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A HEREDITARILY ℓ_1 SUBSPACE OF L_1 WITHOUT THE SCHUR PROPERTY

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ABSTRACT. Let $\infty > p_1 > p_2 > \cdots > 1$. We construct an easily determined 1-symmetric basic sequence in $\left(\sum_{n=1}^{\infty} \oplus \ell_{p_n}\right)_1$, which spans a hereditarily ℓ_1 subspace without the Schur property. An immediate consequence is the existence of hereditarily ℓ_1 subspaces of L_1 without the Schur property.

In our notation and terminology we follow mainly [4]–[6]. Recall that an infinite dimensional Banach space X is said to be hereditarily Y if each infinite dimensional subspace X_0 of X contains a further subspace $Y_0 \subseteq X_0$ which is isomorphic to Y. A Banach space X is said to have the Schur property provided weak convergence of sequences in X implies their norm convergence. It is well known that ℓ_1 has the Schur property. As was shown by J. Bourgain and H. P. Rosenthal in [3], there exists a subspace of L_1 having the Schur property, but which does not embed in ℓ_1. The first example of a hereditarily ℓ_1 Banach space without the Schur property was constructed by J. Bourgain in [2]. Then P. Azimi and J. N. Hagler in [1] constructed a class of such spaces and investigated their further properties. We construct a class of subspaces of L_1 with the same properties. Actually, we make our construction in the sequence space

$$X = \left(\sum_{n=1}^{\infty} \oplus \ell_{p_n}\right)_1,$$

where $\infty > p_1 > p_2 > \cdots > 1$. Since X embeds isometrically in L_1 for $p_1 \leq 2$ (this can be easily deduced from [6, p. 212]), the construction gives examples of hereditarily ℓ_1 subspaces of L_1 without the Schur property. Of course, each of them must be uncomplemented since every complemented subspace of L_1 without the Schur property contains an isomorphic copy of ℓ_2 [7].

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The construction

Fix a sequence of reals $p_1 > p_2 > \cdots > 1$, and consider the sequence space X defined above. For each $n \geq 1$, by $\{\overline{e}_{i,n}\}_{i=1}^{\infty}$ denote the unit vector basis of ℓ_{p_n} and by $\{e_{i,n}\}_{i=1}^{\infty}$ its natural copy in X:

$$e_{i,n} = (\underbrace{0, \cdots, 0}_{n-1}, \overline{e}_{i,n}, 0, \cdots) \in X.$$

Let $\delta_n > 0$ be such that $\sum\limits_{n=1}^{\infty} \delta_n = 1$. For $i \geq 1$ put $z_i = \sum\limits_{n=1}^{\infty} \delta_n e_{i,n}$. Evidently, $\|z_i\| = 1$ for each i.

Denote by Z the closed linear span of $\{z_i\}_{i=1}^{\infty}$. We show that Z posseses the desired properties. Since X embeds isometrically in L_1 for $p_1 \leq 2$, then so does Z, and our construction can be applied to subspaces of L_1.

The sketch

For each $I \subseteq \mathbb{N}$, by P_I we denote the natural projection of X onto $[e_{i,n} : \ i \in \mathbb{N}, \ n \in I]$ (i.e. with the kernel $[e_{i,n} : \ i \in \mathbb{N}, \ n \notin I]$). Of course, $\|P_I\| = \|Id - P_I\| = 1$. The main step here is to prove that Z is hereditarily ℓ_1. Given an infinite dimensional subspace Z_0 of Z, we find a sequence $\{x_s\}_{s=1}^{\infty}$ in Z_0 and a block basic subsequence $\{u_s\}_{s=1}^{\infty}$ of $\{z_i\}_{i=1}^{\infty}$ having "almost disjoint supports" and which is close enough to $\{x_s\}_{s=1}^{\infty}$. (Here by "almost disjoint supports" we mean that for each $\varepsilon > 0$ there are disjoint subsets I_s of $\mathbb{N}$ with $\|P_{I_s} u_s\| \geq (1-\varepsilon)\|u_s\|$.) Hence $\{x_s\}_{s=1}^{\infty}$ contains a subsequence equivalent to the unit vector basis of ℓ_1.

The proof

Proposition 1. *For all scalars $\{a_i\}_{i=1}^{m}$ and each permutation of integers $\tau : \mathbb{N} \to \mathbb{N}$ one has*

$$\| \sum_{i=1}^{m} a_i z_{\tau(i)} \| = \sum_{n=1}^{\infty} \delta_n \Big(\sum_{i=1}^{m} |a_i|^{p_n} \Big)^{\frac{1}{p_n}}.$$

Hence, $\{z_i\}_{i=1}^{\infty}$ is a 1-symmetric basic sequence.

Proof. The proof is straightforward:

$$\| \sum_{i=1}^{m} a_i z_{\tau(i)} \| = \sum_{n=1}^{\infty} \delta_n \, \| \sum_{i=1}^{m} a_i e_{\tau(i),n} \| = \sum_{n=1}^{\infty} \delta_n \Big(\sum_{i=1}^{m} |a_i|^{p_n} \Big)^{\frac{1}{p_n}}.$$

□

Thus, if a series $\sum\limits_{i=1}^{\infty} a_i z_i$ converges, then $\sum\limits_{i=1}^{\infty} |a_i|^{p_n} < \infty$ for each n.

Proposition 2. *$\{z_i\}_{i=1}^{\infty}$ tends weakly to zero.*

Proof. Let $z^* \in Z^*, \ \|z^*\| = 1$, and $\varepsilon > 0$. Pick an integer N such that

$$\sum_{n=N+1}^{\infty} \delta_n < \frac{\varepsilon}{2}.$$

Since $e_{i,n} \longrightarrow 0$ weakly as $i \longrightarrow \infty$ for each $n \geq 1$, we may pick $K \in \mathbb{N}$ so that if $n = 1, \cdots, N$ and $k \geq K$, then

$$|z^*(e_{k,n})| < \frac{\varepsilon}{2}.$$

Since $\sum_{n=1}^{N} \delta_n < 1$, we have that for $k \geq K$,

$$|z^*(z_k)| = \Big|z^*\Big(\sum_{n=1}^{\infty} \delta_n e_{k,n}\Big)\Big| < \sum_{n=1}^{N} \delta_n |z^*(e_{k,n})| + \sum_{n=N+1}^{\infty} \delta_n < \varepsilon.$$

□

Lemma 3. *Let Z_0 be an infinite dimensional subspace of Z, $n, m, j \in \mathbb{N}$ $(n > 1)$ and $\varepsilon > 0$. Then there are $\{x_i\}_{i=1}^m \subset Z_0$ and $\{u_i\}_{i=1}^m \subset Z$ of the form*

$$u_i = \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} z_s \ \ \textit{where}\ \ j = j_1 < j_2 < ... < j_{m+1}$$

such that

$$\sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}} = 1 \ \ \textit{and}\ \ \|u_i - x_i\| < \frac{\varepsilon}{m} \|u_i\|$$

for each $i = 1, ..., m$.

Proof. Put $Z_1 = Z_0 \cap \ [z_i]_{i=j+1}^{\infty}$. Since Z_0 is infinite dimensional and $[z_i]_{i=j+1}^{\infty}$ has finite codimension in Z, Z_1 is infinite dimensional as well. Put $j_1 = j$ and choose any

$$\overline{x}_1 = \sum_{s=j_1+1}^{\infty} \overline{a}_{1,s} z_s \in Z_1 \setminus \{0\}.$$

Without lost of generality we may assume that

$$\sum_{s=j_1+1}^{\infty} |\overline{a}_{1,s}|^{p_{n-1}} = 1$$

(otherwise we multiply $\overline{x}_1$ by a suitable number). Then choose $j_2 > j_1$ so that for

$$\overline{u}_1 = \sum_{s=j_1+1}^{j_2} \overline{a}_{1,s} z_s$$

we have

$$\|\overline{u}_1 - \overline{x}_1\| < \frac{\varepsilon \|\overline{x}_1\|}{4m}, \ \ \lambda_1 = \Big(\sum_{s=j_1+1}^{j_2} |\overline{a}_{1,s}|^{p_{n-1}} \Big)^{\frac{1}{p_{n-1}}} \geq \frac{1}{2} \quad \text{and} \quad \|\overline{u}_1\| \geq \frac{\|\overline{x}_1\|}{2}.$$

Hence,

$$\|\overline{u}_1 - \overline{x}_1\| < \frac{\varepsilon \|\overline{u}_1\|}{2m}.$$

Now put $a_{1,s} = \lambda_1^{-1} \overline{a}_{1,s}$, $x_1 = \lambda_1^{-1} \overline{x}_1$ and $u_1 = \lambda_1^{-1} \overline{u}_1$. Then

$$\sum_{s=j_1+1}^{j_2} |a_{1,s}|^{p_{n-1}} = \frac{1}{\lambda_1^{p_{n-1}}} \sum_{s=j_1+1}^{j_2} |\overline{a}_{1,s}|^{p_{n-1}} = 1$$

and

$$\|u_1 - x_1\| = \frac{1}{\lambda_1} \|\overline{u}_1 - \overline{x}_1\| < \frac{\varepsilon \|\overline{u}_1\|}{2\lambda_1 m} \leq \frac{\varepsilon \|\overline{u}_1\|}{m} \leq \frac{\varepsilon \|u_1\|}{m}.$$

Continuing the procedure in the obvious manner, we construct the desired sequences. □

For $n \in \mathbb{N}$ denote $Q_n = P_{\{n,\, n+1,\, \cdots\}}$.

Lemma 4. *Let Z_0 be an infinite dimensional subspace of Z, $j, n \in \mathbb{N}$ and $\varepsilon > 0$. There exist an $x \in Z_0$, $x \neq 0$ and a $u \in Z$ of the form*

$$u = \sum_{i=j+1}^{l} a_i z_i, \quad \text{where } l > j,$$

such that
(i) $\|Q_n u\| \geq (1-\varepsilon)\,\|u\|$;
(ii) $\|x - u\| < \varepsilon\,\|u\|$.

Proof. Choose m so that

$$\frac{1}{\delta_n}\ m^{\frac{1}{p_{n-1}} - \frac{1}{p_n}} < \varepsilon.$$

Using Lemma 3 choose $\{x_i\}_{i=1}^m \subset Z_0$ and $\{u_i\}_{i=1}^m \subset Z$ to satisfy the claims of the lemma and put

$$x = \sum_{i=1}^m x_i \quad \text{and} \quad u = \sum_{i=1}^m u_i.$$

First, we prove (ii). Since $\{z_s\}_{s=1}^\infty$ is 1-symmetric, then $\|u_i\| \leq \|u\|$ for $i = 1, ..., m$ and

$$\|x - u\| \leq \sum_{i=1}^m \|x_i - u_i\| < \sum_{i=1}^m \frac{\varepsilon\,\|u_i\|}{m} \leq \sum_{i=1}^m \frac{\varepsilon\,\|u\|}{m} = \varepsilon\,\|u\|.$$

To prove (i), we first show that $\|u\| - \|Q_n u\| < m^{\frac{1}{p_{n-1}}}$. To see this,

$$\|u\| - \|Q_n u\| = \sum_{k=1}^{n-1} \delta_k\,\|\sum_{i=1}^m P_{\{k\}} u_i\| = \sum_{k=1}^{n-1} \delta_k\,\|\sum_{i=1}^m \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} e_{s,k}\|$$

$$= \sum_{k=1}^{n-1} \delta_k \Big(\sum_{i=1}^m \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_k}\Big)^{\frac{1}{p_k}} \leq \sum_{k=1}^{n-1} \delta_k \Big(\sum_{i=1}^m \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}}\Big)^{\frac{1}{p_{n-1}}}$$

$$= \sum_{k=1}^{n-1} \delta_k \Big(\sum_{i=1}^m 1\Big)^{\frac{1}{p_{n-1}}} = m^{\frac{1}{p_{n-1}}} \sum_{k=1}^{n-1} \delta_k < m^{\frac{1}{p_{n-1}}}.$$

On the other hand,

$$\|u\| = \sum_{k=1}^{\infty} \delta_k\,\|\sum_{i=1}^m \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} e_{s,k}\| \geq \delta_n\,\|\sum_{i=1}^m \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} e_{s,n}\|$$

$$= \delta_n \Big(\sum_{i=1}^m \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_n}\Big)^{\frac{1}{p_n}} \geq \delta_n \Big(\sum_{i=1}^m \Big(\sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}}\Big)^{\frac{p_n}{p_{n-1}}}\Big)^{\frac{1}{p_n}}$$

$$= \delta_n \Big(\sum_{i=1}^m 1\Big)^{\frac{1}{p_n}} = \delta_n m^{\frac{1}{p_n}}.$$

Hence

$$1 - \frac{\|Q_n u\|}{\|u\|} \leq \frac{1}{\delta_n}\ m^{\frac{1}{p_{n-1}} - \frac{1}{p_n}} < \varepsilon,$$

and $\|Q_n u\| \geq (1-\varepsilon)\,\|u\|$. $\square$

Theorem 5. *Z is hereditarily ℓ_1 and does not have the Schur property.*

Proof. By Proposition 2, Z does not have the Schur property. To prove that Z is hereditarily ℓ_1, let Z_0 be an infinite dimensional subspace of Z. Using Lemma 4, construct inductively sequences $\{x_s\}_{s=1}^\infty \subset Z_0$, $\{u_s\}_{s=1}^\infty \subset Z$ of the form

$$u_s = \sum_{i=j_s+1}^{j_{s+1}} a_i z_i,$$

where $j_1 < j_2 < ...$ and $\|u_s\| = 1$ and a sequence $1 \le n_1 < n_2 < \cdots$ so that

$$\text{(i)} \qquad \|Q_{n_s} u_s\| \ge \frac{7}{8}\,,$$

$$\text{(ii)} \qquad \|u_s - x_s\| \le 2^{-s-1}\,,$$

$$\text{(iii)} \qquad \|Q_{n_{s+1}} u_s\| < \frac{1}{8}\,.$$

To see that this can be done, let $j_1 = n_1 = 1$ and $\varepsilon_1 = \frac{1}{8}$. Choose by Lemma 4 an $x_1 \in Z \setminus \{0\}$ and

$$u_1 = \sum_{i=j_1+1}^{j_2} a_i z_i$$

such that $\|u_1\| = 1$, $\|Q_{n_1} u_1\| \ge 1 - \varepsilon_1 = \frac{7}{8}$ and $\|x_1 - u_1\| < \varepsilon_1 < \frac{1}{4}$. Then choose $n_2 > n_1$ so that $\|Q_{n_2} u_1\| < \frac{1}{8}$. Continuing the procedure in the obvious manner, we construct the desired sequences.

Evidently, (i) yields

$$\text{(i}'\text{)} \qquad \|u_s - Q_{n_s} u_s\| \le \frac{1}{8}.$$

Conditions (i$'$) and (iii) imply the equivalence of $\{u_s\}_{s=1}^\infty$ to the unit vector basis of ℓ_1. Indeed, for $s \ge 1$ put $v_s = Q_{n_s} u_s - Q_{n_{s+1}} u_s$. Then $\|v_s\| \ge \frac{3}{4}$, and for each sequence of scalars $\{a_s\}_{s=1}^m$ one has

$$\|\sum_{s=1}^m a_s v_s\| = \sum_{s=1}^m |a_s|\,\|v_s\| \ge \frac{3}{4}\sum_{s=1}^m |a_s|,$$

and hence

$$\|\sum_{s=1}^m a_s u_s\| \ge \|\sum_{s=1}^m a_s v_s\| - \|\sum_{s=1}^m a_s(u_s - v_s)\| \ge \frac{3}{4}\sum_{s=1}^m |a_s| - \|\sum_{s=1}^m a_s(u_s - Q_{n_s} u_s) + \sum_{s=1}^m a_s Q_{n_{s+1}} u_s\| \ge \frac{3}{4}\sum_{s=1}^m |a_s| - \frac{1}{8}\sum_{s=1}^m |a_s| - \frac{1}{8}\sum_{s=1}^m |a_s| = \frac{1}{2}\sum_{s=1}^m |a_s|.$$

Thus, $\{u_s\}_{s=1}^\infty$ is equivalent to the unit vector basis of ℓ_1 and so is $\{x_s\}_{s=1}^\infty$ by (ii) and the stability of basic sequences property [5, p. 5]. □

Since for $p_1 \le 2$ the space X embeds isometrically in L_1 (this can be easily deduced from [6, p. 212]), we obtain

Corollary 6. *L_1 contains hereditarily ℓ_1 subspaces without the Schur property.*

Finally, we remark that it does not matter whether p_n tends to 1 or not. Furthermore, if we repeat the construction with the space X replaced by $X_p = \left(\sum_{n=1}^{\infty} \oplus \ell_{p_n}\right)_p$, with $1 < p < \infty$ and an arbitrary decreasing sequence p_n (not necessarily satisfying $p \le \inf_n p_n$), then Z becomes hereditarily ℓ_p, but in this case it is clear that Z does not have the Schur property.

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