

## FREE PRODUCTS ARISING FROM ELEMENTS OF FINITE ORDER IN SIMPLE RINGS

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(Communicated by Martin Lorenz)

ABSTRACT. Our main result implies that, if  $R$  is a simple artinian ring which is not a matrix ring over an absolute field, then any noncentral element of  $R$ , of prime order not dividing the characteristic, is a factor in a free product with a unit which has infinite order in  $R$ . Unexpected consequences follow for division rings and group algebras.

### 1. THE MAIN THEOREM AND ITS CONSEQUENCES

Let  $R$  be an associative algebra over a field  $k$ . If the group of units  $\mathcal{U}(R)$  contains a free product  $C_n * C_\infty$  of cyclic groups, then obviously  $R$  contains a non-central element of order  $n$ . The converse is evidently false in this degree of generality. For example, let  $P$  be a finite non-abelian  $p$ -group, and consider the group ring  $R = k[P]$  over a field  $k$  of characteristic  $p$ . Then  $R$  contains non-central  $p$ -power elements, and it is well known that  $\mathcal{U}(R)$  is nilpotent. Any possible converse also needs to take into account the situation encountered in the following example: Let  $D$  be a division ring the center  $k$  of which contains a primitive  $p$ -th root of unity (for some prime  $p$ ). Then any element of  $D$  of multiplicative order  $p$  is central, so  $D^* = D \setminus \{0\}$  cannot contain a product  $C_p * C_\infty$ . In particular, if  $a$  is a non-central element of  $D$  of order  $p^2$ , then  $D^*$  does not contain  $C_{p^2} * C_\infty$ , and the best one can hope for is a group  $G = \langle a, u \rangle \subseteq D^*$ , with a central subgroup  $Z$  of order  $p$ , such that  $G/Z \cong C_p * C_\infty$ .

For simple artinian rings  $R$ , the situation is not completely clear when the characteristic divides the order of the element  $a$ . Nevertheless, we have the following result (recall that an absolute field is a subfield of the algebraic closure of a finite field).

**Theorem 1.1.** *Let  $R = D^{t \times t}$ , where  $D$  is a division ring with center  $k$ , and assume that  $D$  is not an absolute field. Let  $a \in R$  be a non-central element of finite order*

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Received by the editors April 6, 2003 and, in revised form, March 2, 2004.

2000 *Mathematics Subject Classification.* Primary 16S36; Secondary 16K40, 16P90.

*Key words and phrases.* Simple ring, division ring, element of finite order, free product over the center.

The research of the first author was partially supported by NSERC, Canada, and Fapesp (Projeto Temático 00/07.291-0).

The research of the second author was partially supported by CNPq-Brazil (Grant 302.756/82-5) and Fapesp (Projeto Temático 00/07.291-0).

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$n$ . Set  $\langle a \rangle \cap k^* = \langle a^r \rangle$ , where  $r \mid n$ . Then  $R$  contains infinitely many units  $u$  of infinite order such that  $\langle a, u \rangle / \langle a^r \rangle \cong C_r * C_\infty$  in the following cases:

(a)  $\text{char } D$  does not divide  $n$  (in particular, this is the case when either  $\text{char } D = 0$ , or  $t = 1$ ).

(b)  $D$  is commutative.

(c)  $D$  is locally finite-dimensional over its center.

(We refer to the above situation by saying that  $a$  is largely free in  $R$ , and that  $\langle a, u \rangle$  is a free product modulo the center.)

Plainly, the result cannot hold when  $D$  is an absolute field, because  $\text{GL}(n, D)$  is then locally finite. Also, any attempt to generalize Theorem 1.1 can fail in the presence of non-trivial central idempotents.

**Lemma 1.2.** *Let  $R = \bigoplus_{i=1}^m R_i$  be semisimple artinian with  $m \geq 2$ , and assume that none of the simple rings  $R_i$  is a matrix ring over an absolute field. Then any element of finite prime-power order not dividing  $\text{char } R$  is largely free in  $R$ . The conclusion can fail (a) if the element has composite order; or (b) if one of the  $R_i$  is a non-commutative matrix ring over an absolute field.*

*Proof.* Let  $a \in R$  have order  $n = p^m$ , where  $p$  does not divide  $\text{char } R_i$  for any  $i$ . If  $R_i = Re_i$ , where  $e_i$  is an idempotent, then there exists an  $i$  such that  $ae_i$  has order exactly  $n$  in  $R_i$ . By Theorem 1.1, there exists a unit  $u_i$  of infinite order in  $R_i$  such that  $\langle ae_i, u_i \rangle$  is a free product modulo center. If  $u = u_i + (1 - e_i)$ , then it is trivial to verify that  $\langle a, u \rangle$  is also a free product over the center.

As an example for (a), let  $R = R_1 \oplus R_2$  for ease of notation, and suppose that for  $i = 1, 2$ , the subring  $R_i$  contains an element  $a_i$  of order  $n_i$  modulo the center of  $R_i$ , and that the least common multiple of  $n_1$  and  $n_2$  is  $n > \max\{n_1, n_2\}$ . Then  $a = a_1 + a_2$  has finite order  $n$  modulo the center of  $R$ . Let  $u$  be any unit of  $R$ , and let  $w$  be the group commutator  $(a^{n_1}, u^{-1}a^{n_2}u)$ . Then the projection of  $w$  into each  $R_i$  is  $e_i$ , so in fact  $w = 1$  in  $R$ . But  $w = a^{-n_1}u^{-1}a^{-n_2}ua^{n_1}u^{-1}a^{n_2}u$ , so  $\langle a, u \rangle$  is not a free product modulo its center  $\langle a^n \rangle$ .

For (b), suppose that some  $R_i = k^{t \times t}$  is a matrix ring over an absolute field  $k$ . Let  $\alpha$  be any non-central element of  $R_i$ , and set  $a = \alpha e_i + (1 - e_i)$ . Then, for any unit  $u$  of  $R$ , we have  $\langle a, u \rangle e_j = \langle u e_j \rangle$  if  $j \neq i$ , and  $\langle a, u \rangle e_i \subseteq \mathcal{U}(R_i)$ , which is locally finite. Thus,  $\langle a, u \rangle$  cannot be a free product modulo the center for any  $u$ .  $\square$

We derive a number of consequences of Theorem 1.1 first. Our first corollary is the following counterpart to a recent result of Passman and Gonçalves on integral units [3].

**Corollary 1.3.** *Let  $k$  be a non-absolute field of characteristic  $p \geq 0$ , and let  $G$  be a finite group. Let  $a$  be an element of prime order in  $G$ . Then  $a$  is largely free in the group ring  $k[G]$  if and only if  $a$  is non-central in  $G/O_p(G)$  (recall that  $O_p(G)$  is the largest normal  $p$ -subgroup of the group  $G$ , and  $O_0(G) = 1$ ).*

*Proof.* Let  $J$  denote the Jacobson radical of  $S = k[G]$ . If  $x \in G$  becomes central in  $S/J$ , then the group commutator  $(x, G) \subseteq (1 + J) \cap G = O_p(G)$ , so  $x$  is central in  $G/O_p(G)$ . Therefore, if  $a$  is non-central modulo  $O_p(G)$ , then it remains non-central in the semisimple ring  $S/J$ , and hence in at least one of the simple components  $R$  of  $S/J$ . All such components contain the non-absolute field  $k$  in their center. By Theorem 1.1, the projection  $b$  of  $a$  in  $R$  is largely free in  $R$ , so there exists a unit  $v \in R$  of infinite order such that  $\langle b, v \rangle = \langle b \rangle * \langle v \rangle$  is a free product. After replacing

$v$  by its extension to a unit of  $S/J$ , if necessary,  $v$  can be lifted to a unit  $u$  of  $S$  over the nilpotent ideal  $J$ . The elements  $u$  and  $a$  generate a free product in  $S$ , so  $a$  is largely free.

Conversely, suppose that  $a$  is central in  $G/O_p(G)$ . If  $u$  is any unit of  $S$ , then the commutator  $(a, u) = 1$  in  $S/J$ , since  $a$  is central modulo  $J$ . But the group  $1 + J$  is nilpotent, and so  $\langle a, u \rangle$  is soluble. In other words, no free product in  $S$  can contain  $\langle a \rangle$  as a factor.  $\square$

In another direction, Theorem 1.1 has the following striking corollary.

**Corollary 1.4.** *Let  $R$  be as in Theorem 1.1, and let  $N$  be a normal subgroup of  $\mathcal{U}(R)$ . Then  $N$  contains a non-central element of prime order  $p \neq \text{char } R$  if and only if  $N$  contains a free product of a countable number of copies of  $C_p$ .*

*Proof.* Suppose  $N$  contains a non-central element  $a$  of finite order  $n$  coprime to  $\text{char } D$ . Then, in the notation of Theorem 1.1,  $N$  contains the subgroup  $\langle a^{u^i} : i \in \mathbb{Z} \rangle$ , which, modulo its central subgroup  $\langle a^r \rangle$ , is a free product of countably many copies of  $C_r$ . The stated result is the case  $n = r = p$ .  $\square$

The previous corollary provides a much-simpler proof of a result of Gorbounov *et al.* ([4], Proposition 3.7) regarding the existence of free products in the Morava stabilizer group (a certain normal subgroup of the  $p$ -adic division algebra of index  $n = p - 1$  and Hasse invariant  $1/n$ ). Starting with a non-central element  $\alpha$  of prime order, the authors obtain a free product of a finite number of conjugates of  $\langle \alpha \rangle$  by using Nagao’s Theorem on the amalgamated free product structure of  $\text{GL}(2, K[t])$ , and the uncountability of the  $p$ -adic field.

We note in passing that Corollary 1.4 is also a generalization of Herstein’s well-known result that a non-central element of finite order in a division ring has an infinite number of conjugates (see, e.g., [8], Theorem 13.26).

## 2. PROOF OF THE MAIN THEOREM

The proof of Theorem 1.1 is based on two special cases. The first requires the following technical result.

**Lemma 2.1.** *Let  $R = \sum_{i=0}^{m-1} y^i K$  be a cyclic algebra, where  $\text{Gal}(K/E) = \langle \sigma \rangle$  has order  $m$ ,  $y^m = b$ , and  $E$  is not a finite field. Let  $a \in K$  such that  $a, a^\sigma, \dots, a^{\sigma^{m-1}}$  are distinct. Then there exists a non-zero rational function  $\phi_a \in E(x_0, \dots, x_{m-1})$ , where  $x_0, \dots, x_{m-1}$  are commuting indeterminates over the field  $K$ , with the following property: If  $w = \sum y^i w_i \in E[y]$ , then  $w^{-1}aw$  has full support if and only if  $\phi_a(w_0, \dots, w_{m-1}) \neq 0$  (as in [2], we say that an element  $\sum_{i=0}^{m-1} y^i b_i \in R$  has full support if every  $b_i \neq 0$ ).*

*Proof.* Set  $S = R \otimes_E E(x_0, \dots, x_{m-1})$ , and let  $x = \sum_{i=0}^{m-1} y^i x_i$  be the generic element of  $S$ . Write  $y^m = b \in E$ . It is convenient to have  $b^{m!} \neq 1$ . If this is not the case, replace  $y$  by  $\alpha y$ , where  $\alpha \in E^*$  is such that  $\alpha^{m!} \neq 1$ . This replaces  $y^i$  by  $\alpha^i y^i$ , and so has no effect on the size of the support of the elements of  $R$ .

Write  $x^{-1} = f^{-1} \sum_{i=0}^{m-1} y^i f_i$ , where  $f, f_i \in E[x_0, \dots, x_{m-1}]^*$ . The equation  $1 = x^{-1}x$  immediately implies that

$$(2.1) \quad x_0 f_0 + b \sum_{i=1}^{m-1} x_i f_{m-i} = f,$$

and for  $k = 1, \dots, m - 1$ ,

$$(2.2) \quad \sum_{i=0}^k x_i f_{k-i} + b \sum_{i=k+1}^{m-1} x_i f_{m+k-i} = 0.$$

If we write  $x^{-1}ax = f^{-1} \sum_{t=0}^{m-1} y^t g_t$ , then it is trivial to verify that

$$(2.3) \quad g_t = \sum_{i=0}^t a^{\sigma^i} x_i f_{t-i} + b \sum_{i=t+1}^{m-1} a^{\sigma^i} x_i f_{m+t-i}$$

for  $t = 0, \dots, m - 1$ . Use (2.1) or (2.2) as appropriate to eliminate the term involving  $x_0$  from (2.3). This yields

$$(2.4) \quad g_0 = af + b \sum_{i=1}^{m-1} (a^{\sigma^i} - a)x_i f_{m-i},$$

and for  $t \geq 1$ ,

$$(2.5) \quad g_t = \sum_{i=1}^t (a^{\sigma^i} - a)x_i f_{t-i} + b \sum_{i=t+1}^{m-1} (a^{\sigma^i} - a)x_i f_{m+t-i}.$$

We claim that  $\phi_a = fg_0 \dots g_{m-1} \neq 0$ . The easiest way to see this is to find suitable specializations of the values  $x_i$  for which the various  $g_t \neq 0$ . For example, consider the values  $x_0 = 1, x_i = 0$  for  $i \geq 1$ . This specializes  $x, x^{-1} \mapsto 1$ , and by (2.4), specializes  $g_0$  to  $af(1, 0, \dots, 0) = a \neq 0$ . In particular,  $g_0$  is not identically zero. Similarly, consider some  $t \in [1, m - 1]$ . Let the least common multiple of  $m$  and  $t$  be  $n$ , so  $n = tu = mv$ . Then  $1 - b^v = 1 - y^n = 1 - (y^t)^u$ . Therefore,  $(1 - b^v)(1 - y^t)^{-1} = (1 - (y^t)^u)(1 - y^t)^{-1} = \sum_{j=0}^{u-1} y^{jt}$ . Consider the specialization  $x_0 = 1, x_t = -1$ , and all other  $x_i = 0$ . This yields  $x \mapsto 1 - y^t$ , and from (2.5) we obtain  $g_t \mapsto (a^{\sigma^t} - a)x_t f_0 \mapsto -(a^{\sigma^t} - a)(1 - b^v)^{-1} \neq 0$ . The result follows.  $\square$

The following is the first special case of Theorem 1.1.

**Corollary 2.2.** *Let  $R = (K/E, \sigma, b)$ , be a cyclic algebra, where  $Gal(K/E) = \langle \sigma \rangle$  has order  $m$ , and the field  $K = E[a]$  is generated by a primitive  $n$ -th root of unity  $a$  over the center  $E$ . Assume that  $E$  is the quotient field of a Dedekind domain distinct from  $E$ . Then  $a$  is largely free in  $R$ .*

*Proof.* Write  $R = \bigoplus_{i=0}^{m-1} y^i K$  and  $b = y^m \in E$ . As in [2], for a fixed non-trivial discrete valuation  $\nu$  of  $K$ , write  $V_\nu \subset K$  for the set of those elements  $x \in K$  such that the set of values  $\{\nu(x^{\sigma^i}) : i = 0, \dots, m - 1\}$  has a unique maximum and a unique minimum, and let  $T_\nu \subset R$  be the set of those non-zero elements  $w = \sum_{i=0}^{m-1} y^i w_i$  of  $R$  such that the value  $\nu(w_i^{\sigma^j})$  is independent of  $i$  and  $j$ .

The elements  $a, a^2, \dots, a^{r-1}$  are all distinct (recall that  $r$  is the least integer such that  $a^r$  is central). Let  $\phi$  be the product of the finitely many rational functions  $\phi_a, \phi_{a^2}, \dots, \phi_{a^{r-1}}$  provided by Lemma 2.1. Since the field  $E$  is infinite, there exist  $m$ -tuples  $(e_0, \dots, e_{m-1}) \in E^{(m)}$  for which  $\phi(e_0, \dots, e_{m-1})$  is defined and non-zero. Setting  $z = \sum y^i e_i$ , we find that all the conjugates  $c^i = z^{-1} a^i z, i = 1, \dots, r - 1$ , have full support.

By Lemma 7 of [2], there exists a valuation  $\nu$  of  $K$  such that  $V = V_\nu$  is infinite and all the  $c^i$  belong to  $T = T_\nu$ . If  $v \in V$ , then any word of the form  $c^{r_1} v^{s_1} \dots c^{r_d} v^{s_d}$ , with  $1 \leq r_1, \dots, r_d \leq r - 1$ , belongs to  $TVTV \dots TV$ . By Lemma 6 of [2], the

latter product is contained in  $TV$ . Since  $TV$  consists of elements of full support in  $R$ , it has empty intersection with  $K$ , so the product  $TVT\cdots TV$  cannot be trivial. In other words,  $\langle c, v \rangle$  is a free product over its center, and hence so is  $\langle c, v \rangle^{z^{-1}} = \langle a, v^{z^{-1}} \rangle$ .  $\square$

The other special case of Theorem 1.1 is, in some sense, at the other extreme, and deals with the case where the underlying division ring is commutative. We need a number of preliminary results.

**Lemma 2.3.** *Let  $K$  be a field with a non-trivial discrete valuation  $\nu$ , and let  $t \geq 2$ . Fix a set  $\{e_{ij}\}$  of  $t^2$  matrix units in  $R = K^{t \times t}$ . Consider the subsets*

$$V = V_\nu = \left\{ \text{diag}(r_i) \in GL(t, K) : \text{the set } \{\nu(r_i)\} \text{ has a unique minimum value} \right\},$$

$$T = T_\nu = \left\{ \sum_{i,j} t_{ij}e_{ij} : \text{the value } \nu(r_{ij}) \text{ is finite and independent of } i \text{ and } j \right\}$$

of  $R$ . Then

- (i)  $TVT \subseteq T$ .
- (ii)  $TV$  and  $VT$  are sub-semigroups of  $R$ , neither of which contains a scalar matrix.

*Proof.* For  $t = \sum_{i,j} t_{ij}e_{ij} \in T$ , write  $\nu(t) = \nu(t_{ij})$ . Similarly, for  $w = \text{diag}(w_i) \in V$ , write  $\nu(w) = \min\{\nu(w_i)\}$ . For any  $i, j$ , we have  $(wt)_{i,j} = \sum_m w_{im}t_{mj} = w_it_{ij} \neq 0$ , so the final assertion of (ii) follows. If also  $t' \in T$ , then for any  $i, j$  we have  $(twt')_{i,j} = \sum_{m,n} t_{im}w_{mn}t'_{nj} = \sum_m t_{im}w_mt'_{mj}$ . For a fixed  $m$ , we have  $\nu(t_{im}w_mt'_{mj}) = \nu(t) + \nu(t') + \nu(w_m)$ . Since every  $m$  occurs in the sum for  $(twt')_{i,j}$ , it is evident that  $\nu((twt')_{i,j}) = \nu(t) + \nu(t') + \nu(w)$ . By definition, this means that  $twt' \in T$ , so (i) is established. The rest of (ii) now follows because  $(TV)(TV) = (TVT)V \subseteq TV$ , and similarly  $VTVT \subseteq VT$ .  $\square$

**Corollary 2.4.** *Let  $K$  be the quotient field of a Dedekind domain not equal to  $K$ , and let  $t \geq 2$ . If  $S$  is any finite set of non-scalar matrices in  $GL(t, K)$ , then there exists a matrix  $x \in GL(t, K)$ , and a discrete valuation  $\nu$  of  $K$ , such that  $S^x$  is contained in the set  $T_\nu$  defined in Lemma 2.3. If  $K$  is a finite Galois extension of a subfield  $\Phi$ , then we may choose  $x \in GL(t, \Phi)$ .*

*Proof.* Let  $g \in GL(t, K)$  be non-scalar, and let  $x = (x_{ij})$  be the generic  $t \times t$  matrix in  $t^2$  commuting indeterminates  $x_{ij}$  over  $K$ . It is well known that  $x^{-1}gx$  has all entries non-zero (see, e.g., [10], page 36, Point 1). Let  $\phi_g \neq 0$  be the product of the entries of  $x^{-1}gx$ , so  $\phi_g \in K(x_{ij} : 1 \leq i, j \leq t)$ . If  $Q(g)$  is the Zariski open subset of  $K^{t \times t}$  where  $\phi_g$  is defined and non-zero, then, for every  $w \in Q(g)$ , the conjugate  $g^w = \sum_{i,j} t_{ij}e_{ij}$  has every  $t_{ij} \neq 0$ . In particular, any  $w \in \bigcap_{g \in S} Q(g)$  has the property that all the  $g^w$  have non-zero entries. The existence of  $\nu$  now follows from Lemma 7 of [2].

To obtain the result over  $\Phi$ , simply replace  $\phi_g$  by  $\psi_g = \prod_{\gamma \in \Gamma} (\phi_g)^\gamma$ , where  $\Gamma = \text{Gal}(K/\Phi)$ . Then  $\psi_g \in \Phi(x_{ij} : 1 \leq i, j \leq t)^*$ , and the result follows as before.  $\square$

This allows us to prove another special case of Theorem 1.1.

**Corollary 2.5.** *Let  $K$  be a non-absolute field. If  $a$  is any non-scalar matrix of finite order in  $GL(t, K)$ , then there exist infinitely many elements  $u \in GL(t, K)$ , of*

infinite order, such that  $\langle a, u \rangle$  is a free product over the center. If  $K/k$  is a finite Galois extension, then the elements  $u$  may be chosen to belong to  $GL(t, k)$ .

*Proof.* Let  $\Lambda$  be the subring of  $K$  generated by the finitely many entries of  $a$  (and, if necessary, an element of  $K$  which is not a root of unity). Then  $\Lambda$  is a Dedekind domain which is not a field, and we may replace  $K$  by the quotient field of  $\Lambda$  in what follows.

Let  $S$  consist of the finitely many non-scalar powers of  $a$ . Choose a valuation  $\nu$  as in Corollary 2.4, and then choose any  $w \in V_\nu \cap V_\nu^{-1}$ , where  $V_\nu$  is defined in Lemma 2.3. By part (ii) of Lemma 2.3, there exists a conjugate  $b$  of  $a$  such that the group  $\langle b, w \rangle$  is a free product modulo its center. Conjugating back, we obtain the result for  $a$ .

Finally, if  $k$  is given, then the sets  $T_\nu$  and  $V_\nu$  may be chosen to lie in  $k^{t \times t}$ , and the conjugating element may also be chosen in  $GL(t, k)$ , by Corollary 2.4.  $\square$

*Proof of Theorem 1.1.* Part (b) is Corollary 2.5. For part (c), we may obviously assume that  $D$  is finite-dimensional over  $k$ . Let  $K/k$  be a Galois splitting field for  $R$ , so  $R_K = R \otimes_k K \cong K^{t \times t}$  for some  $t$ . Consider the natural action of  $G = \text{Gal}(K/k)$  on  $R_K$ . By the last statement of Corollary 2.5, the unit  $u$  may be chosen to belong to the fixed ring of  $R_K$  under  $G$ . This fixed ring is obviously  $R$ , as required. We now turn to part (a), and we distinguish several cases.

*Case 1.  $F = k[a]$  is a subfield of  $R$ :* The extension  $F = k(a)$  of  $k$  is non-trivial and cyclotomic, so we can write  $\text{Gal}(F/k) = \langle \sigma \rangle$ , where  $\sigma$  has finite order  $m \geq 2$ . By the (generalized) Skolem-Noether Theorem (e.g., [5], Theorem 4.3.1),  $\sigma$  is induced by conjugation by an element  $y \in D$ . In particular,  $y^m = b \in C_R(a)$ .

Let  $K$  be the quotient field of  $k_0[a, b]$ , where  $k_0$  is the prime field. Then  $x$  acts on  $K$  as the automorphism  $\sigma$ , and  $K[x] \equiv (K/E, \sigma, b)$  is a classical cyclic algebra, where  $E$  is the fixed field of  $K$ . The conclusion in this case follows from Corollary 2.2.

In the remaining cases,  $F = k[a]$  is not a field. It is, however, semisimple, because the order of  $a$  is coprime to  $\text{char } k$ . Let  $1 = e_1 + \dots + e_d$  be the decomposition of 1 into primitive idempotents in  $F$ .

*Case 2.  $k$  contains a primitive  $n$ -th root of unity  $\zeta$ :* In this case, every  $F e_i = k e_i$ . Let  $V \cong D^{(t)}$  be a simple  $R$ -module. Then  $V = \bigoplus_{j=1}^d V e_j$ . The decomposition of  $V e_j$  into one-dimensional  $D$ -subspaces is equivalent to the decomposition of each  $e_j$  into a sum of orthogonal idempotents  $f_{ji}$ . These can be chosen to satisfy  $f_{ji} e_j = e_j f_{ji} = f_{ji}$  for all  $i$ , and then the  $f_{ji}$  also commute with  $a e_j$ , since by assumption  $a e_j \in k_j$ . This means that the matrix of  $a$  determined by the idempotents  $f_{ji}$  is block diagonal  $\text{diag}(A_1, \dots, A_d)$ , where each  $A_j = \zeta^{r_j} I$  is a scalar matrix, and the least common multiple of the exponents  $r_j$  is  $n$  (what makes  $a$  non-central is, of course, the fact that not all the  $r_j$  are the same).

We also observe that, if  $k$  is an absolute field, then  $D$  contains an element  $b$  transcendental over  $k$  (since algebraic division algebras over absolute fields are commutative). Therefore,  $D$  always contains a non-absolute subfield  $K$ , and so  $R$  contains the subring  $R_1 = K^{t \times t}$ . Note that  $a \in R_1$  by construction. The existence of a free product now follows from Corollary 2.5 and completes the proof of Theorem 1.1.

*Case 3.  $F = k[a]$  is not a field, and  $k$  does not contain a primitive  $n$ -th root of unity:* Let  $\Phi = k(\zeta)$ , where  $\zeta$  is a primitive  $n$ -th root of unity, and set  $S = R \otimes_k \Phi$ .

Then  $S = \Delta^{s \times s}$  is simple artinian (see, e.g., [1], p. 364). By Case 2, there exists a subfield  $K \supseteq \Phi$  of  $S$  which is not absolute, and such that  $a \in K^{s \times s}$ . Replacing  $K$  by the intersection of its finitely many conjugates under  $\text{Gal}(\Phi/k)$ , we may assume that  $K$  is stable under  $\text{Gal}(\Phi/k)$ . The conclusion follows from Corollary 2.5, by Galois descent.  $\square$

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