THE INITIAL VALUE PROBLEM FOR A THIRD ORDER DISPERSED EQUATION ON THE TWO-DIMENSIONAL TORUS

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ABSTRACT. We present the necessary and sufficient conditions for the $L^2$-well-posedness of the initial problem for a third order linear dispersive equation on the two-dimensional torus. Birkhoff’s method of asymptotic solutions is used to prove necessity. Some properties of a system for quadratic algebraic equations associated to the principal symbol play a crucial role in proving sufficiency.

1. introduction

This paper is concerned with the initial value problem of the form

$$Lu = f(t, x) \text{ in } \mathbb{R} \times \mathbb{T}^2,$$

$$u(0, x) = u_0(x) \text{ in } \mathbb{T}^2,$$

where $u(t, x)$ is a real-valued unknown function of $(t, x) = (t, x_1, x_2) \in \mathbb{R} \times \mathbb{T}^2$, $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, $u_0(x)$ and $f(t, x)$ are given real-valued functions,

$$L = \partial_t + p(\partial) + \sum_{|\alpha| = 2} \frac{2!}{\alpha!} a_{\sigma(\alpha)}(x) \partial^\alpha + \tilde{b}(x) \cdot \nabla + c(x),$$

$\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\partial = \nabla = (\partial_1, \partial_2)$, $p(\xi) = \xi_1\xi_2(\xi_1 + \xi_2)$, $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2$, $\alpha! = \alpha_1!\alpha_2!$, $\sigma(\alpha) = (\alpha_1 - \alpha_2)/2$, $\tilde{b}(x) = (b_1(x), b_2(x))$, and $a_{\sigma(\alpha)}$, $b_j(x)$, $c(x)$ are real-valued smooth functions on $\mathbb{T}^2$. Such operators arise in the study of gravity waves of deep water. See [1], [2], [4] and the references therein.

The purpose of this paper is to present the necessary and sufficient conditions for the existence of a unique solution to (1)-(2). To state our results, we introduce notation and the definition of $L^2$-well-posedness. $C^\infty(\mathbb{T}^2)$ is the set of all smooth functions on $\mathbb{T}^2$. $L^2(\mathbb{T}^2)$ is the set of all square-integrable functions on $\mathbb{T}^2$. For $g \in L^2(\mathbb{T}^2)$, set

$$\|g\| = \left( 2\int_{\mathbb{T}^2} |g(x)|^2 dx \right)^{1/2}.$$

$C(\mathbb{R}; L^2(\mathbb{T}^2))$ is the set of all $L^2(\mathbb{T}^2)$-valued continuous functions on $\mathbb{R}$, and similarly, $L^1_{loc}(\mathbb{R}; L^2(\mathbb{T}^2))$ is the set of all $L^2(\mathbb{T}^2)$-valued locally integrable functions on $\mathbb{R}$. Here we state the definition of $L^2$-well-posedness.
Definition 1. The initial value problem (1)-(2) is said to be $L^2$-well-posed if for any $u_0 \in L^2(\mathbb{T}^2)$ and $f \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{T}^2))$, (1)-(2) possesses a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{T}^2))$.

In view of Banach’s closed graph theorem, if (1)-(2) is $L^2$-well-posed, then for any $T > 0$, there exists $C_T > 0$ such that all the solutions satisfy the energy inequality

$$
\|u(t)\| \leq C_T \left( \|u_0\| + \left\| \int_0^t \|f(s)\| ds \right\| \right) \quad (|t| \leq T).
$$

Our results are the following.

Theorem 1. The following conditions are mutually equivalent.

I. (1)-(2) is $L^2$-well-posed.

II. For any $x \in \mathbb{T}^2$ and for any $\xi \in \Lambda$,

$$
\int_{0}^{2\pi} \sum_{|\alpha| = 2} \frac{\partial_t}{\partial t} a_{\sigma(\alpha)}(x + t\xi)(\xi^\alpha) dt = 0,
$$

where $p'(\xi) = \nabla_x p(\xi) = (\xi_2(2\xi_1 + \xi_2), \xi_1(2\xi_2 + \xi_1))$ and $\Lambda = \{ \xi \in \mathbb{R}^2 | p'(\xi) \in \mathbb{Z}^2 \}$.

III. $a_0(x) = a_1(x) + a_{-1}(x)$, and there exists $\phi(x) \in C^\infty(\mathbb{T}^2)$ such that $\nabla \phi(x) = (a_{-1}(x), a_1(x))$.

Here we explain the background of our problem. There are many papers dealing with the well-posedness of the initial value problem for dispersive equations. Generally speaking, it is difficult to characterize the well-posedness. In fact, results on the characterization are very limited. In [7] Mizohata studied the initial value problem for Schrödinger-type operators of the form

$$
S = \partial_t - iq(t) + \bar{b}(x) \cdot \nabla + c(x)
$$
on $\mathbb{R}^n$, where $i = \sqrt{-1}$ and $q(\xi) = \xi_1^2 + \cdots + \xi_n^2$. He gave the necessary condition for the $L^2$-well-posedness:

$$
\sup_{(T, x, \xi) \in \mathbb{R} \times \mathbb{R}^n} \left| \int_0^T \text{Im} \bar{b}(x + t\xi(\xi)) \cdot \xi dt \right| < +\infty.
$$

[6] gives the upper bound of the admissible bad first order term ($\text{Im} \bar{b}(x) \cdot \nabla$). In other words, [6] is necessary so that ($\text{Im} \bar{b}(x) \cdot \nabla$) can be controlled by the local smoothing effect of $e^{itq(t)}$. In [5] Ichinose generalized [6] for the Schrödinger-type equation on a complete Riemannian manifold. When $n = 1$, [4] is also the sufficient condition for $L^2$-well-posedness. Moreover, in [8] and [9] Tarama characterized the $L^2$-well-posedness for third order equations on $\mathbb{R}$.

For equations on compact manifolds, the local smoothing effect breaks down everywhere in the manifold. Then, the restriction on the bad lower order terms becomes stronger, and the characterization of $L^2$-well-posedness seems to be relatively easier. In fact, the $L^2$-well-posedness for $S$ on $\mathbb{T}^n$ is characterized. See [2], [6] and [10].

$L$ is the simplest example of higher order dispersive operators on higher-dimensional spaces. $p(\xi)$ satisfies $p'(\xi) \neq 0$ and $\det p''(\xi) \neq 0$ for $\xi \neq 0$. The symbol of the Laplacian $q(\xi)$ also satisfies the same conditions. It is interesting that our method of proof does not work if we replace $p(\xi)$ by $r(\xi) = \xi_1^3 + \xi_2^3$. This seems to be due to the degeneracy of $\det r''(\xi) = 36\xi_1^3\xi_2$ for $\xi \neq 0$. 


The organization of this paper is as follows. In Sections 2 and 3 we prove \( I \Rightarrow II \) and \( II \Rightarrow III \), respectively. We omit the proof of \( III \Rightarrow I \). Indeed, under the condition \( a_0 = a_1 + a_{-1} \), \( L \) becomes
\[
L = \partial_t + p(\partial) + a_1(x) \frac{\partial p}{\partial \xi_1}(\partial) + a_1(x) \frac{\partial p}{\partial \xi_2}(\partial) + \tilde{b}(x) \cdot \nabla + c(x).
\]
In view of \( \nabla \phi = (a_{-1}, a_1) \), we have
\[
e^\phi L e^{-\phi} = \partial_t + A, \quad A = p(\partial) + \tilde{b}_1(x) \partial_1 + \tilde{b}_2(x) \partial_2 + \tilde{c}(x),
\]
where \( \tilde{b}_1(x), \tilde{b}_2(x) \) and \( \tilde{c}(x) \) are real-valued smooth functions on \( \mathbb{T}^2 \). It is easy to see that the initial value problem for \( e^\phi L e^{-\phi} \) is \( L^2 \)-well-posed since
\[
A + A^* = -\frac{\partial \tilde{b}_1}{\partial x_1}(x) - \frac{\partial \tilde{b}_2}{\partial x_2}(x).
\]

2. The proof of \( I \Rightarrow II \)

We begin with the reduction of (1). Set
\[
a(x, \xi) = \sum_{|\alpha|=2} \frac{2^{|\alpha|}}{\alpha!} a_{\sigma(\alpha)}(x) \xi^\alpha
\]
for short. If
\[
(6) \quad \sup_{(T, x, \xi) \in \mathbb{R} \times \mathbb{T}^2 \times \mathbb{Q}^2} \left| \int_0^T a(x + tp'(\xi), \xi) dt \right| < +\infty,
\]
then (1) holds since \( a(x + tp'(\xi), \xi) \) is a \( 2\pi \)-periodic function in \( t \in \mathbb{R} \) for any \((x, \xi) \in \mathbb{T}^2 \times A \). (6) is equivalent to an apparently weaker condition
\[
(7) \quad \sup_{(T, x, \xi) \in \mathbb{R} \times \mathbb{T}^2 \times \mathbb{Q}^2} \left| \int_0^T a(x + tp'(\xi), \xi) dt \right| < +\infty.
\]
For \( \xi \in \mathbb{Q}^2 \), there exist \( \alpha \in \mathbb{Z}^2 \) and \( l \in \mathbb{N} \) such that \( \xi = \alpha/l \). Changing the variable by \( t = l^2 s \), we have
\[
\int_0^T a(x + tp'(\xi), \xi) dt = \int_0^{T/l^2} a(x + sp'(\alpha), \alpha) dt.
\]
Then, (7) is equivalent to
\[
(8) \quad \sup_{(T, x, \alpha) \in \mathbb{R} \times \mathbb{T}^2 \times \mathbb{Z}^2} \left| \int_0^T a(x + tp'(\alpha), \alpha) dt \right| < +\infty.
\]
To prove \( I \Rightarrow II \), we shall show the contraposition of \( I \Rightarrow (8) \).

Suppose that \( (8) \) fails to hold; that is, for any \( n \in \mathbb{N} \), there exist \( T_n \in \mathbb{R}, x_n \in \mathbb{T}^2 \) and \( \alpha \in \mathbb{Z}^2 \) such that
\[
\int_0^{T_n} a(x_n + tp'(\alpha), \alpha) dt \geq 2n.
\]
We shall show that the energy inequality \( (8) \) fails to hold. First, we consider the case that \( T_n >> 0 \) and
\[
\int_0^{T_n} a(x_n + tp'(\alpha), \alpha) dt \geq 2n.
\]
Since \([0, T_n] \times \mathbb{T}^2\) is compact and

\[
(s, x) \in [0, T_n] \times \mathbb{T}^2 \mapsto \int_0^s a(x + tp'(\alpha), \alpha)dt
\]

is continuous, there exists \((T, y) \in (0, T_n] \times \mathbb{T}^2\) such that

\[
\max_{(s, x) \in [0, T_n] \times \mathbb{T}^2} \int_0^s a(x + tp'(\alpha), \alpha)dt = \int_0^T a(y + tp'(\alpha), \alpha)dt.
\]

Pick up \(\psi \in C^\infty(\mathbb{T}^2)\) such that \(\|\psi\| = 1\) and

\[
\int_0^T a(x + tp'(\alpha), \alpha)dt \geq n \quad \text{in} \quad \text{supp}[\psi].
\]

Let \(u\) be a complex-valued solution to (1) with a complex-valued given function \(f(t, x)\). Then, \(L \text{Re} u = \text{Re} f\) and \(L \text{Im} u = \text{Im} f\) since all the coefficients in \(L\) are real-valued. We construct a sequence of complex-valued asymptotic solutions to \(Lu = 0\). For \(l \in \mathbb{N}\), set

\[
u_l(t, x) = e^{i tp'(\alpha) + il\alpha x + \phi_l(t, x)} \psi(x + (t - T/|l|^2)p'(\alpha)),
\]

\[
\phi_l(t, x) = \int_0^{t/|l|^2} a(x + sp'(\alpha), \alpha)ds.
\]

Then, \(u_l \in C^\infty(\mathbb{R} \times \mathbb{T}^2)\),

\[
\|u_l(0)\| = \|\psi\| = 1,
\]

\[
\|u_l(T/|l|^2)\| = \|\exp(\phi_l(T/|l|^2, \cdot))\psi(\cdot)\| \geq e^n.
\]

Next we compute \(Lu_l\). Set \(b(x, \xi) = \vec{b}(x) \cdot \xi\) and \(v_l(t, x) = e^{-itp'(\alpha) - il\alpha x} u_l(t, x)\) for short. We deduce

\[
e^{-itp'(\alpha) - il\alpha x} Lu_l = (\partial_\xi + ip(\alpha))v_l + p(\bar{\partial} + il\alpha)v_l
\]

\[
+ a(x, \partial + il\alpha)v_l + b(x, \partial + il\alpha)v_l + c(x)v_l,
\]

\[
(\partial_\xi + ip(\alpha))v_l = ip(\alpha)v_l + a(x + tp'(\alpha), \alpha)v_l
\]

\[
+ e^{\phi_1p'(\alpha)} \nabla \psi(x + (t - T/|l|^2)p'(\alpha)),
\]

\[
p(\bar{\partial} + il\alpha)v_l = (p(\bar{\partial}) + il\alpha \cdot p'(\bar{\partial}) - p'(\alpha) \cdot \nabla - ip(\alpha))v_l
\]

\[
= (p(\bar{\partial}) + il\alpha \cdot p'(\bar{\partial}))v_l - ip(\alpha)v_l
\]

\[
- \left( \int_0^{t/|l|^2} p'(\alpha) \cdot \nabla (a(x + sp'(\alpha), \alpha))ds \right) v_l
\]

\[
- e^{\phi_2} p'(\alpha) \cdot \nabla \psi(x + (t - T/|l|^2)p'(\alpha))
\]

\[
= (p(\bar{\partial}) + il\alpha \cdot p'(\bar{\partial}))v_l - ip(\alpha)v_l
\]

\[
- \left( \int_0^{t/|l|^2} \frac{d}{ds} (a(x + sp'(\alpha), \alpha))ds \right) v_l
\]

\[
- e^{\phi_2} p'(\alpha) \cdot \nabla \psi(x + (t - T/|l|^2)p'(\alpha))
\]

\[
= (p(\bar{\partial}) + il\alpha \cdot p'(\bar{\partial}))v_l - ip(\alpha)v_l
\]

\[
- a(x + tp'(\alpha), \alpha)v_l + a(x, \alpha)v_l
\]

\[
- e^{\phi_1} p'(\alpha) \cdot \nabla \psi(x + (t - T/|l|^2)p'(\alpha)),
\]

\[
(13)
\]
\( a(x, \partial + il\alpha)v_l = a(x, \partial)v_l - a(x, l\alpha)v_l \)
\[ (14) \]
\[ + 2il(\alpha_1(x) + \alpha_2a_0(x))\partial_1v_l \]
\[ + 2il(\alpha_1a_0(x) + \alpha_2a_{-1}(x))\partial_2v_l, \]
\[ (15) \]
\[ b(x, \partial + il\alpha)v_l = b(x, \partial)v_l - ib(x, l\alpha)v_l. \]

Substituting (12), (13), (14) and (15) into (11), we obtain
\[ |Lu_l(t, x)| \leq C|\alpha| \sum_{|\beta| \leq 3} |\partial^\beta v_l(t, x)|. \]

Then, we deduce
\[ |Lu_l(t, x)| \leq C_0|\alpha|(1 + T|\alpha|^2) \exp \left( \int_0^T a(y + sp'(\alpha), \alpha)ds \right) \]
for \( t \in [0, T/l^2] \). Integrating the \( L^2(T^2) \)-norm of (16) over \( [0, T/l^2] \), we have
\[ \int_0^{T/l^2} \|Lu_l(t)\|dt \leq \frac{A_\alpha}{l}, \]

\[ A_\alpha = 2\pi C_0 T|\alpha|(1 + T|\alpha|^2) \exp \left( \int_0^T a(y + sp'(\alpha), \alpha)ds \right). \]

If we take \( l \) satisfying \( A_\alpha \leq l \), then
\[ \int_0^{T/l^2} \|Lu_l(t)\|dt \leq 1. \]

Combining (9), (10) and (17), we obtain
\[ \|u_l(T/l^2)\| \geq e^{\alpha l} \geq \|u_l(0)\| + \int_0^{T/l^2} \|Lu_l(t)\|dt, \]
which breaks the energy inequality (3).

When \( T_n > 0 \) and
\[ \int_0^{T_n} a(x_n + tp(\alpha), \alpha)dt \leq -2n, \]
we employ a sequence of asymptotic solutions of the form
\[ u_l(t, x) = e^{-itp(\alpha)} - il\alpha x - \psi(t, x) \psi(x + (t - T/l^2)p'(l\alpha)). \]

When \( T_n < 0 \), the proof above works also in \( [T, 0] \) for some \( T \in [T_n, 0] \). The proof of \( I \Rightarrow II \) is finished.

3. THE PROOF OF \( II \Rightarrow III \)

To prove \( II \Rightarrow III \), we need to know the properties of \( \Lambda \).

**Lemma 2.** For any \( \alpha \in \mathbb{Z}^2 \), there exists \( \xi(\alpha) \in \Lambda \) such that \( p'(\pm \xi(\alpha)) = \alpha \).
Moreover, \( \xi(0) = 0 \), and for \( \alpha \neq 0 \), \( \xi(\alpha) \neq 0 \) and
\[ \xi_1(\alpha)\xi_2(\alpha) + \xi_1(-\alpha)\xi_2(-\alpha) \neq 0. \]
Proof. For the sake of intelligibility, we express two vectors by the entries \((\xi, \eta) \in \Lambda\) and \((\alpha, \beta) \in \mathbb{Z}^2\). We solve a system of quadratic algebraic equations:

\[ \eta(2\xi + \eta) = \alpha, \quad \xi(2\eta + \xi) = \beta. \]

**Case** \((\alpha, \beta) = (0, 0)\). Suppose \(\eta(2\xi + \eta) = 0\) and \(\xi(2\eta + \xi) = 0\). Then \(\eta = 0\) or \(2\xi + \eta = 0\), and \(\xi = 0\) or \(2\eta + \xi = 0\). In any case, \((\xi, \eta) = 0\) is a unique solution.

**Case** \(\alpha = 0, \beta \neq 0\). Suppose \(\beta \neq 0\), \(\eta(2\xi + \eta) = 0\) and \(\xi(2\eta + \xi) = \beta\). Then, \(\eta = 0\) or \(\eta = -2\xi\), and \(\xi(2\eta + \xi) = \beta\). If \(\eta = 0\), then \(\xi^2 = \beta\), which implies \(\beta > 0\) and \((\xi, \eta) = (\pm\sqrt{\beta}, 0)\). If \(\eta = -2\xi\), then \(-3\xi^2 = \beta\), which implies \(\beta < 0\) and \((\xi, \eta) = \pm(\sqrt{-\beta}/3, -2\sqrt{-\beta}/3)\). Then, we have

\[
(\xi(0, \beta), \eta(0, \beta)) = \begin{cases}
(\pm(\sqrt{\beta}, 0)) & \text{if } \beta > 0, \\
\pm\left(\frac{-\beta}{3}, -2\sqrt{-\beta}ight) & \text{if } \beta < 0,
\end{cases}
\]

\[
\xi(0, \beta)\eta(0, \beta) + \xi(0, -\beta)\eta(0, -\beta) = -\frac{2}{3}|\beta| \neq 0.
\]

**Case** \(\alpha \neq 0, \beta = 0\). In the same way as the case \(\alpha = 0\) and \(\beta \neq 0\), we have

\[
(\xi(\alpha, 0), \eta(\alpha, 0)) = \begin{cases}
(\pm(0, \sqrt{\alpha})) & \text{if } \alpha > 0, \\
\pm\left(-2\sqrt{-\alpha}, \sqrt{-\frac{\alpha}{3}}\right) & \text{if } \alpha < 0,
\end{cases}
\]

\[
\xi(\alpha, 0)\eta(\alpha, 0) + \xi(\alpha, 0)\eta(0, -\alpha) = -\frac{2}{3}|\alpha| \neq 0.
\]

**Case** \(\alpha \beta \neq 0\). Suppose \(\alpha \beta \neq 0\), \(\eta(2\xi + \eta) = \alpha\) and \(\xi(2\eta + \xi) = \beta\). \(\xi\eta \neq 0\) since \(\alpha\beta = \xi\eta(2\eta + \xi)(2\xi + \eta) \neq 0\). Substituting \(\eta = -\xi/2 + \beta/(2\xi)\) into \(\eta(2\xi + \eta) = \beta\), we have \(3\xi^2 + 2(2\alpha - \beta)\xi^2 - \beta^2 = 0\). Then,

\[
\xi^2 = \frac{\beta - 2\alpha \pm \sqrt{\beta(2\alpha)^2 + 3\beta^2}}{3}.
\]

Since \(|\beta - 2\alpha| < \sqrt{\beta(2\alpha)^2 + 3\beta^2}\) and \(\xi^2 > 0\),

\[
\xi^2 = \frac{\beta - 2\alpha + \sqrt{\beta(2\alpha)^2 + 3\beta^2}}{3}.
\]

Then, we have

\[
\xi(\alpha, \beta) = \pm\sqrt{\beta - 2\alpha + \sqrt{\beta(2\alpha)^2 + 3\beta^2}}.
\]

Using \(2\xi\eta = \beta - \xi^2 = \alpha - \eta^2\), we get

\[
2\xi\eta = \frac{2(\alpha + \beta) - \sqrt{\beta(2\alpha)^2 + 3\beta^2}}{3}, \quad (19)
\]

\[
\eta^2 = \frac{\alpha - 2\beta + \sqrt{\beta(2\alpha)^2 + 3\beta^2}}{3}. \quad (20)
\]

Here we remark that \(\eta^2 > 0\) is satisfied in \((20)\) since \((\sqrt{\beta(2\alpha)^2 + 3\beta^2})^2 - (\alpha - 2\beta)^2 = 3\alpha^2 > 0\).
Using (19) and (20), we deduce
\[ 2|\xi\eta| = \chi(\alpha, \beta) \frac{2(\alpha + \beta) - \sqrt{(\beta - 2\alpha)^2 + 3\beta^2}}{3}, \]
\[ \chi(\alpha, \beta) = \text{sgn} \left( 2(\alpha + \beta) - \sqrt{(\beta - 2\alpha)^2 + 3\beta^2} \right). \]

\( \chi(\alpha, \beta) \) makes sense for \( \alpha \beta \neq 0 \) since
\[ 4(\alpha + \beta)^2 - \left( \sqrt{(\beta - 2\alpha)^2 + 3\beta^2} \right)^2 = 12\alpha\beta \neq 0. \]

Thus, we have
\[ \left[ \xi(\alpha, \beta) \right] = \pm \left[ \begin{array}{c} \sqrt{\beta - 2\alpha + \sqrt{(\beta - 2\alpha)^2 + 3\beta^2}} \\ \chi(\alpha, \beta) \sqrt{\alpha - 2\beta + \sqrt{(\beta - 2\alpha)^2 + 3\beta^2}} \end{array} \right]. \]

It follows from (19) that
\[ \xi(\alpha, \beta)\eta(\alpha, \beta) + \xi(-\alpha, -\beta)\eta(-\alpha, -\beta) = -\frac{1}{3} \sqrt{(\beta - 2\alpha)^2 + 3\beta^2} \neq 0. \]

This completes the proof. \( \square \)

Finally, we prove II \( \Rightarrow \) III. Express \( a_{\sigma(\alpha)}(x) \) by the Fourier series of the form
\[ a_{\sigma(\alpha)}(x) = \sum_{\beta \in \mathbb{Z}^2} a_{\sigma(\alpha), \beta} \exp(i\beta \cdot x). \]

Substitute the Fourier series into (21). Then, for \( (x, \xi) \in \mathbb{T}^2 \times \Lambda, \)
\[ 0 = \sum_{\beta \in \mathbb{Z}^2} \sum_{|\alpha|=2} \frac{2!}{\alpha!} a_{\sigma(\alpha), \beta} \xi^\alpha e^{i\beta \cdot x} \int_0^{2\pi} e^{it\beta \cdot p'(\xi)} dt 
= 2\pi \sum_{\beta \in \mathbb{Z}^2} \sum_{\beta \cdot p'(\xi) = 0} (a_{1, \beta} \xi_1^2 + 2a_{0, \beta} \xi_1 \xi_2 + a_{-1, \beta} \xi_2^2) e^{i\beta \cdot x} 
= 2\pi \sum_{\beta \in \mathbb{Z}^2} \left\{ (a_{-1, \beta} - a_{1, \beta}) \cdot p'(\xi) + 2(a_{0, \beta} - a_{1, \beta} - a_{-1, \beta}) \xi_1 \xi_2 \right\} e^{i\beta \cdot x}. \]

It follows that if \( \beta \cdot p'(\xi) = 0 \) and \( \xi \in \Lambda, \) then
\[ (a_{-1, \beta} - a_{1, \beta}) \cdot p'(\xi) + 2(a_{0, \beta} - a_{1, \beta} - a_{-1, \beta}) \xi_1 \xi_2 = 0. \]

In view of Lemma 2 we have
\[ (a_{-1, \beta} - a_{1, \beta}) \cdot \alpha + 2(a_{0, \beta} - a_{1, \beta} - a_{-1, \beta}) \xi_1(\alpha) \xi_2(\alpha) = 0, \]
\[ -(a_{-1, \beta} - a_{1, \beta}) \cdot \alpha + 2(a_{0, \beta} - a_{1, \beta} - a_{-1, \beta}) \xi_1(-\alpha) \xi_2(-\alpha) = 0, \]
for \( \alpha \in \mathbb{Z}^2 \) satisfying \( \alpha \cdot \beta = 0. \) The sum of (21) and (22) is
\[ 2(a_{0, \beta} - a_{1, \beta} - a_{-1, \beta})(\xi_1(\alpha)\xi_2(\alpha) + \xi_1(-\alpha)\xi_2(-\alpha)) = 0. \]

In view of (13), we get \( a_{0, \beta} = a_{1, \beta} + a_{-1, \beta} \) for all \( \beta \in \mathbb{Z}^2. \) Thus, \( a_0(x) = a_1(x) + a_{-1}(x), \) and (21) becomes
\[ (a_{-1, \beta} - a_{1, \beta}) \cdot \alpha = 0 \quad \text{if} \quad \alpha \cdot \beta = 0. \]
$(a_{-1,0}, a_{1,0}) = 0$ since $\alpha \cdot 0 = 0$ for all $\alpha \in \mathbb{Z}^2$. For $\beta \neq 0$, (23) implies that there exists $\phi_\beta \in \mathbb{C}$ such that $(a_{-1,\beta}, a_{1,\beta}) = i\phi_\beta \beta$. If we set

$$\phi(x) = \sum_{\beta \neq 0} \phi_\beta e^{i\beta \cdot x},$$

then

$$\nabla \phi(x) = \sum_{\beta \neq 0} i\phi_\beta \beta e^{i\beta \cdot x} = \sum_{\beta \neq 0} (a_{-1,\beta}, a_{1,\beta}) e^{i\beta \cdot x} = (a_{-1}(x), a_{1}(x)),$$

which is desired. The proof of $\text{II} \implies \text{III}$ is finished.

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