PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 7, Pages 1925–1928
S 0002-9939(05)07763-4
Article electronically published on January 31, 2005

THE ARTIN-STAFFORD GAP THEOREM

AGATA SMOKTUNOWICZ

(Communicated by Lance W. Small)

ABSTRACT. Let K be an algebraically closed field, and let R be a finitely graded K-algebra which is a domain. We show that R cannot have Gelfand-Kirillov dimension strictly between 2 and 3.

In [1] Artin and Stafford proved that if K is a field and R is a finitely graded domain with $2 \leq GKDim(R) < \frac{11}{5}$, then $GKDim(R) = 2$. They also described the structure of such algebras [1]. Artin, Stafford and Van der Bergh conjectured [1, 3] that a finitely graded domain cannot have Gelfand-Kirillov dimension strictly between 2 and 3. The purpose of this paper is to show that if K is an algebraically closed field, then this conjecture is true. We slightly change Corollary 1.5 from [1] and leave the rest of the proof the same as in [1]. We use the same terminology as in [1]. We call a graded K-algebra $R = \bigoplus_{n\geq 0} R_n$ *finitely graded* if it is a finitely generated algebra and if R_0 is a finite-dimensional vector space over K. Let $GKDim(R)$ denote the Gelfand-Kirillov dimension of R. Let $K(x)$ denote the field of rational functions in the variable x.

Lemma 1. *Let K be an infinite field, and let D be a K-algebra which is an Ore domain with $GKDim(D) \geq 2$. Let $a_1, \dots, a_n$ be generators of D, and let $x \in D$ be not algebraic over K. Then for every natural number s there are $c_1, \dots, c_s \in span_K\{a_1, \dots, a_n\}$ such that, for all $t, j \leq s$, elements $c_1c_2 \dots c_t x^j$ are linearly independent over K.*

Proof. Let $V = span_K\{a_1, a_2, \dots, a_n\}$. We first show that if, for some t, $V^t \subseteq \sum_{i=0}^{t-1} V^i K(x)$, then $GKDim(D) \leq 1$. Since V^t is a finite-dimensional vector space over K, then there is $g(x) \in K(x)$ and a number $p > \deg g(x)$ such that $V^t \subseteq \sum_{i<t, j\leq p} KV^i \frac{x^j}{g(x)}$. Consequently (by induction) for every $m > t$, $V^m \subseteq \sum_{i<t, j\leq mp} KV^i \frac{x^j}{g(x)^m}$, which implies $GKDim(D) \leq 1$.

Thus, we may assume that V^t is not a subset of $\sum_{i=0}^{t-1} V^i K(x)$ for all t. We induce the following order on the generators of D: $a_1 > a_2 > \dots > a_n$. Let $>$ be the corresponding deg-lex order on the monoid generated by $a_1, \dots, a_n$. Let W be the least subset of D consisting of monomials and such that every $v \in D$ can be written as a finite sum $v = \sum_{u\leq v, u\in W, f_u(x)\in K(x)} u f_u(x)$. Note that $W = W_1 \cup W_2 \cup \dots$ where $W_i \neq \emptyset$ and $W_i \subseteq V^i$ for all i. Denote $\widetilde{W_i} = W_i K(x) = \{\sum wr : w \in W_i, r \in K(x)\}$. The minimality of W ensures that $\widetilde{W_i} \cap \sum_{j=1}^{i-1} \widetilde{W_j} = \emptyset$. Let s be a natural number.

Received by the editors February 24, 2004 and, in revised form, March 15, 2004.
2000 *Mathematics Subject Classification.* Primary 16D90, 16P40, 16S80.
Key words and phrases. Graded domains, Gelfand–Kirillov dimension.

Suppose that there exist $c_1, \dots, c_s \in span_K V$ such that, for each $i \le s$, $c_1c_2\dots c_i \notin \sum_{k<i} \widetilde{W}_k$. Then $c_1c_2\dots c_i x^j$ are linearly independent over K, for all $i, j \le s$. We aim to show that there is $e \in K$ such that if $c_j = \sum_{i=1}^n e^{2^{2^{nj+i}}} a_i$, for $j \le s$, then $c_1c_2\dots c_i \notin \sum_{k<i} \widetilde{W}_k$, for all $i \le s$. Given $d \in K$, let $\bar{c}_j = \sum_{i=1}^n d^{2^{2^{nj+i}}} a_i$, for $j = 1, 2, \dots, s$. Let $t \le s$. Then $\bar{c}_1\bar{c}_2\dots\bar{c}_t = \sum_{1\le i_1,\dots,i_t\le n} d^{h(i_1,\dots,i_t)} a_{i_1}a_{i_2}\dots a_{i_t}$, for some natural numbers $h(i_1, \dots, i_t)$. Observe that $h(i_1, \dots, i_t) < 2^{2^{n(s+3)}}$. Moreover, if $(i_1, \dots, i_t) \ne (j_1, \dots, j_t)$, then $h(i_1, \dots, i_t) \ne h(j_1, \dots, j_t)$. Given $1 \le i_1, \dots, i_t \le n$ write $a_{i_1}\dots a_{i_t} = \sum_{w_{j,t}\in W_t} w_{j,t} f_{j,(i_1,\dots,i_t)}(x) + e_{i_1,\dots,i_t}$ for some $e_{i_1,\dots,i_t} \in \sum_{j=1}^{t-1} \widetilde{W}_j$ and some $f_{j,(i_1,\dots,i_t)}(x) \in K(x)$. Note that if $a_{j_1}\dots a_{j_t} = w_{1,t}$, then $f_{1,(j_1,\dots,j_t)}(x) = 1$. Observe that if $\sum_{1\le i_1,\dots,i_t\le n} d^{h(i_1,\dots,i_t)} f_{1,(i_1,\dots,i_t)}(x) \ne 0$, then $\bar{c}_{i_1}\dots\bar{c}_{i_t} \notin \sum_{j=1}^{t-1} \widetilde{W}_j$ by the minimality of W. Suppose that $d_1, \dots, d_l \in K$ where $l > 2^{2^{n(s+3)}}$ are pairwise distinct elements such that, for all $j \le l$, we have $\sum_{1\le i_1,\dots,i_t\le n} d_j^{h(i_1,\dots,i_t)} f_{1,(i_1,\dots,i_t)}(x) = 0$. We can rewrite this system of equations as $Bv = 0$ where

$$B = \begin{pmatrix} 1 & d_1 & d_1^2 & \dots & d_1^{l-1} \\ 1 & d_2 & d_2^2 & \dots & d_2^{l-1} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & d_l & d_l^2 & \dots & d_l^{l-1} \end{pmatrix} \tag{1}$$

and v is a column vector with coefficients from the set $\{0, f_{1,(i_1,\dots,i_t)}(x) : 1 \le i_1, \dots, i_t \le n\}$. Since a Vandermonde matrix is invertible (for pairwise distinct elements d_i), this implies that $v = 0$. Consequently, all $f_{1,(i_1,\dots,i_t)}(x) = 0$, a contradiction. Hence $\sum_{1\le i_1,\dots,i_t\le n} d^{h(i_1,\dots,i_t)} f_{1,(i_1,\dots,i_t)}(x) \ne 0$ for almost all $d \in K$. This is true for every $t \le s$. Since K is infinite we get that there is $e \in K$ such that $\sum_{1\le i_1,\dots,i_t\le n} e^{h(i_1,\dots,i_t)} f_{1,(i_1,\dots,i_t)}(x) \ne 0$ for every $t \le s$. Now $c_i = \sum_{i=1}^n e^{2^{2^{nj+1}}} a_i$ are as required, which completes the proof. □

Fix a finitely graded domain R with $GKDimR < 3$. The graded ring of fractions $Q = Q(R)$ of R is the ring obtained by inverting homogeneous elements from R. It is described in [2] as a skew Laurent polynomial ring $D[z, z^{-1}; \sigma]$, in which σ is an automorphism of a division ring D, and multiplication is defined by $zd = d^\sigma z$.

The following lemma was proved by Artin and Stafford in [1].

Lemma 2. *Let R be a finitely graded K-algebra and assume that R is an Ore domain with graded quotient division ring $Q(R) = D_0[z, z^{-1}; \sigma]$. If $GKDimD_0 < 2$, then $GKDimR \le 2$.*

Proof. This follows from Theorem 1.15 [1], Bergman's Gap Theorem, the Warfield-Small Theorem [4] and Theorem 0.5 [1], [1, p. 242]. □

Lemma 3. *Let $R = \bigoplus_{n\ge 0} R_n$ be a graded K algebra with $GKDimR < 3$, and let α be a number. Then $dim_K R_{\alpha s} > \frac{s}{s+2} dim_K R_{\alpha(s+1)}$ for infinitely many s.*

Proof. Conversely, suppose that there is m_0 such that for all $p > m_0$ we have

$$dim_K R_{\alpha(p+1)} \ge \frac{p+2}{p} dim_K R_{\alpha p} \ge \frac{p+2}{p}\frac{p+1}{p-1} dim_K R_{\alpha(p-1)} \ge \dots .$$

Continuing in this way we get that $dim_K R_{\alpha(p+1)} \geq \frac{(p+2)(p+1)}{(m_0+2)(m_0+1)}$, which is impossible since $GKDim(R) < 3$. □

Theorem 1. *Let K be an infinite field, and let R be a finitely graded domain with $GKDim(R) < 3$ and graded quotient ring $Q(R) = D_0[z, z^{-1}; \sigma]$. Then either $GKDim(R) \leq 2$ or D_0 is algebraic over K.*

Proof. It suffices to show that if $GKDim(D_0) \geq 2$ and D_0 contains element x not algebraic over K, then $GKDim(R) \geq 3$ (by Lemma 2). Suppose that $GKDim(D_0) \geq 2$. Then, by Bergman's Gap Theorem, there is a finitely generated subring D of D_0 such that $x \in D$ and $GKDim(D) \geq 2$. Let $a_0, \dots, a_n$ be generators of D. Since R is graded we can write $R = \bigoplus_{n \geq 0} R_n$. There is a natural number α and $q, q_0, p_1, \dots, p_n \in R_\alpha$ such that $x = q^{-1}q_0$ and $a_i = p_i q^{-1}$ for $1 \leq i \leq n$. By Lemma 3 there is a natural number s such that $dim_K R_{\alpha s} > \frac{s}{2}(dim_K R_{\alpha(s+1)} - dim_K R_{\alpha s})$. By Lemma 1, there are $c_1, \dots, c_s$ such that for every $t, j \leq s$ elements, $c_1 c_2 \dots c_t x^j$ are linearly independent over K and $c_i = q_i q^{-1}$ for some $q_i \in R_\alpha$, $1 \leq i \leq n$.

Define inductively sets $R_{\alpha s} = U_0 \supseteq U_1 \supseteq U_2 \supseteq \dots$ as follows: $U_1 = \{r \in R_{\alpha s} : rc_1 \in R\}$, ..., $U_{i+1} = \{r \in U_i : rc_1 c_2 \dots c_{i+1} \in R\}$. Observe now that $dim_K U_{i+1} = dim_K(U_i \cap R_{\alpha s}(c_1 \dots c_{i+1})^{-1}) = dim_K(E_i q_{i+1} \cap R_{\alpha s} q)$, where $E_i = U_i c_1 \dots c_i$. Consequently,

$$\begin{aligned} dim_K U_{i+1} &= dim_K E_i q_{i+1} + dim_K R_{\alpha s} q - dim_K(E_i q_{i+1} \cup R_{\alpha s} q) \\ &\geq dim_K U_i + dim_K R_{\alpha s} - dim_K R_{\alpha(s+1)}. \end{aligned}$$

Hence for all l we have $dim_K U_l \geq dim_K U_0 - l(dim_K R_{\alpha(s+1)} - dim_K R_{\alpha s}) = dim_K R_{\alpha s} - l(dim_K R_{\alpha(s+1)} - dim_K R_{\alpha s})$. Similarly define $R_{\alpha s} = \bar{U}_0 \supseteq \bar{U}_1 \supseteq \bar{U}_2 \supseteq \dots$ as follows: $\bar{U}_1 = \{r \in R_{\alpha s} : xr \in R\}$, ... $\bar{U}_{i+1} = \{r \in \bar{U}_i : x^{i+1} r \in R\}$. Since $x = q^{-1} q_0$ similarly as before $dim_K \bar{U}_l \geq dim_K R_{\alpha s} - l(dim_K R_{\alpha(s+1)} - dim_K R_{\alpha s})$. Now let $\frac{s-2}{2} \leq l < \frac{s}{2}$. Then $dim_K U_l > 0$ and $dim_K \bar{U}_l > 0$. Let $b \in U_l, e \in \bar{U}_l$; then for all $i, j \leq l$, elements $bc_1 \dots c_i x^j e \in R_{2\alpha s}$. Since R is a domain, then all these elements are linearly independent over K. This yields $dim_K R_{2\alpha s} \geq l^2$. Let $a \geq 1$ be such that R_a contains a non-zero element. Since R is a domain, then $dim_K R_i \leq dim_K R_{i+a}$. Consequently, if $i \geq 0$, then $dim_K R_{2\alpha s + ia} \geq (\frac{s-2}{2})^2$. Now $dim_K \sum_{i=1}^{2s(\alpha+a)} R_i \geq \sum_{i=0}^{2s} dim_K R_{2\alpha s + ia} \geq 2s(\frac{s-2}{2})^2$. This holds for infinitely many s by Lemma 3, which is impossible since $GKDim(R) < 3$. □

Corollary 1. *Let K be an algebraically closed field, and let R be a finitely graded K-algebra which is a domain. Then R cannot have Gelfand-Kirillov dimension strictly between 2 and 3.*

Corollary 2. *Let K be a finite field, and let R be a finitely graded K-algebra which is a domain. Then R cannot have Gelfand-Kirillov dimension strictly between 2 and 3.*

Proof. Let $Q(R) = D[z, z^{-1}, \sigma]$ be the graded quotient ring of R. By Jacobson's theorem, if D is algebraic over K, then D is commutative. By Theorem 1.15 [1], D is finitely generated as a division ring over K. Then $GKDimR \leq 2$. Let $R' = R \otimes_K K(x)$, where $K(x)$ is the field of rational functions in the variable x. Denote the Gelfand-Kirillov dimension of R' as a $K(x)$ algebra by c. Then $c = GKDimR$. Since R' is a domain and $K(x)$ is an infinite field, it follows by Theorem 1 that c cannot be strictly between 2 and 3. □

References

1. M. Artin and J. T. Stafford, *Noncommutative graded domains with quadratic growth*, Invent. Math. **122** (1995), 231–276. MR1358976 (96g:16027)
2. C. Năstăsescu and F. Van Oystaeyen, *Graded Ring Theory*, North Holland, Amsterdam, 1982. MR0676974 (84i:16002)
3. J. T. Stafford and M. Van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 2, 171–216. MR1816070 (2002d:16036)
4. L. W. Small and R. B. Warfield, Jr., *Prime affine algebras of Gelfand-Kirillov dimension one*, J. Algebra **91**, 386–389 (1984). MR0769581 (86h:16006)

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, P.O. Box 21, 00-956 Warsaw, Poland

E-mail address: agatasm@impan.gov.pl