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ON THE CLASSES OF $\mathcal{L}^\lambda$, QUASI-$\mathcal{L}^E$ AND $\mathcal{L}^{\lambda,g}$ SPACES

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Abstract. The two better-known ways of understanding the notion of local unconditional structure allow us to define successive extensions of the well-known class of the $\mathcal{L}^p$ spaces of Lindenstrauss and Pelczyǹski. This paper also studies stability properties of these classes under ultrapowers, biduals and complemented subspaces.

1. Introduction

The ℓ_p spaces play a central role in the classical theory of operators and tensor norms and in this setting, Lindenstrauss and Pelczyǹski [12] in 1968 introduced the class of the $\mathcal{L}^p$ spaces. Although the definition of the class $\mathcal{L}^p$ is finite dimensional, it has many implications in the global structure of the involved spaces, and it is clear that in general the nice behavior of the $\mathcal{L}^p$ spaces is a consequence of the good properties of the ℓ_p spaces. Anyway, if we extend the class replacing the space ℓ_p for another sequence space λ, some new problems appear. Throughout this paper we will answer these types of questions. As we will see the natural extension of the $\mathcal{L}^p$ spaces denoted here by $\mathcal{L}^\lambda$ is not satisfactory if λ is not an ℓ_p space, and this draws our attention toward two successive extensions of the $\mathcal{L}^\lambda$ spaces, the classes of quasi-$\mathcal{L}^E$ and the class of generalized $\mathcal{L}^\lambda$ spaces here denoted by $\mathcal{L}^{\lambda,g}$. These definitions are also of local type and we investigate some interesting aspects of its local and global structure. In this paper we restrict our attention to the study of the relationships between some local properties of λ and the stability of the classes under ultrapowers, biduals, and complemented subspaces. These properties have been applied in [15] in the study of the operator ideals associated to the tensor norm defined by a sequence space λ.

The notation is standard. Let ω be the vector space of all real sequences and φ its subspace of sequences with finitely many nonzero coordinates. A Banach sequence space is an ideal in ω provided with a norm which makes it a Banach lattice. A Banach sequence space λ will be called regular whenever the sequence of all $e_i := (\delta_{ij})_j$ forms an unconditional Schauder basis in λ. Observe that every infinite dimensional Banach sequence space λ has a solid and regular subspace $\lambda_r := \overline{\varphi}^\lambda$ such that λ is regular if and only if $\lambda = \lambda_r$. In the paper $S_k(\lambda)$ represents the sectional subspace of λ generated for $\{e_i, i = 1, .., k\}$ and if λ is regular,

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$P_k : \lambda \to S_k(\lambda)$ represents the canonical projection map. A Banach space X is said to contain $S_k(\lambda)$ uniformly provided there is a sequence $G_n, n \in \mathbb{N}$ of subspaces of X for which $\sup_{n\in\mathbb{N}} d(G_n, S_n(\lambda)) < \infty$, where $d(E, F)$ represents the Banach-Mazur distance between the Banach spaces E and F. Recall that a Banach space E is said to be finitely representable in a Banach space F if for every finite dimensional subspace M of E and every $\varepsilon > 0$ there is a finite dimensional subspace N of F such that $d(M, N) \leq 1 + \varepsilon$. *With the same notation if there is $b > 0$ not depending on M such that $d(M, N) \leq b$ we say in brief that X is b-finitely representable in F.* Finally if E is a subspace of F, we denote by $I_{E,F}$ the corresponding inclusion map.

2. DPR-local unconditional structure and ultrapowers: The class of the $\mathcal{L}^\lambda$ spaces

A notion of local unconditional structure was introduced by Dubinsky, Pelczyǹski and Rosenthal in [2] motivated by the fact that certain classical Banach spaces admit a family of finite dimensional subspaces having unconditional basis with respect to the same constant, which is dense in the family of all its finite dimensional subspaces, or equivalently:

Definition 2.1. We say that a Banach space X has *DPR-local unconditional structure* if there is $c \geq 1$ such that for each finite dimensional subspace F of X there are a finite dimensional subspace Y containing F and a finite dimensional Banach lattice L such that $d(Y, L) \leq c$.

Remark 2.2. 1) Every Banach lattice has DPR-local unconditional structure for all $c > 1$.

2) Every finite dimensional Banach space has DPR-local unconditional structure since it is isomorphic to a Banach lattice, but, as we can read in [1], it is the uniform bound on the isomorphisms of all finite dimensional subspaces of an infinite dimensional Banach space X which restricts the class of the Banach spaces having local unconditional structure.

3) Moreover from [8], a Banach space with DPR-local unconditional structure is either super-reflexive or it contains uniformly isomorphic copies of $S_n(\ell_1)$ or $S_n(\ell_\infty)$ for all $n \in \mathbb{N}$.

A strong version of DPR-local unconditional structure with respect to the sections of ℓ_p characterizes the intensively studied class of the $\mathcal{L}^p$ spaces of Lindenstrauss and Pelczyǹski [12], see also [13], and we extend this notion to general Banach sequence spaces:

Definition 2.3. Let λ be a Banach sequence space. We say that a Banach space X has $DPR - S_k(\lambda)$*-local unconditional structure* (or DPR-local unconditional structure with respect to the sections of λ) if there exists a real constant $c \geq 1$ such that for every finite dimensional subspace F of X, there are a finite dimensional subspace E containing F and $n \in \mathbb{N}$ such that $d(E, S_n(\lambda)) \leq c$.

Definition 2.4. Given a Banach sequence space λ, a Banach space X is said to be an $\mathcal{L}^\lambda$ space if it has $DPR - S_k(\lambda)$-local unconditional structure.

The ℓ_p spaces are in $\mathcal{L}^p$ and in general if λ is a regular Banach sequence space, as an immediate consequence of theorem 6, §16 in [11], λ is an $\mathcal{L}^\lambda$ space.

The standard reference on ultraproducts of Banach spaces is [6], and we refer the reader to it for concrete definitions. We only recall the main features useful for us and set the notation we will use.

Let D be a nonempty index set and $\mathcal{U}$ a non-trivial ultrafilter in D. Given a family $\{X_d,\ d \in D\}$ of Banach spaces, $(X_d)_\mathcal{U}$ denotes the corresponding ultraproduct Banach space. If every $X_d,\ d \in D$, coincides with a fixed Banach space X the corresponding ultraproduct is named an ultrapower of X and is denoted by $(X)_\mathcal{U}$. Remark that if every $X_d,\ d \in D$, is a Banach lattice, $(X_d)_\mathcal{U}$ has a canonical order which makes it a Banach lattice. If we have another family of Banach spaces $\{Y_d,\ d \in D\}$ and a family of operators $\{T_d \in \mathcal{L}(X_d, Y_d),\ d \in D\}$ such that $\sup_{d\in D} \|T_d\| < \infty$, then $(T_d)_\mathcal{U} \in \mathcal{L}((X_d)_\mathcal{U}, (Y_d)_\mathcal{U})$ denotes the canonical ultraproduct operator.

The structure and local structure of the ultrapowers have been studied for many authors. The construction of ultraproducts is an essential tool in operator theory, and we remark that ultraproducts and finite representability are in the heart of the local theory, i.e. the study of the Banach spaces through or in terms of its finite dimensional subspaces. Obviously the $\mathcal{L}^\lambda$ spaces are characterized by a very strong form of b-finite representability in λ for some $b > 0$.

The next proposition shows that the stability of the class $\mathcal{L}^\lambda$ under ultraproducts depends strongly on λ.

Proposition 2.5. *Given a Banach sequence space λ, the following conditions are equivalent:*

(1) *Every ultrapower $(\lambda)_\mathcal{U}$ is an $\mathcal{L}^\lambda$ space.*

(2) *λ satisfies the following property (**P**): there is $c > 0$ such that given a positive integer n, there exists some positive integer $m = m(n)$ such that any n-dimensional subspace N of λ is contained in an m-dimensional subspace M of λ with $d(M, S_m(\lambda)) < c$.*

Proof. (1) $\to$ (2) : Let $\mathcal{U}$ be an ultrafilter on an index set D, and let $\{N_d, d \in D\}$ be an arbitrary family of n-dimensional subspaces of λ. If for every $d \in D$, $\{x_d^i, i = 1, .., n\}$ is a basis of norm one vectors in N_d, $\{x^i = (x_d^i)_\mathcal{U}, i = 1, .., n\}$ is a basis in $N = (N_d)_\mathcal{U}$; hence N is an n-dimensional subspace of $(\lambda)_\mathcal{U}$. By hypothesis, $(\lambda)_\mathcal{U}$ is an $\mathcal{L}^\lambda$-space. Then, there are $c > 0$ and an m-dimensional subspace M of $(\lambda)_\mathcal{U}$ containing N such that $d(M, S_m(\lambda)) < c$. Let $\{(x_d^i)_\mathcal{U}, i = n+1, .., m\}$ be a basis of an algebraic complement of N in M. Then $M = (M_d)_\mathcal{U}$ where $M_d = span(x_d^i, i = 1, .., m)$. But since $\{(x_d^i)_\mathcal{U}, i = 1, .., m\}$ is a basis in M, there is an $I_1 \in \mathcal{U}$ such that if $d \in I_1$, $\{x_d^i, i = 1, .., m\}$ are linearly independent; hence $dim(M_d) = m$ for every $d \in I_1$. But also there is an $I_0 \in \mathcal{U}$ such that $d(M, M_d) \leq 1 + \varepsilon$ for every $d \in I_0$; hence for every $d \in I_0 \cap I_1$, $N_d \subset M_d$ with $d(M_d, S_m(\lambda)) < c(1 + \varepsilon)$. The result follows because $D, \mathcal{U}$ and $\{N_d, d \in D\}$ are arbitrary.

(2) $\to$ (1) : If N is an n-dimensional subspace of $(\lambda)_\mathcal{U}$ and $\varepsilon > 0$, from [6], proposition 6.1, $N = (N_d)_\mathcal{U}$ with $dim(N_d) = n$. By hypothesis, λ satisfies (**P**); hence there are $c > 0$ and $m = m(n, c)$ such that for every $d \in D$ there is an m-dimensional subspace M_d of λ containing N_d with $d(M_d, S_m(\lambda)) < c$. Then fixing a basis in every M_d, $M = (M_d)_\mathcal{U}$ is an m-dimensional subspace of $(\lambda)_\mathcal{U}$ containing N. Moreover there is $I_0 \in \mathcal{U}$ such that $d(M, M_d) < 1 + \varepsilon$ for every $d \in I_0$. Then $d(M, S_m(\lambda)) \leq c(1 + \varepsilon)$. $\square$

Remark 2.6. We emphasize that (**P**) is a very strong property that ℓ_p spaces satisfy [16], and Yves Raynaud pointed out to us that $\ell_p(\ell_q)$ also satisfies (**P**). But we don't know any Banach sequence space λ outside of the ℓ_p setting with this property.

3. The class of quasi-$\mathcal{L}^E$ spaces

The limitations concerning the stability of the $\mathcal{L}^\lambda$ spaces under ultrapowers manifested in Proposition 2.5 are the main motivation for considering another class of Banach spaces. The "soul" of the new class is the so-called uniform projection property, which was defined also in 1975 by Pelczyǹski and Rosenthal [16]. It is another local type property which is weaker than the property (**P**) but with a similar spirit.

Definition 3.1. A Banach space X has the *uniform projection property* if there is a $b > 0$ such that for each natural number n there is a natural number $m = m(n)$ only depending on n such that for every n-dimensional subspace $M \subset X$ there exists a k-dimensional and b-complemented subspace Z of X containing M with $k \leq m$.

Remark 3.2. 1) The class of Banach spaces with the uniform projection property is quite large and includes some rearrangement invariant function spaces, for example the reflexive Orlicz spaces and modular spaces; see [14]. The Bochner spaces $L_p(\mu, E)$ satisfy the uniform projection property if E does, $1 \leq p \leq \infty$ (see [6]), and the Hardy space H^1 also satisfies the uniform projection property; see [9].

2) It is known that the uniform projection property is stable under ultrapowers (see [6]) and duals [7].

Definition 3.3. Let E be a Banach space. We say that a Banach space X is a quasi-$\mathcal{L}^E$ space if there are positive real numbers a and b, $a \geq 1$, such that for every finite dimensional subspace $F \subset X$ there are a finite dimensional subspace $G \subset X$ containing F and a b-complemented finite dimensional subspace Y of E with $d(G, Y) \leq a$.

Obviously every quasi-$\mathcal{L}^E$ space X is a-finitely representable in E for some $a > 0$, but in general E itself is not a quasi-$\mathcal{L}^E$ space. Moreover $\mathcal{L}^\lambda \subset$ quasi-$\mathcal{L}^\lambda$. As in the case of the $\mathcal{L}^\lambda$ spaces, we are interested in the Banach spaces E having the property that every ultrapower of E is a quasi-$\mathcal{L}^E$ space.

As in Proposition 2.5 we prove:

Theorem 3.4. *Let E be a Banach space satisfying the uniform projection property. Then every ultrapower of E is a quasi-$\mathcal{L}^E$ space.*

Proof. Let M be an n-dimensional subspace of $(E)_\mathcal{U}$. Without loss of generality we can suppose that $M = (M_d)_\mathcal{U}$ where every M_d, $d \in D$, is an n-dimensional subspace of E. From the hypothesis, there are a $b > 0$, a natural number $m(n)$, and a family of b-complemented subspaces $\{Y_d, d \in D\}$ in E such that $M_d \subset Y_d$ for every $d \in D$ with its respective family of projections denoted by $\{P_d, d \in D\}$ such that $\|P_d\| \leq b$ and $dim(Y_d) \leq m(n)$ for every $d \in D$. Let $P = (P_d)_\mathcal{U}$, which is a projection of $(E)_\mathcal{U}$ onto a k-dimensional subspace Y of $(E)_\mathcal{U}$ such that $M \subset Y$, with $\|P\| \leq b$ and $k \leq m(n)$.

For every $d \in D$, let $\{z_d^i, i = 1, .., m(n)\}$ be an Auerbach basis in an $m(n)$-dimensional subspace W_d of E containing Y_d, i.e. $\| \sum_{i=1}^n c_i z_d^i \| \geq \max_{i=1,..,m(n)} |c_i|$.

Then for every $x = (x_d)_\mathcal{U} \in (E)_\mathcal{U}$, if

$$P_d(x_d) = \sum_{i=1}^{n} a_d^i z_d^i$$

for some scalars a_d^i such that $|a_d^i| \leq b\ \|x\|$, then

$$P(x) = (\sum_{i=1}^{n} a_d^i z_d^i)_\mathcal{U} = \sum_{i=1}^{n} a^i (z_d^i)$$

where $a^i = \lim_\mathcal{U} a_d^i$, $i = 1, .., m(n)$. We denote by W the subspace of $(E)_\mathcal{U}$ generated by $\{(z_d^i)_\mathcal{U}, i = 1, .., m(n)\}$. It is clear that $W = (W_d)_\mathcal{U}$.

From [5], lemma 4.1 and proposition 4.2, given $\varepsilon > 0$ there is $d_0 \in D$ such that for every $x = (x_d)_\mathcal{U} \in W$,

$$(1 - \varepsilon)\ \|x\| \leq \|x_{d_0}\| \leq (1 + \varepsilon)\ \|x\|.$$

We denote $C_{d_0} : W \to W_{d_0}$ such that $C_{d_0}((x_d)_\mathcal{U}) = x_{d_0}$, which is an isomorphism satisfying $\|C_{d_0}\| \leq 1 + \varepsilon$ and $\|C_{d_0}^{-1}\| \leq \frac{1}{1-\varepsilon}$. Then $(C_{d_0})_{|Y}$ is an isomorphism from Y onto Y_{d_0} such that $\|(C_{d_0})_{|Y}\| \|((C_{d_0})_{|Y})^{-1}\| \leq \frac{1+\varepsilon}{1-\varepsilon}$. □

An immediate consequence of the well-known principle of local reflexivity is that if X'' is a quasi-$\mathcal{L}^E$ space, then X has the same property. Moreover it is known that, also by local reflexivity, every Banach space X satisfies that X'' is a 1-complemented subspace of some ultrapower of X. In this direction and with a similar proof we have the following useful property.

Proposition 3.5. *Let X be a quasi-$\mathcal{L}^E$ space. Then X'' is isomorphic to a complemented subspace of some ultrapower of E.*

Proof. For every $\varepsilon > 0$, let D be the set of tuples (M, N, ε), where M and N are finite dimensional subspaces of X'' and X' respectively, endowed with the canonical order: $d_1 = (M_1, N_1, \varepsilon_1) \leq d_2 = (M_2, N_2, \varepsilon_2)$ iff $M_1 \subset M_2$, $N_1 \subset N_2$, $\varepsilon_1 \geq \varepsilon_2$. We denote $d = (M_d, N_d, \varepsilon_d)$. We will consider the following standard construction of a filter basis on D: fix $d_0 \in D$ and put

$$R(d_0) = \{d \in D : d_0 \leq d\};$$

then

$$\mathcal{R} := \{R(d), d \in D\}$$

is a filter basis of subsets of D. According to Zorn's lemma, there is an ultrafilter in D containing $\mathcal{R}$. An ultrafilter with this property is denoted by $\mathcal{U}$.

From the principle of local reflexivity, for every M_d, there is a finite dimensional subspace F_d of X and an isomorphic map $T_d : M_d \to F_d$ such that $(T_d)_{|M_d \cap G_d} = id_{M_d \cap G_d}$, $\langle x', T_d(x'')\rangle = \langle x', x''\rangle$ for every $x'' \in M_d$ and every $x' \in N_d$, and $\|T_d\|\ \|T_d^{-1}\| \leq 1 + \varepsilon$. Then X'' is isometric to a subspace of $(F_d)_\mathcal{U}$ under the map $A : X'' \to (F_d)_\mathcal{U}$ such that for every $x \in X''$, $A(x'') = (x_d)_\mathcal{U}$ with $x_d = T_d(x'')$ if $x'' \in M_d$, and $x_d = 0$ if $x'' \notin M_d$.

But by hypothesis, for every $d \in D$ there are positive constants a, b, $a \geq 1$, another finite dimensional subspace Y_d of X containing F_d, a finite dimensional and a b-complemented subspace G_d of E with projection map P_d and an isomorphism $S_d : Y_d \to G_d$ with $\|S_d\|\ \|S_d^{-1}\| \leq a + \varepsilon$. Then $S = (S_d)_\mathcal{U}$ defines an isomorphism between $(Y_d)_\mathcal{U}$ and $(G_d)_\mathcal{U}$, and X'' is isometric to a subspace of $(Y_d)_\mathcal{U}$ under the map $B = I_F\ A$ where $I_F : (F_d)_\mathcal{U} \to (Y_d)_\mathcal{U}$ is the inclusion map. Moreover $(G_d)_\mathcal{U}$

is a b-complemented subspace of $(E)_{\mathcal{U}}$; we denote by $I_E : (G_d)_{\mathcal{U}} \to (E)_{\mathcal{U}}$ and $P_E = (P_d)_{\mathcal{U}} : (E)_{\mathcal{U}} \to (G_d)_{\mathcal{U}}$ the inclusion and projection map respectively.

Introduce the map $P : (E)_{\mathcal{U}} \to X''$ such that for every $x' \in X'$ and for every $(e_d)_{\mathcal{U}} \in (E)_{\mathcal{U}}$,

$$\langle P((e_d)_{\mathcal{U}}), x' \rangle = \lim_{\mathcal{U}} \langle S_d^{-1}\, P_d(e_d), x' \rangle.$$

But if $x'' \in M_{d_0}$ and $x' \in N_{d_0}$, then

$$\langle P\ I_E\ S\ B(x''), x' \rangle = \lim_{\mathcal{U}} \langle S_d^{-1}\ S_d\ T_d(x''), x' \rangle = \langle x'', x' \rangle;$$

hence $P\ I_E\ S\ B = id_{X''}$. □

The following definition is given in [10].

Definition 3.6. Let $X = Y + Z$. Then Z is said to be locally disjoint from Y if there are positive constants a, b such that for each pair of finite dimensional subspaces $M \subset Y$, $N \subset Z$ there is a finite dimensional subspace $S_{M,N} \subset Y$ and an isomorphism $\tau : N \to S_{M,N}$ such that $\|\tau\|\|\tau^{-1}\| \leq a$ and $\|y + \tau(z)\| \geq b\|y\|$ for all $y \in M$ and $z \in N$.

Proposition 3.7. *Let X be a quasi-$\mathcal{L}^E$ space (an $\mathcal{L}^\lambda$ space) and Y a complemented subspace of X with projection P. If $ker(P)$ is locally disjoint from Y, then Y is a quasi-$\mathcal{L}^E$ space (an $\mathcal{L}^\lambda$ space).*

Proof. Put $X = Y \oplus ker(P)$. There is a constant $c > 0$ such that if A is a finite dimensional subspace of Y, there are a finite dimensional subspace B of X containing A and a complemented subspace C of E (C a section of λ) such that $d(B, C) \leq c$. Moreover $B \subset B_1 \oplus B_2$, $B_1 = P(B) \subset Y$, and $B_2 = (I_X - P)(B) \subset ker(P)$. Because $ker(P)$ is locally disjoint from Y, given $\varepsilon > 0$ there are positive constants a, b depending only on Y and $ker(P)$, a subspace S_{B_1,B_2} of Y and an isomorphism $\tau : B_2 \to S_{B_1 B_2}$ such that $\|\tau\| = 1$, $\|\tau^{-1}\| \leq a$ and $\|b_1 + \tau(b_2)\| \geq b\|b_1\|$, for all $b_1 \in B_1$ and $b_2 \in B_2$. Let $T : B_1 \oplus B_2 \to Y$ be such that $T(b_1 + b_2) = b_1 + \tau(b_2)$, for all $b_1 \in B_1$, $b_2 \in B_2$. Then

$$\begin{aligned}\|T(b_1 + b_2)\| &\leq \|b_1\| + \|\tau\|\|b_2\| = \|b_1\| + \|b_2\| = \|P(b_1 + b_2)\| + \|(I - P)(b_1 + b_2)\| \\ &\leq (2\|P\| + 1)\|b_1 + b_2\|.\end{aligned}$$

Moreover $\|T(b_1 + b_2)\| = \|b_1 + \tau(b_2)\| \geq b\|b_1\|$ and $\|b_1 + \tau(b_2)\| \geq \|\tau(b_2)\| - \|b_1\| \geq \frac{1}{a}\|b_2\| - \frac{1}{b}\|b_1 + \tau(b_2)\|$; hence $(1 + \frac{1}{b})\|b_1 + \tau(b_2)\| \geq \frac{1}{a}\|b_2\|$. Then

$$\|T(b_1 + b_2)\| = \|b_1 + \tau(b_2)\| \geq \frac{b}{2a(1 + b)}\|b_1 + b_2\|.$$

If $D = T(B)$, $d(B, D)$ is a constant depending only on $\|P\|$, a and b; hence $d(C, D)$ is a constant depending only on $\|P\|$, a, b and c. Since D is a finite dimensional subspace of Y containing A, the result follows. □

Remark 3.8. In [10], theorem 5, Lacey proves that if X is a Banach space, $X = Y \oplus Z$ and X is finitely representable in Y, then Z is locally disjoint from Y, but, looking at his proof, the same result follows if X is b-finitely representable in Y for some constant $b \geq 1$. In particular, since the quasi-$\mathcal{L}^E$ spaces are b-finitely representable in E, we have the following consequences.

Corollary 3.9. a) *Let X be a quasi-$\mathcal{L}^E$ space, and let Y be a complemented subspace of X containing uniformly copies of the finite dimensional subspaces of E. Then Y is also a quasi-$\mathcal{L}^E$ space.*

b) *Let λ be a regular Banach sequence space, and let X be a quasi-$\mathcal{L}^\lambda$ space (an $\mathcal{L}^\lambda$ space). If Y is a complemented subspace of X containing copies of $S_k(\lambda)$ uniformly, then Y is also a quasi-$\mathcal{L}^\lambda$ space (an $\mathcal{L}^\lambda$ space).*

Corollary 3.10. a) *Let E be a Banach space with the uniform projection property. If X is a quasi-$\mathcal{L}^E$ space, then X'' is also a quasi-$\mathcal{L}^E$ space.*

b) *Let λ be a Banach sequence space with the uniform projection property (the property (**P**)). If X is a quasi-$\mathcal{L}^\lambda$ space (an $\mathcal{L}^\lambda$ space), then X'' is also a quasi-$\mathcal{L}^\lambda$ space (an $\mathcal{L}^\lambda$ space).*

Proof. The result follows from 3.5, 3.9 and 3.4 (2.5). □

4. GL-local unconditional structure and quasi-$\mathcal{L}^E$ spaces: The class of the $\mathcal{L}^{\lambda,g}$ spaces

The class of the quasi-$\mathcal{L}^\lambda$ spaces if λ has the uniform projection property is very satisfactory with respect to the construction of ultrapowers and biduals, and quite satisfactory with respect to the complemented subspaces, but in this class we have lost the sectional subspaces of λ. For this reason we consider now an, at the moment, less restrictive notion of local unconditional structure than the DPR-local unconditional structure, defined by Gordon and Lewis [4], which has been a basic tool in the study of many important topics in the theory of Banach spaces, Banach algebras and operators.

Definition 4.1. We say that a Banach space X has GL-local unconditional structure if there exists a real constant $c > 0$ such that for every finite dimensional subspace F of X, there are a finite dimensional lattice L and linear operators $u : F \to L$ and $v : L \to X$ such that $\|u\|\|v\| \leq c$ and $v\ u = I_{F,X}$.

In fact from [3] (see also [10]) a Banach space X has GL-local unconditional structure if and only if X'' is isomorphic to a complemented subspace of a Banach lattice.

Proposition 4.2. *Let E be a Banach space having GL-local unconditional structure. Then every quasi-$\mathcal{L}^E$ space X has GL-local unconditional structure.*

Proof. From the definition of the class of the quasi-$\mathcal{L}^\lambda$ spaces, there are positive real numbers $a \geq 1$ and b such that for every finite dimensional subspace $F \subset X$ there are a finite dimensional subspace $G \subset X$ containing F and a b-complemented finite dimensional subspace Y of E with $d(G, Y) < a$. Since E has GL-local unconditional structure there is $c > 0$ such that the inclusion map of Y into E factors through a finite dimensional Banach lattice L; hence there are $u : Y \to L$ and $v : L \to E$ such that $I_{Y,E} = v\ u$ with $\|u\|\ \|v\| \leq c$. Given $\varepsilon > 0$, let $C : G \to Y$ be such that $\|C\|\|C^{-1}\| \leq a + \varepsilon$ and let $P_Y : E \to Y$ be projections with norm less than or equal to b. We denote $A : F \to L$ such that $A = u\ C\ I_{F,G}$, and $B : L \to X$ such that $B = I_{G,X}\ C^{-1}\ P_Y\ v$. The desired result follows because $B\ A = I_{F,X}$ with $\|A\|\ \|B\| \leq (a + \varepsilon)\ b\ c$. □

If λ is a Banach sequence space we also can consider the GL-local unconditional structure with respect to the sections of a Banach sequence space λ.

Definition 4.3. Let λ be a Banach sequence space. We say that a Banach space X has $GL-S_k(\lambda)$-local unconditional structure (or GL-local unconditional structure with respect to the sections of λ) if there exists a real constant $c>0$ such that for every finite dimensional subspace F of X, there are $n\in\mathbb{N}$ and linear operators $u:F\to S_n(\lambda)$ and $v:S_n(\lambda)\to X$ such that $\|u\|\|v\|\leq c$ with $v\,u=I_{F,X}$.

Also, associated to the $GL-S_k(\lambda)$-local unconditional structure, we define the corresponding class of Banach spaces:

Definition 4.4. Given a Banach sequence space λ, a Banach space X is said to be a generalized $\mathcal{L}^\lambda$ space if X has $GL-S_k(\lambda)$-local unconditional structure.

The class of *generalized $\mathcal{L}^\lambda$ spaces* is denoted by $\mathcal{L}^{\lambda,g}$. We recall that, for a wide class of Banach sequence spaces λ, this class can be as useful in the operator spaces related to λ as the $\mathcal{L}^p$ spaces in the classical operator theory which is connected with the ℓ_p spaces; see for instance [15]. The following proposition gives examples of $\mathcal{L}^{\lambda,g}$ spaces.

Proposition 4.5. *Let λ be a Banach sequence space. Then:*
a) *Complemented subspaces of $\mathcal{L}^{\lambda,g}$ spaces are $\mathcal{L}^{\lambda,g}$ spaces.*
b) *If $X''\in\mathcal{L}^{\lambda,g}$, X has the same property.*
c) *If λ is regular, quasi-$\mathcal{L}^\lambda\subset\mathcal{L}^{\lambda,g}$.*

Proof. a) is immediate from the definition and b) follows for local reflexivity.

c) Since X is a quasi-$\mathcal{L}^\lambda$ space, there are $a\geq 1$ and $b>0$ such that given $\varepsilon>0$, for every finite dimensional subspace F of X, there are a subspace G of X containing F, a complemented subspace Y of λ with projection map $P:\lambda\to Y$ with $\|P\|\leq b$ and an isomorphism $T:G\to Y$ with $\|T\|\|T^{-1}\|\leq a+\varepsilon$. Since λ is an $\mathcal{L}^\lambda$ space, there are a finite dimensional subspace H of λ containing Y and an isomorphism $S:H\to S_{dim(H)}$ with $\|S\|\|S^{-1}\|\leq 1+\varepsilon$. The following "controlled norm" scheme finishes the proof:

$$F\hookrightarrow G\overset{T}{\to}Y\hookrightarrow H\overset{S}{\to}S_{dim(H)}(\lambda)\overset{S^{-1}}{\to}H\overset{P_{|H}}{\to}Y\overset{T^{-1}}{\to}G\hookrightarrow X.$$

□

Proposition 4.6. *Let λ be a Banach sequence space, and let X be a Banach space. Consider the following statements:*
i) $X\in\mathcal{L}^{\lambda,g}$.
ii) *$I_{X,X''}$ factors through an ultrapower of λ_r.*
iii) *$id_{X''}$ factors through an ultrapower of λ_r.*
Then,
a) i) $\to$ ii) $\leftrightarrow$ iii).
b) *If λ_r satisfies the uniform projection property, all are equivalent.*

Proof. a) Consider the ultrafilter of Proposition 3.5. By hypothesis, there is $b>0$ such that for every $d\in D$, there are $n_d\in\mathbb{N}$, $u_d:M_d\to S_{n_d}(\lambda)$ and $v_d:S_{n_d}(\lambda)\to X$ such that $v_du_d=I_{M_d,X}$ and $\|u_d\|\|v_d\|\leq b$. Then i) $\to$ ii) because X is isometric to a subspace of $(M_d)_\mathcal{U}$, by local reflexivity X'' is isometric to a 1-complemented subspace of $(X)_\mathcal{U}$ and $(S_{n_d}(\lambda))_\mathcal{U}$ is a complemented subspace of $(\lambda_r)_\mathcal{U}$ with projection norm less than or equal to one. ii) $\to$ iii): Since $I_{X,X''}=S_2S_1$ with $S_1:X\to(\lambda_r)_\mathcal{U}$ and $S_2:(\lambda_r)_\mathcal{U}\to X''$, then by bidualization we have $(I_{X,X''})''=S_2''S_1''$, which provides a factorization of the biconjugate of the inclusion

map through $((\lambda_r)_\mathcal{U})''$. Note that although $(I_{X,X''})''$ is an into isometry $X'' \to X''''$, which differs generally from the natural inclusion map $I_{X'',X''''}$, it has nevertheless the same left inverse $P = (I_{X',X'''})'$. Hence $I_{X''} = (PS_2'')S_1''$ provides a factorization of the identity of X'' through $((\lambda_r)_\mathcal{U})''$. Now the usual local reflexivity argument gives a factorization of $((\lambda_r)_\mathcal{U})''$ through an iterated ultrapower $((\lambda_r)_\mathcal{U})_\mathcal{V}$, which is isometric with $(\lambda_r)_{\mathcal{U}\times\mathcal{V}}$.

b) If λ_r satisfies the uniform projection property, $(\lambda_r)_\mathcal{U}$ is a $GL - S_k(\lambda)$ space. If $I_{X,X''} = S_2S_1$, $S_1 : X \to (\lambda_r)_\mathcal{U}$ and $S_2 : (\lambda_r)_\mathcal{U} \to X''$, given a finite dimensional subspace M of X such that $S_1(M)$ is a finite dimensional subspace of $(\lambda_r)_\mathcal{U}$, then there are $n \in \mathbb{N}$, $u : S_1(M) \to S_n(\lambda)$ and $v : S_n(\lambda) \to (\lambda_r)_\mathcal{U}$ such that $I_{M,(\lambda)_\mathcal{U}} = vu$ with $\|u\|\|v\| \leq b$ for some constant b independent of M. The result follows for local reflexivity because $S_2v(S_n(\lambda))$ is a finite dimensional subspace of X'' and $S_2S_1(X) = X$. □

Proposition 4.7. *Let λ be a Banach sequence space such that λ_r satisfies the uniform projection property. Then:*

a) *The class $\mathcal{L}^{\lambda,g}$ is stable under ultrapowers.*

b) *X is an $\mathcal{L}^{\lambda,g}$ space if and only if X'' is, if and only if X'' is isomorphic to a complemented subspace of some $(\lambda_r)_\mathcal{U}$.*

Proof. a) Let $\mathcal{D}$ be an ultrafilter on an index set D, and let X be an $\mathcal{L}^{\lambda,g}$ space. Given a finite dimensional subspace $M = (M_d)_\mathcal{D}$ of $(X)_\mathcal{D}$, there is $a > 0$ such that for every $d \in D$ there are $n_d \in \mathbb{N}$ and linear and continuous maps $u_d : M_d \to S_{n_d}(\lambda)$, $v_d : S_{n_d}(\lambda) \to X$ with $\|u_d\|\ \|v_d\| \leq a$ and $v_d\ u_d = I_{M_d,X}$. Then we construct in the canonical way the maps $u := (u_d)_\mathcal{D}$, $u : M \to (S_{n_d}(\lambda))_\mathcal{D}$ and $v := (v_d)_\mathcal{D}$, $v : (S_{n_d}(\lambda))_\mathcal{D} \to (X)_\mathcal{D}$. From Theorem 3.4, $(\lambda_r)_\mathcal{D}$ is a quasi-$\mathcal{L}^{\lambda_r}$-space, and since $u(M)$ is a finite dimensional subspace of $(S_{n_d}(\lambda))_\mathcal{D}$, hence of $(\lambda_r)_\mathcal{D}$, there are positive real numbers b and c, a finite dimensional subspace G of $(\lambda_r)_\mathcal{D}$ containing $u(M)$, a finite dimensional and b-complemented subspace Y of λ_r with projection $P : \lambda_r \to Y$ and an isomorphism $C : G \to Y$ with $\|C\|\ \|C^{-1}\| \leq c$. Since λ_r is an $\mathcal{L}^{\lambda_r}$ space, there are a constant $h > 0$, a finite dimensional subspace Z of λ_r containing Y and an isomorphism $H : Z \to S_{dim(Z)}(\lambda)$ with $\|H\|\ \|H^{-1}\| \leq h$. Finally we denote by Q the natural projection map from $(\lambda_r)_\mathcal{D}$ onto $(S_{n_d}(\lambda))_\mathcal{D}$. Then we have the following schemes:

$$M \xrightarrow{u} u(M) \hookrightarrow G \xrightarrow{C} Y \hookrightarrow Z \xrightarrow{H} S_{dim(Z)}(\lambda),$$

$$S_{dim(Z)}(\lambda) \xrightarrow{H^{-1}} Z \xrightarrow{P_{|Z}} Y \xrightarrow{C^{-1}} G \hookrightarrow (\lambda_r)_\mathcal{D} \xrightarrow{Q} (S_{n_d}(\lambda))_\mathcal{D} \xrightarrow{v} X.$$

If we denote $A := H\ I_{Y,Z}\ C\ I_{u(M),G}\ u$ and $B := v\ Q\ C^{-1}\ P_{|Z}\ H^{-1}$, then $B\ A = I_{M,(X)_\mathcal{D}}$ and $\|A\|\ \|B\| \leq a\ b\ c\ h$.

b) We only claim that if $id_{X''} = S_2S_1$, $S_1 : X'' \to (\lambda_r)_\mathcal{U}$ and $S_2 : (\lambda_r)_\mathcal{U} \to X''$, then $S_1(X'')$ is a complemented subspace of $(\lambda_r)_\mathcal{U}$. □

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