - \int_0^t h^2 db(s)) f(x, t).
\]

If \(-A < t < A\), then by the mean value theorem, for each \( x \in C_{a,b}[0,T] \) there exists \( \theta \equiv \theta(x)\), \( 0 < \theta < 1 \), such that

\[
\frac{1}{t} (f(x, t) - f(x, 0)) = \frac{\partial f}{\partial t}(x, \theta t).
\]

From this fact and the inequalities \(|\int_0^T h d\tilde{X}| \leq e^{\int_0^T h d\tilde{X}}\) and \(|e^u| \leq e^{|u|}\) for \( u \in \mathbb{R} \), we have the following inequalities:

\[
\left\{ e(\int_0^T h d\tilde{X}) - 1 \right\} - \int_0^T h d\tilde{X}(s) \leq e^{\int_0^T h d\tilde{X}} - \int_0^T h d\tilde{X}(s) \leq \alpha e^{\beta |\int_0^T h d\tilde{X}|}
\]

for some constants \( \alpha > 0 \) and \( \beta > 0 \). Since \( \alpha e^{\beta |\int_0^T h d\tilde{X}|} \in L^p(X) \) for all \( p \geq 1 \), by the dominated convergence theorem,

\[
\lim_{t \to 0} E[\left\{ e(\int_0^T h d\tilde{X}) - 1 \right\} - \int_0^T h d\tilde{X}|^p] = 0
\]

and the lemma is proved. \( \square \)

**Theorem 4.3.** Let \( F \in \mathcal{D}^p(X) \) and \( \gamma(t) = \int_0^t g(u) db(u) \) with \( g \in L^{1, 2}_{a,b}[0,T] \). Then

\[
E[D(\gamma)F] = E[F \cdot \int_0^T g dX] - \int_0^T g da \cdot E[F].
\]

**Proof.** Since \( F \in \mathcal{D}^p \), \( \frac{1}{p} (F(\cdot + \gamma(t)) - F(\cdot)) \) converges to \( D(\gamma)F \) in \( L^p(X) \). Hence it converges to \( D(\gamma)F \) in \( L^1(X) \). From Theorem 3.6 and Lemma 4.2 it follows that

\[
E[D(\gamma)F] = \lim_{t \to 0} E[\frac{1}{t} (F(\cdot + t\gamma) - F(\cdot))]
\]

\[
= \lim_{t \to 0} E[F(\cdot) \left\{ \frac{1}{t} (\epsilon(tg) - 1) \right\}]
\]

\[
E[F \cdot \left( \int_0^T g dX - \int_0^T g da \right)]
\]

and the theorem is proved. \( \square \)

**Theorem 4.4** (Integration by parts formula). Let \( F \in \mathcal{D}^p(X) \), \( G \in \mathcal{D}^q(X) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Then we have, for \( \gamma(t) = \int_0^t g(u) db(u) \) with \( g \in L^{1, 2}_{a,b}[0,T] \),

\[
E[G \cdot D(\gamma)F] = E[F \cdot G \cdot \int_0^T g dX] - \int_0^T g da \cdot E[F \cdot G] - E[F \cdot D(\gamma)G].
\]
Proof. We first note that for $F \in L^p(X)$ and $G \in L^q(X)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have $F \cdot G \in L^1(X)$. Now we shall show that the limit
\[
\lim_{t \to 0} \frac{1}{t} \{ F(\cdot + t\gamma) \cdot G(\cdot + t\gamma) - F(\cdot) \cdot G(\cdot) \}
\]
exists in $L^1(X)$ and the limit is equal to $G \cdot \mathcal{D}_t F + F \cdot \mathcal{D}_t G$. This assertion follows from the following inequality:
\[
E[\frac{1}{t}(F(\cdot + t\gamma) \cdot G(\cdot + t\gamma) - F(\cdot) \cdot G(\cdot)) - (G \cdot \mathcal{D}_t F + F \cdot \mathcal{D}_t G)]
\leq \|\{\frac{1}{t}(F(\cdot + t\gamma) - F(\cdot)) - \mathcal{D}_t F\}G(\cdot + t\gamma)\|_1 + \|\mathcal{D}_t F\{G(\cdot + t\gamma) - G(\cdot)\}\|_1
\]
\[
+ \|F\|_p \|\frac{1}{t}(G(\cdot + t\gamma) - G(\cdot)) - \mathcal{D}_t G\|_q
\]
since all three terms on the right-hand side above go to zero as $t$ goes to zero. It then follows that
\[
(4.4) \quad E[\mathcal{D}_t (F \cdot G)] = E[G \cdot \mathcal{D}_t F] + E[F \cdot \mathcal{D}_t G].
\]
But this implies that $F \cdot G \in \mathcal{D}^1(X)$, and hence by Theorem 1.3,
\[
E[\mathcal{D}_t (F \cdot G)] = E[(F \cdot G) \cdot \int_0^T g dX] - \int_0^T g da E[F \cdot G].
\]
Therefore, we conclude that
\[
E[G \cdot \mathcal{D}_t F] = E[(F \cdot G) \cdot \int_0^T g dX] - \int_0^T g da E[F \cdot G] - E[F \cdot \mathcal{D}_t G]
\]
and the theorem is proved. \qed

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References


Department of Mathematics, Sogang University, Seoul 121-742, Korea
E-mail address: rouge@sogang.ac.kr

Current address: Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, New York 10012