A HEREDITARILY $\ell_1$ SUBSPACE OF $L_1$ WITHOUT THE SCHUR PROPERTY

M. M. POPOV

(Communicated by N. Tomczak-Jaegermann)

Abstract. Let $\infty > p_1 > p_2 > \cdots > 1$. We construct an easily determined 1-symmetric basic sequence in $(\sum_{n=1}^{\infty} \oplus \ell_{p_n})_1$, which spans a hereditarily $\ell_1$ subspace without the Schur property. An immediate consequence is the existence of hereditarily $\ell_1$ subspaces of $L_1$ without the Schur property.

In our notation and terminology we follow mainly [4]–[6]. Recall that an infinite dimensional Banach space $X$ is said to be hereditarily $Y$ if each infinite dimensional subspace $X_0$ of $X$ contains a further subspace $Y_0 \subseteq X_0$ which is isomorphic to $Y$. A Banach space $X$ is said to have the Schur property provided weak convergence of sequences in $X$ implies their norm convergence. It is well known that $\ell_1$ has the Schur property. As was shown by J. Bourgain and H. P. Rosenthal in [3], there exists a subspace of $L_1$ having the Schur property, but which does not embed in $\ell_1$. The first example of a hereditarily $\ell_1$ Banach space without the Schur property was constructed by J. Bourgain in [2]. Then P. Azimi and J. N. Hagler in [1] constructed a class of such spaces and investigated their further properties. We construct a class of subspaces of $L_1$ with the same properties. Actually, we make our construction in the sequence space

$$X = \left( \sum_{n=1}^{\infty} \oplus \ell_{p_n} \right)_1,$$

where $\infty > p_1 > p_2 > \cdots > 1$. Since $X$ embeds isometrically in $L_1$ for $p_1 \leq 2$ (this can be easily deduced from [6] p. 212), the construction gives examples of hereditarily $\ell_1$ subspaces of $L_1$ without the Schur property. Of course, each of them must be uncomplemented since every complemented subspace of $L_1$ without the Schur property contains an isomorphic copy of $\ell_2$ [7].

I would like to thank the referee for clarification of some arguments (the proof of Proposition 2 and especially for the idea of considering our example in the setting of the sequence space $X$, which makes the construction more general, clear and natural) and a number of corrections.
THE CONSTRUCTION

Fix a sequence of reals $p_1 > p_2 > \cdots > 1$, and consider the sequence space $X$ defined above. For each $n \geq 1$, by $\{e_{i,n}\}_{i=1}^\infty$ denote the unit vector basis of $\ell_{p_n}$ and by $\{e_i\}_{i=1}^\infty$ its natural copy in $X$:

$$e_{i,n} = (0, \cdots, 0, e_{i,n}, 0, \cdots) \in X.$$

Let $\delta_n > 0$ be such that $\sum_{n=1}^\infty \delta_n = 1$. For $i \geq 1$ put $z_i = \sum_{n=1}^\infty \delta_ne_{i,n}$. Evidently, $\|z_i\| = 1$ for each $i$.

Denote by $Z$ the closed linear span of $\{z_i\}_{i=1}^\infty$. We show that $Z$ possesses the desired properties. Since $X$ embeds isometrically in $L_1$ for $p_1 \leq 2$, then so does $Z$, and our construction can be applied to subspaces of $L_1$.

THE SKETCH

For each $I \subseteq \mathbb{N}$, by $P_I$ we denote the natural projection of $X$ onto $[e_{i,n} : i \in \mathbb{N}, n \in I]$ (i.e. with the kernel $[e_{i,n} : i \in \mathbb{N}, n \notin I]$). Of course, $\|P_I\| = \|Id - P_I\| = 1$. The main step here is to prove that $Z$ is hereditarily $\ell_1$. Given an infinite dimensional subspace $Z_0$ of $Z$, we find a sequence $\{x_n\}_{n=1}^\infty$ in $Z_0$ and a block basic subspace $\{u_k\}_{k=1}^\infty$ of $\{z_i\}_{i=1}^\infty$ having “almost disjoint supports” and which is close enough to $\{x_n\}_{n=1}^\infty$. (Here by “almost disjoint supports” we mean that for each $\varepsilon > 0$ there are disjoint subsets $I_0$ of $\mathbb{N}$ with $\|P_{I_0}u_k\| \geq (1 - \varepsilon)\|u_k\|$.) Hence $\{x_n\}_{n=1}^\infty$ contains a subsequence equivalent to the unit vector basis of $\ell_1$.

THE PROOF

**Proposition 1.** For all scalars $\{a_i\}_{i=1}^m$ and each permutation of integers $\tau : \mathbb{N} \to \mathbb{N}$ one has

$$\left\| \sum_{i=1}^m a_i z_{\tau(i)} \right\| = \sum_{n=1}^\infty \delta_n \left( \sum_{i=1}^m |a_i| p_n \right)^{\frac{1}{p_n}}.$$

Hence, $\{z_i\}_{i=1}^\infty$ is a 1-symmetric basic sequence.

**Proof.** The proof is straightforward:

$$\left\| \sum_{i=1}^m a_i z_{\tau(i)} \right\| = \sum_{n=1}^\infty \delta_n \left( \sum_{i=1}^m |a_i| p_n \right)^{\frac{1}{p_n}}.$$  

Thus, if a series $\sum_{i=1}^\infty a_i z_i$ converges, then $\sum_{i=1}^\infty |a_i| p_n < \infty$ for each $n$.

**Proposition 2.** $\{z_i\}_{i=1}^\infty$ tends weakly to zero.

**Proof.** Let $z^* \in Z^*$, $\|z^*\| = 1$, and $\varepsilon > 0$. Pick an integer $N$ such that

$$\sum_{n=N+1}^\infty \delta_n < \frac{\varepsilon}{2}.$$

Since $e_{i,n} \rightharpoonup 0$ weakly as $i \to \infty$ for each $n \geq 1$, we may pick $K \in \mathbb{N}$ so that if $n = 1, \cdots, N$ and $k \geq K$, then

$$|z^*(e_{k,n})| < \frac{\varepsilon}{2}.$$
Since $\sum_{n=1}^{N} \delta_n < 1$, we have that for $k \geq K$,

$$|z^*(z_k)| = \left|z^*\left(\sum_{n=1}^{\infty} \delta_n e_{k,n}\right)\right| < \sum_{n=1}^{N} \delta_n |z^*(e_{k,n})| + \sum_{n=N+1}^{\infty} \delta_n < \varepsilon.$$ 

\[\square\]

**Lemma 3.** Let $Z_0$ be an infinite dimensional subspace of $Z$, $n, m, j \in \mathbb{N}$ ($n > 1$) and $\varepsilon > 0$. Then there are $\{x_i\}_{i=1}^{m} \subset Z_0$ and $\{u_i\}_{i=1}^{m} \subset Z$ of the form

$$u_i = \sum_{s=j_i-1}^{j_i+1} a_{i,s} z_s \text{ where } j_1 < j_2 < \ldots < j_{m+1}$$

such that

$$\sum_{s=j_i-1}^{j_i+1} |a_{i,s}|^{p-1} = 1 \text{ and } \|u_i - x_i\| < \frac{\varepsilon}{m} \|u_i\|$$

for each $i = 1, \ldots, m$.

**Proof.** Put $Z_1 = Z_0 \cap [z_i]_{i=j+1}^{\infty}$. Since $Z_0$ is infinite dimensional and $[z_i]_{i=j+1}^{\infty}$ has finite codimension in $Z$, $Z_1$ is infinite dimensional as well. Put $j_1 = j$ and choose any

$$\mathbf{x}_1 = \sum_{s=j_1-1}^{\infty} \mathbf{x}_{1,s} z_s \in Z_1 \setminus \{0\}.$$

Without lost of generality we may assume that

$$\sum_{s=j_1-1}^{\infty} |\mathbf{x}_{1,s}|^{p-1} = 1$$

(otherwise we multiply $\mathbf{x}_1$ by a suitable number). Then choose $j_2 > j_1$ so that for

$$\mathbf{y}_1 = \sum_{s=j_2-1}^{j_2} \mathbf{y}_{1,s} z_s$$

we have

$$\|\mathbf{y}_1 - \mathbf{x}_1\| < \frac{\varepsilon}{4m}, \quad \lambda_1 = \left(\sum_{s=j_1-1}^{j_2} |\mathbf{y}_{1,s}|^{p-1}\right)^{\frac{1}{p-1}} \geq \frac{1}{2} \text{ and } \|\mathbf{y}_1\| \geq \frac{\|\mathbf{x}_1\|}{2}.$$

Hence,

$$\|\mathbf{y}_1 - \mathbf{x}_1\| < \frac{\varepsilon}{2m}.$$ 

Now put $a_{1,s} = \lambda_1^{-1} \mathbf{y}_{1,s}$, $x_1 = \lambda_1^{-1} \mathbf{x}_1$ and $u_1 = \lambda_1^{-1} \mathbf{y}_1$. Then

$$\sum_{s=j_1-1}^{j_2} |a_{1,s}|^{p-1} = \frac{1}{\lambda_1^{p-1}} \sum_{s=j_1-1}^{j_2} |\mathbf{y}_{1,s}|^{p-1} = 1$$

and

$$\|u_1 - x_1\| = \frac{1}{\lambda_1} \|\mathbf{y}_1 - \mathbf{x}_1\| < \frac{\varepsilon}{\lambda_1 m} \leq \frac{\varepsilon}{m} \leq \frac{\varepsilon}{m} \|u_1\|.$$ 

Continuing the procedure in the obvious manner, we construct the desired sequences. \[\square\]
For $n \in \mathbb{N}$ denote $Q_n = P_{\{n, n+1, \ldots\}}$.

**Lemma 4.** Let $Z_0$ be an infinite dimensional subspace of $Z$, $j, n \in \mathbb{N}$ and $\varepsilon > 0$. There exist an $x \in Z_0$, $x \neq 0$ and a $u \in Z$ of the form

$$u = \sum_{i=j+1}^{l} a_i z_i,$$

where $l > j$,

such that

(i) $\|Q_n u\| \geq (1 - \varepsilon) \|u\|$;

(ii) $\|x - u\| < \varepsilon \|u\|$.

**Proof.** Choose $m$ so that

$$\frac{1}{\delta_n} \left( \frac{1}{m^{\frac{1}{p_n - 1}}} - \frac{1}{p_n} \right) < \varepsilon.$$

Using Lemma 3 choose $\{x_i\}_{i=1}^{m} \subset Z_0$ and $\{u_i\}_{i=1}^{m} \subset Z$ to satisfy the claims of the lemma and put

$$x = \sum_{i=1}^{m} x_i \quad \text{and} \quad u = \sum_{i=1}^{m} u_i.$$

First, we prove (ii). Since $\{z_s\}_{s=1}^{\infty}$ is 1-symmetric, then $\|u_i\| \leq \|u\|$ for $i = 1, \ldots, m$ and

$$\|x - u\| \leq \sum_{i=1}^{m} \|x_i - u_i\| < \sum_{i=1}^{m} \varepsilon \|u_i\| \leq \sum_{i=1}^{m} \varepsilon \|u\| = \varepsilon \|u\|.$$

To prove (i), we first show that $\|u\| - \|Q_n u\| < m^{\frac{1}{p_n - 1}}$. To see this,

$$\|u\| - \|Q_n u\| = \sum_{k=1}^{n-1} \delta_k \| P_{\{k\}} u\| = \sum_{k=1}^{n-1} \delta_k \| \sum_{s=j+1}^{j+i} a_{i,s} e_{s,k}\|$$

$$= \sum_{k=1}^{n-1} \delta_k \left( \sum_{i=1}^{m} \sum_{s=j+1}^{j+i} |a_{i,s}|^{p_k} \right)^{\frac{1}{p_k}} \leq \sum_{k=1}^{n-1} \delta_k \left( \sum_{i=1}^{m} \sum_{s=j+1}^{j+i} |a_{i,s}|^{p_{n-1}} \right)^{\frac{1}{p_{n-1}}}$$

$$= \sum_{k=1}^{n-1} \delta_k \left( \sum_{i=1}^{m} \right)^{\frac{1}{p_{n-1}}} = m^{\frac{1}{p_{n-1}}} \sum_{k=1}^{n-1} \delta_k < m^{\frac{1}{p_{n-1}}}.$$

On the other hand,

$$\|u\| = \sum_{k=1}^{\infty} \delta_k \| \sum_{i=1}^{m} \sum_{s=j+1}^{j+i} a_{i,s} e_{s,k}\| \geq \delta_n \| \sum_{i=1}^{m} \sum_{s=j+1}^{j+i} a_{i,s} e_{s,n}\|$$

$$= \delta_n \left( \sum_{i=1}^{m} \sum_{s=j+1}^{j+i} |a_{i,s}|^{p_n} \right)^{\frac{1}{p_n}} \geq \delta_n \left( \sum_{i=1}^{m} \left( \sum_{s=j+1}^{j+i} |a_{i,s}|^{p_{n-1}} \right)^{\frac{1}{p_{n-1}}} \right)^{\frac{1}{p_n}}$$

$$= \delta_n \left( \sum_{i=1}^{m} \right)^{\frac{1}{p_n}} = \delta_n m^{\frac{1}{p_n}}.$$

Hence

$$1 - \frac{\|Q_n u\|}{\|u\|} \leq \frac{1}{\delta_n} m^{\frac{1}{p_{n-1}}} - \frac{1}{p_n} < \varepsilon,$$

and $\|Q_n u\| \geq (1 - \varepsilon) \|u\|$.

\[\square\]
Theorem 5. $Z$ is hereditarily $\ell_1$ and does not have the Schur property.

Proof. By Proposition 2, $Z$ does not have the Schur property. To prove that $Z$ is hereditarily $\ell_1$, let $Z_0$ be an infinite dimensional subspace of $Z$. Using Lemma 4, construct inductively sequences $\{x_s\}_{s=1}^{\infty} \subset Z_0$, $\{u_s\}_{s=1}^{\infty} \subset Z$ of the form

$$u_s = \sum_{i=j_s+1}^{j_{s+1}} a_i z_i,$$

where $j_1 < j_2 < \ldots$ and $\|u_s\| = 1$ and a sequence $1 \leq n_1 < n_2 < \ldots$ so that

(i) $\|Q_{n_s} u_s\| \geq \frac{7}{8}$,

(ii) $\|u_s - x_s\| \leq 2^{-s-1}$,

(iii) $\|Q_{n_{s+1}} u_s\| < \frac{1}{8}$.

To see that this can be done, let $j_1 = n_1 = 1$ and $\varepsilon_1 = \frac{1}{8}$. Choose by Lemma 4 an $x_1 \in Z \setminus \{0\}$ and

$$u_1 = \sum_{i=j_1+1}^{j_2} a_i z_i$$

such that $\|u_1\| = 1$, $\|Q_{n_1} u_1\| \geq 1 - \varepsilon_1 = \frac{7}{8}$ and $\|x_1 - u_1\| < \varepsilon_1 < \frac{1}{2}$. Then choose $n_2 > n_1$ so that $\|Q_{n_1} u_1\| < \frac{1}{2}$. Continuing the procedure in the obvious manner, we construct the desired sequences.

Evidently, (i) yields

$$\|u_s - Q_{n_s} u_s\| \leq \frac{1}{8}.$$

Conditions (i′) and (iii) imply the equivalence of $\{u_s\}_{s=1}^{\infty}$ to the unit vector basis of $\ell_1$. Indeed, for $s \geq 1$ put $v_s = Q_{n_s} u_s - Q_{n_{s+1}} u_s$. Then $\|v_s\| \geq \frac{3}{4}$, and for each sequence of scalars $\{a_s\}_{s=1}^{m}$ one has

$$\left\| \sum_{s=1}^{m} a_s v_s \right\| = \sum_{s=1}^{m} |a_s| \|v_s\| \geq \frac{3}{4} \sum_{s=1}^{m} |a_s|,$$

and hence

$$\left\| \sum_{s=1}^{m} a_s u_s \right\| \geq \left\| \sum_{s=1}^{m} a_s v_s \right\| - \left\| \sum_{s=1}^{m} a_s (u_s - v_s) \right\| \geq \frac{3}{4} \sum_{s=1}^{m} |a_s| - \left\| \sum_{s=1}^{m} a_s (u_s - Q_{n_s} u_s) \right\| + \left\| \sum_{s=1}^{m} a_s Q_{n_{s+1}} u_s \right\| \geq \frac{3}{4} \sum_{s=1}^{m} |a_s| - \frac{1}{8} \sum_{s=1}^{m} |a_s| - \frac{1}{8} \sum_{s=1}^{m} |a_s| = \frac{1}{2} \sum_{s=1}^{m} |a_s|.$$

Thus, $\{u_s\}_{s=1}^{\infty}$ is equivalent to the unit vector basis of $\ell_1$ and so is $\{x_s\}_{s=1}^{\infty}$ by (ii) and the stability of basic sequences property [5], p. 5).

Since for $p_1 \leq 2$ the space $X$ embeds isometrically in $L_1$ (this can be easily deduced from [3], p. 212), we obtain

Corollary 6. $L_1$ contains hereditarily $\ell_1$ subspaces without the Schur property.
Finally, we remark that it does not matter whether $p_n$ tends to 1 or not. Furthermore, if we repeat the construction with the space $X$ replaced by $X_p = \left( \sum_{n=1}^{\infty} \ell_{p_n} \right)_p$, with $1 < p < \infty$ and an arbitrary decreasing sequence $p_n$ (not necessarily satisfying $p \leq \inf_n p_n$), then $Z$ becomes hereditarily $\ell_p$, but in this case it is clear that $Z$ does not have the Schur property.

References