

BASES OF THE CONTACT-ORDER FILTRATION OF DERIVATIONS OF COXETER ARRANGEMENTS

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ABSTRACT. In a recent paper, we constructed a basis for the contact-order filtration of the module of derivations on the orbit space of a finite real reflection group acting on an ℓ -dimensional Euclidean space. Recently M. Yoshinaga constructed another basis for the contact-order filtration. In this note we give an explicit formula relating Yoshinaga's basis to the basis we constructed earlier. The two bases turn out to be equal (up to a constant matrix).

1. THE SETUP AND THE MAIN RESULT

Let V be an ℓ -dimensional Euclidean vector space with inner product I . Then its dual space V^* is equipped with the inner product I^* , which is induced by I . Let S be the symmetric algebra of V^* over \mathbb{R} . Identify S with the algebra of polynomial functions on V . Let Der_S be the S -module of \mathbb{R} -linear derivations of S . When X_1, \dots, X_ℓ denote a basis for V^* , the partial derivations $\partial_i := \partial/\partial X_i$ with respect to X_i ($1 \leq i \leq \ell$) naturally form a basis for Der_S over S . Let K be the field of quotients of S and Der_K be the K -vector space of \mathbb{R} -linear derivations of K . Then the partial derivations ∂_i ($1 \leq i \leq \ell$) naturally form a basis for Der_K over K .

Let W be a finite irreducible orthogonal reflection group (a **Coxeter group**) acting on V . The Coxeter group W naturally acts on V^* , S and Der_S . The W -invariant subring of S is denoted by R . Then it is classically known [1, V.5.3, Theorem 3] that there exist algebraically independent homogeneous polynomials $P_1, \dots, P_\ell \in R$ with $\deg P_1 \leq \dots \leq \deg P_\ell$, which are called **basic invariants**, such that

$$R = \mathbb{R}[P_1, \dots, P_\ell].$$

The **primitive derivation** $D \in \text{Der}_R$ is characterized by

$$DP_i = \begin{cases} 1 & \text{for } i = \ell, \\ 0 & \text{otherwise.} \end{cases}$$

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Write $\mathbf{P} = (P_1, \dots, P_\ell)$. Let $J(\mathbf{P})$ denote the Jacobian matrix $[\partial P_j / \partial X_i]_{ij}$. Define $\Delta := \det J(\mathbf{P})$. Then Δ is an anti-invariant [1, V.5.4] and $\Delta^2 \in R$. Let

$$I : \text{Der}_R \times \text{Der}_R \rightarrow \frac{1}{\Delta^2} R$$

be the symmetric R -bilinear form induced by I . Let

$$\begin{aligned} \nabla : \text{Der}_R \times \text{Der}_R &\longrightarrow \frac{1}{\Delta^2} \text{Der}_R, \\ (X, Y) &\longmapsto \nabla_X Y \end{aligned}$$

be the **Levi-Civita connection** with respect to I . Since $\frac{\partial}{\partial P_j} \in \frac{1}{\Delta} \text{Der}_S$ by Cramer's rule, one can embed Der_R into Der_K . Extend the Levi-Civita connection naturally to

$$\nabla : \text{Der}_K \times \text{Der}_K \longrightarrow \text{Der}_K.$$

Note that $\nabla_\xi(\eta) = \sum_i \xi(\eta(X_i)) \partial_i$ because each $I(\partial_i, \partial_j)$ is constant.

In [6], we introduced derivations $\xi_1^{(m)}, \dots, \xi_\ell^{(m)} \in \text{Der}_S$ for $m \geq 0$ by

$$(\xi_1^{(m)}, \dots, \xi_\ell^{(m)}) := \begin{cases} (\frac{\partial}{\partial P_1}, \dots, \frac{\partial}{\partial P_\ell}) J(\mathbf{P})^T A J(D^k[\mathbf{X}])^{-1} & \text{if } m = 2k, \\ (\frac{\partial}{\partial P_1}, \dots, \frac{\partial}{\partial P_\ell}) J(\mathbf{P})^T A J(D^k[\mathbf{X}])^{-1} J(\mathbf{P}) & \text{if } m = 2k + 1. \end{cases}$$

Here $A := [I^*(X_i, X_j)]_{ij}$ and $J(D^k[\mathbf{X}]) := [\partial(D^k(X_j)/\partial X_i)]_{ij}$ ($D^k := D \circ D \circ \dots \circ D$ (k times)). Let \mathcal{A} be the Coxeter arrangement determined by the Coxeter group W : \mathcal{A} is the set of reflecting hyperplanes. Choose for each hyperplane $H \in \mathcal{A}$ a linear form $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. The derivations $\xi_1^{(m)}, \dots, \xi_\ell^{(m)} \in \text{Der}_S$ were constructed in [6] so that they may form a basis for the S -module [10]

$$D^{(m)}(\mathcal{A}) := \{\theta \in \text{Der}_S \mid \theta(\alpha_H) \in S\alpha_H^m \text{ for any } H \in \mathcal{A}\}$$

for each nonnegative integer m . The filtration

$$\text{Der}_S = D^{(0)}(\mathcal{A}) \supset D^{(1)}(\mathcal{A}) \supset D^{(2)}(\mathcal{A}) \supset \dots$$

of Der_S is called the **contact-order filtration**. Define

$$\mathcal{H}^{(k)} = D^{(2k-1)}(\mathcal{A}) \cap \text{Der}_R$$

for each $k \geq 1$. Let $\mathcal{H}^{(0)} = \text{Der}_R$. Then we have a filtration

$$\text{Der}_R = \mathcal{H}^{(0)} \supset \mathcal{H}^{(1)} \supset \mathcal{H}^{(2)} \supset \dots$$

of Der_R . This filtration is known [7] to be equal to the **Hodge filtration** introduced by K. Saito [4]. The derivations $\xi_1^{(2k-1)}, \dots, \xi_\ell^{(2k-1)} \in \text{Der}_R$ form an R -basis for $\mathcal{H}^{(k)}$ for $k \geq 1$. Therefore the contact-order filtration, restricted to Der_R , gives the Hodge filtration. Define

$$T := \{f \in R \mid Df = 0\} = \mathbb{R}[P_1, \dots, P_{\ell-1}].$$

Then the covariant derivative $\nabla_D : \text{Der}_K \rightarrow \text{Der}_K$ is T -linear. The Hodge filtration was originally defined so that the T -linear map

$$\nabla_D^k := \nabla_D \circ \dots \circ \nabla_D \text{ (} k \text{ times)} : \mathcal{H}^{(k)} \longrightarrow \mathcal{H}^{(0)} = \text{Der}_R$$

is bijective [4]. Thus we may define $\nabla_D^{-k} \xi \in \mathcal{H}^{(k)}$ for any $\xi \in \text{Der}_R$ and $k \geq 0$. In [8] M. Yoshinaga proved the following theorem.

Theorem 1.1. ([8, Theorem 6]) *Let $k \geq 0$ and*

$$E := \sum_i X_i \partial_i = \sum_i (\deg P_i)^{-1} P_i (\partial / \partial P_i)$$

be the Euler derivation. Suppose that ξ_1, \dots, ξ_ℓ are a basis for $D^{(1)}(\mathcal{A})$. Then

- (1) *the derivations $\nabla_{\xi_1} \nabla_D^{-k} E, \dots, \nabla_{\xi_\ell} \nabla_D^{-k} E$ form a basis for $D^{(2k-1)}(\mathcal{A})$ over S , and*
- (2) *the derivations $\nabla_{\partial_1} \nabla_D^{-k} E, \dots, \nabla_{\partial_\ell} \nabla_D^{-k} E$ form a basis for $D^{(2k)}(\mathcal{A})$ over S .*

It thus seems natural to ask how the basis above constructed by Yoshinaga is related to the basis $\xi_1^{(m)}, \dots, \xi_\ell^{(m)}$ given in [6]. The following theorem answers this question:

Theorem 1.2. *Let $k \geq 0$. Then*

$$(\xi_1^{(2k+1)}, \dots, \xi_\ell^{(2k+1)}) = (-1)^k (\nabla_{\xi_1^{(1)}} \nabla_D^{-k} E, \dots, \nabla_{\xi_\ell^{(1)}} \nabla_D^{-k} E)$$

and

$$(\xi_1^{(2k)}, \dots, \xi_\ell^{(2k)}) = (-1)^k (\nabla_{\partial_1} \nabla_D^{-k} E, \dots, \nabla_{\partial_\ell} \nabla_D^{-k} E)A.$$

So the two bases turn out to be equal up to a constant matrix.

The significance of Theorem 1.2 is as follows: Recently M. Yoshinaga [9] affirmatively settled the Edelman-Reiner conjecture, which asserts that the cones of the extended Shi/Catalan arrangements are free. In other words, there exist basic derivations for each extended Shi/Catalan arrangement. However, a formula for basic derivations is still unknown. Since the derivations $\xi_i^{(m)}$ ($i = 1, \dots, \ell$) are the principal (= highest degree) parts of basic derivations, Theorem 1.2 can be interpreted as a differential-geometric formula for the “principal parts.” So it may suggest the existence of a differential-geometric formula for the *whole* basic derivations, including the “non-principal part.” One may also regard Theorem 1.2 as a very explicit algebraic description of the derivations in the right-hand side. They are, as mentioned above, when m is odd, bases for the Hodge filtration, which is the key to defining the flat structure on the orbit space V/W in [5]. The flat structure is called the *Frobenius manifold structure* from the viewpoint of topological field theory [2].

2. PROOF

We will prove Theorem 1.2 in this section. First we show the following.

Lemma 2.1. *For $k \geq 1$ and $\xi \in \text{Der}_K$, we have*

$$\nabla_D^k \circ \nabla_\xi - \nabla_\xi \circ \nabla_D^k = k \nabla_D^{k-1} \circ \nabla_{[D, \xi]}.$$

Proof. We use an induction on k . When $k = 1$, the lemma asserts that

$$\nabla_D \circ \nabla_\xi - \nabla_\xi \circ \nabla_D = \nabla_{[D, \xi]},$$

which is the integrable property of the Levi-Civita connection ∇ . Let $k > 1$. We have

$$\begin{aligned} \nabla_D^k \circ \nabla_\xi &= \nabla_D^{k-1} \circ (\nabla_\xi \circ \nabla_D + \nabla_{[D, \xi]}) \\ &= (\nabla_D^{k-1} \circ \nabla_\xi) \circ \nabla_D + \nabla_D^{k-1} \circ \nabla_{[D, \xi]} \\ &= (\nabla_\xi \circ \nabla_D^{k-1} + (k-1) \nabla_D^{k-2} \circ \nabla_{[D, \xi]}) \circ \nabla_D + \nabla_D^{k-1} \circ \nabla_{[D, \xi]} \\ &= \nabla_\xi \circ \nabla_D^k + (k-1) \nabla_D^{k-2} \circ \nabla_{[D, \xi]} \circ \nabla_D + \nabla_D^{k-1} \circ \nabla_{[D, \xi]} \end{aligned}$$

by using the induction assumption. Since $[D, [D, \xi]] = 0$, we obtain

$$\nabla_D \circ \nabla_{[D, \xi]} = \nabla_{[D, \xi]} \circ \nabla_D.$$

This implies that

$$\nabla_D^k \circ \nabla_\xi = \nabla_\xi \circ \nabla_D^k + k \nabla_D^{k-1} \circ \nabla_{[D, \xi]}.$$

□

Recall the W -invariant inner product $I^* : V^* \times V^* \rightarrow \mathbb{R}$. Let Ω_R^1 denote the R -module of Kähler differentials. Let

$$I^* : \Omega_R^1 \times \Omega_R^1 \rightarrow R$$

be the symmetric R -bilinear form induced by I^* . Let

$$G := [I^*(dP_i, dP_j)]_{ij} = J(\mathbf{P})^T A J(\mathbf{P}).$$

Define

$$B^{(k)} := -J(\mathbf{P})^T A J(D^k[\mathbf{X}]) J(D^{k-1}[\mathbf{X}])^{-1} J(\mathbf{P})$$

for $k \geq 1$ as in [6]. Then we have

- Lemma 2.2.** (1) Every entry of $B^{(k)}$ lies in T ; $D[B^{(k)}] = 0$,
 (2) $\det B^{(k)} \in \mathbb{R}^*$,
 (3) $D[G] = B^{(1)} + (B^{(1)})^T$,
 (4) $B^{(k+1)} = B^{(1)} + kD[G]$.

Proof. By [6, Lemmas 3.3–3.6, Remark 3.7]. □

The following proposition is one of the main results in [7]:

Proposition 2.3. For $k \geq 1$,

$$(\nabla_D^k \xi_1^{(2k-1)}, \dots, \nabla_D^k \xi_\ell^{(2k-1)}) = (-1)^{k-1} \left(\frac{\partial}{\partial P_1}, \dots, \frac{\partial}{\partial P_\ell} \right) B^{(k)}.$$

In the rest of this note, let (ξ_1, \dots, ξ_ℓ) denote $(\xi_1^{(1)}, \dots, \xi_\ell^{(1)})$ for simplicity. Note that

$$(\xi_1, \dots, \xi_\ell) = \left(\frac{\partial}{\partial P_1}, \dots, \frac{\partial}{\partial P_\ell} \right) G.$$

Lemma 2.4.

$$([D, \xi_1], \dots, [D, \xi_\ell]) = \left(\frac{\partial}{\partial P_1}, \dots, \frac{\partial}{\partial P_\ell} \right) D[G] = (\nabla_D \xi_1, \dots, \nabla_D \xi_\ell) (B^{(1)})^{-1} D[G].$$

Proof. The first equality easily follows from

$$[D, \xi_j](P_i) = D \circ \xi_j(P_i) - \xi_j \circ D(P_i) = D \circ \xi_j(P_i) = D[I^*(dP_i, dP_j)].$$

For the second equality, apply Proposition 2.3 when $k = 1$. □

Lemma 2.5. For any $\eta \in \text{Der}_K$,

$$(\nabla_{\xi_1} \eta, \dots, \nabla_{\xi_\ell} \eta) = (\nabla_{\partial_1} \eta, \dots, \nabla_{\partial_\ell} \eta) A J(\mathbf{P}).$$

Proof. Since both sides are additive with respect to η , we may assume $\eta = f\partial_j$ for some j and $f \in K$. Then we have

$$\begin{aligned} (\nabla_{\xi_1}\eta, \dots, \nabla_{\xi_\ell}\eta) &= (\nabla_{\xi_1}(f\partial_j), \dots, \nabla_{\xi_\ell}(f\partial_j)) \\ &= (\xi_1(f)\partial_j, \dots, \xi_\ell(f)\partial_j) \\ &= ((\partial_1 f)\partial_j, \dots, (\partial_\ell f)\partial_j)AJ(\mathbf{P}) \\ &= (\nabla_{\partial_1}(f\partial_j), \dots, \nabla_{\partial_\ell}(f\partial_j))AJ(\mathbf{P}) \\ &= (\nabla_{\partial_1}\eta, \dots, \nabla_{\partial_\ell}\eta)AJ(\mathbf{P}). \end{aligned}$$

□

Proof of Theorem 1.2. When $k = 0$, the formulas are obviously true. Let $k \geq 1$. Let $\xi \in \{\xi_1, \dots, \xi_\ell\}$. Apply the identity in Lemma 2.1 to $\nabla_D^{-k}E$ to get

$$(\nabla_D^k \circ \nabla_\xi)(\nabla_D^{-k}E) - \nabla_\xi E = k(\nabla_D^{k-1} \circ \nabla_{[D, \xi]})(\nabla_D^{-k}E) = k \nabla_D^{-1}(\nabla_{[D, \xi]}E).$$

Since $\nabla_\eta E = \eta$ for any $\eta \in \text{Der}_K$, one obtains

$$\nabla_D^k(\nabla_\xi \nabla_D^{-k}E) - \xi = k \nabla_D^{-1}[D, \xi]$$

and thus

$$\nabla_D^{k+1}(\nabla_\xi \nabla_D^{-k}E) = \nabla_D \xi + k [D, \xi].$$

On the other hand, Lemma 2.4 asserts

$$([D, \xi_1], \dots, [D, \xi_\ell]) = (\nabla_D \xi_1, \dots, \nabla_D \xi_\ell)(B^{(1)})^{-1}D[G].$$

So we have

$$\begin{aligned} &(\nabla_D^{k+1}(\nabla_{\xi_1} \nabla_D^{-k}E), \dots, \nabla_D^{k+1}(\nabla_{\xi_\ell} \nabla_D^{-k}E)) \\ &= (\nabla_D \xi_1, \dots, \nabla_D \xi_\ell) + k(\nabla_D \xi_1, \dots, \nabla_D \xi_\ell)(B^{(1)})^{-1}D[G] \\ &= (\nabla_D \xi_1, \dots, \nabla_D \xi_\ell)(B^{(1)})^{-1}(B^{(1)} + kD[G]) \\ &= (\nabla_D \xi_1, \dots, \nabla_D \xi_\ell)(B^{(1)})^{-1}B^{(k+1)} \\ &= (\partial/\partial P_1, \dots, \partial/\partial P_\ell)B^{(k+1)} \\ &= (-1)^k(\nabla_D^{k+1}\xi_1^{(2k+1)}, \dots, \nabla_D^{k+1}\xi_\ell^{(2k+1)}) \end{aligned}$$

by Lemma 2.2 and Proposition 2.3. This proves the first formula.

For the second formula, compute

$$(\nabla_{\xi_1} \nabla_D^{-k}E, \dots, \nabla_{\xi_\ell} \nabla_D^{-k}E) = (\nabla_{\partial_1} \nabla_D^{-k}E, \dots, \nabla_{\partial_\ell} \nabla_D^{-k}E)AJ(\mathbf{P})$$

by Lemma 2.5. Applying the first formula, we get

$$\begin{aligned} (\xi_1^{(2k)}, \dots, \xi_\ell^{(2k)})J(\mathbf{P}) &= (\xi_1^{(2k+1)}, \dots, \xi_\ell^{(2k+1)}) \\ &= (-1)^k(\nabla_{\xi_1} \nabla_D^{-k}E, \dots, \nabla_{\xi_\ell} \nabla_D^{-k}E) \\ &= (-1)^k(\nabla_{\partial_1} \nabla_D^{-k}E, \dots, \nabla_{\partial_\ell} \nabla_D^{-k}E)AJ(\mathbf{P}). \end{aligned}$$

This completes the proof of Theorem 1.2.

□

REFERENCES

- [1] Bourbaki, N.: Groupes et Algèbres de Lie. Chapitres 4,5 et 6, Hermann, Paris, 1968 MR0240238 (39:1590)
- [2] Dubrovin, B.: Geometry of 2D topological field theories. In: *“Integrable systems and quantum groups”* (ed. Francaviglia, M., Greco, S.), Lectures at C.I.M.E., 1993, LNM **1620**, Springer, Berlin-Heidelberg-New York, 1996, pp. 120–348 MR1397274 (97d:58038)
- [3] Orlik, P., Terao, H.: *Arrangements of Hyperplanes*. Grundlehren der Math. Wiss. **300**, Springer Verlag, 1992 MR1217488 (94e:52014)
- [4] Saito, K.: On a linear structure of the quotient variety by a finite reflexion group. RIMS Kyoto preprint **288**, 1979 = Publ. Res. Inst. Math. Sci. **29** (1993) 535–579 MR1245441 (94k:32059)
- [5] Saito, K.: Finite reflection groups and related geometry (A motivation to the period mapping for primitive forms). preprint, 2000
- [6] Terao, H.: Multiderivations of Coxeter arrangements. *Inventiones Math.*, **148** (2002) 659–674 MR1908063 (2003h:20074)
- [7] Terao, H.: The Hodge filtration and the contact-order filtration of derivations of Coxeter arrangements. preprint 2002 (math.CO/0205058)
- [8] Yoshinaga, M.: The primitive derivation and freeness of multi-Coxeter arrangements. *Proc. Japan Acad. Ser. A Math. Sci.* **78** (2002), no. 7, 116–119 MR1930214 (2003k:32039)
- [9] Yoshinaga, M.: Characterization of a free arrangement and conjecture of Edelman and Reiner. *Inventiones Math.*, **157** (2004), 449–454. MR2077250
- [10] Ziegler, G. M.: Multiarrangements of hyperplanes and their freeness. In: *Singularities*. Contemporary Math. **90**, Amer. Math. Soc., 1989, pp. 345–359 MR1000610 (90e:32015)

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