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THE INITIAL VALUE PROBLEM FOR A THIRD ORDER DISPERSIVE EQUATION ON THE TWO-DIMENSIONAL TORUS

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ABSTRACT. We present the necessary and sufficient conditions for the L^2-well-posedness of the initial problem for a third order linear dispersive equation on the two-dimensional torus. Birkhoff's method of asymptotic solutions is used to prove necessity. Some properties of a system for quadratic algebraic equations associated to the principal symbol play a crucial role in proving sufficiency.

1. INTRODUCTION

This paper is concerned with the initial value problem of the form

$$Lu = f(t,x) \quad \text{in} \quad \mathbb{R} \times \mathbb{T}^2, \tag{1}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{T}^2, \tag{2}$$

where $u(t,x)$ is a real-valued unknown function of $(t,x) = (t,x_1,x_2) \in \mathbb{R} \times \mathbb{T}^2$, $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, $u_0(x)$ and $f(t,x)$ are given real-valued functions,

$$L = \partial_t + p(\partial) + \sum_{|\alpha|=2} \frac{2!}{\alpha!} a_{\sigma(\alpha)}(x)\partial^\alpha + \vec{b}(x) \cdot \nabla + c(x),$$

$\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\partial = \nabla = (\partial_1, \partial_2)$, $p(\xi) = \xi_1\xi_2(\xi_1 + \xi_2)$, $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2$, $\alpha! = \alpha_1!\alpha_2!$, $\sigma(\alpha) = (\alpha_1 - \alpha_2)/2$, $\vec{b}(x) = (b_1(x), b_2(x))$, and $a_{\sigma(\alpha)}$, $b_j(x)$, $c(x)$ are real-valued smooth functions on $\mathbb{T}^2$. Such operators arise in the study of gravity waves of deep water. See [1], [3], [4] and the references therein.

The purpose of this paper is to present the necessary and sufficient conditions for the existence of a unique solution to (1)-(2). To state our results, we introduce notation and the definition of L^2-well-posedness. $C^\infty(\mathbb{T}^2)$ is the set of all smooth functions on $\mathbb{T}^2$. $L^2(\mathbb{T}^2)$ is the set of all square-integrable functions on $\mathbb{T}^2$. For $g \in L^2(\mathbb{T}^2)$, set

$$\|g\| = \left(\int_{\mathbb{T}^2} |g(x)|^2 dx \right)^{1/2}.$$

$C(\mathbb{R}; L^2(\mathbb{T}^2))$ is the set of all $L^2(\mathbb{T}^2)$-valued continuous functions on $\mathbb{R}$, and similarly, $L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{T}^2))$ is the set of all $L^2(\mathbb{T}^2)$-valued locally integrable functions on $\mathbb{R}$. Here we state the definition of L^2-well-posedness.

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Definition 1. The initial value problem (1)-(2) is said to be L^2-well-posed if for any $u_0 \in L^2(\mathbb{T}^2)$ and $f \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{T}^2))$, (1)-(2) possesses a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{T}^2))$.

In view of Banach's closed graph theorem, if (1)-(2) is L^2-well-posed, then for any $T > 0$, there exists $C_T > 0$ such that all the solutions satisfy the energy inequality

$$\|u(t)\| \leqslant C_T \left(\|u_0\| + \left| \int_0^t \|f(s)\| ds \right| \right) \quad (|t| \leqslant T). \tag{3}$$

Our results are the following.

Theorem 1. *The following conditions are mutually equivalent.*

I. (1)-(2) *is L^2-well-posed.*

II. *For any $x \in \mathbb{T}^2$ and for any $\xi \in \Lambda$,*

$$\int_0^{2\pi} \sum_{|\alpha|=2} \frac{2!}{\alpha!} a_{\sigma(\alpha)}(x + tp'(\xi)) \xi^\alpha dt = 0, \tag{4}$$

where $p'(\xi) = \nabla_\xi p(\xi) = (\xi_2(2\xi_1 + \xi_2), \xi_1(2\xi_2 + \xi_1))$ and $\Lambda = \{\xi \in \mathbb{R}^2 | p'(\xi) \in \mathbb{Z}^2\}$.

III. *$a_0(x) = a_1(x) + a_{-1}(x)$, and there exists $\phi(x) \in C^\infty(\mathbb{T}^2)$ such that $\nabla\phi(x) = (a_{-1}(x), a_1(x))$.*

Here we explain the background of our problem. There are many papers dealing with the well-posedness of the initial value problem for dispersive equations. Generally speaking, it is difficult to characterize the well-posedness. In fact, results on the characterization are very limited. In [7] Mizohata studied the initial value problem for Schrödinger-type operators of the form

$$S = \partial_t - iq(\partial) + \vec{b}(x) \cdot \nabla + c(x)$$

on $\mathbb{R}^n$, where $i = \sqrt{-1}$ and $q(\xi) = \xi_1^2 + \cdots + \xi_n^2$. He gave the necessary condition for the L^2-well-posedness:

$$\sup_{(T,x,\xi) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n} \left| \int_0^T \operatorname{Im} \vec{b}(x + tq'(\xi)) \cdot \xi dt \right| < +\infty. \tag{5}$$

(5) gives the upper bound of the admissible bad first order term $(\operatorname{Im} \vec{b}(x)) \cdot \nabla$. In other words, (5) is necessary so that $(\operatorname{Im} \vec{b}(x)) \cdot \nabla$ can be controlled by the local smoothing effect of $e^{itq(\partial)}$. In [5] Ichinose generalized (5) for the Schrödinger-type equation on a complete Riemannian manifold. When $n = 1$, (5) is also the sufficient condition for L^2-well-posedness. Moreover, in [8] and [9] Tarama characterized the L^2-well-posedness for third order equations on $\mathbb{R}$.

For equations on compact manifolds, the local smoothing effect breaks down everywhere in the manifold. Then, the restriction on the bad lower order terms becomes stronger, and the characterization of L^2-well-posedness seems to be relatively easier. In fact, the L^2-well-posedness for S on $\mathbb{T}^n$ is characterized. See [2], [6] and [10].

L is the simplest example of higher order dispersive operators on higher-dimensional spaces. $p(\xi)$ satisfies $p'(\xi) \neq 0$ and $\det p''(\xi) \neq 0$ for $\xi \neq 0$. The symbol of the Laplacian $q(\xi)$ also satisfies the same conditions. It is interesting that our method of proof does not work if we replace $p(\xi)$ by $r(\xi) = \xi_1^3 + \xi_2^3$. This seems to be due to the degeneracy of $\det r''(\xi) = 36\xi_1\xi_2$ for $\xi \neq 0$.

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The organization of this paper is as follows. In Sections 2 and 3 we prove I $\Longrightarrow$ II and II $\Longrightarrow$ III, respectively. We omit the proof of III $\Longrightarrow$ I. Indeed, under the condition $a_0 = a_1 + a_{-1}$, L becomes

$$L = \partial_t + p(\partial) + a_{-1}(x)\frac{\partial p}{\partial \xi_1}(\partial) + a_1(x)\frac{\partial p}{\partial \xi_2}(\partial) + \vec{b}(x)\cdot\nabla + c(x).$$

In view of $\nabla\phi = (a_{-1}, a_1)$, we have

$$e^{\phi}Le^{-\phi} = \partial_t + A, \quad A = p(\partial) + \tilde{b}_1(x)\partial_1 + \tilde{b}_2(x)\partial_2 + \tilde{c}(x),$$

where $\tilde{b}_1(x)$, $\tilde{b}_2(x)$ and $\tilde{c}(x)$ are real-valued smooth functions on $\mathbb{T}^2$. It is easy to see that the initial value problem for $e^{\phi}Le^{-\phi}$ is L^2-well-posed since

$$A + A^{\star} = -\frac{\partial \tilde{b}_1}{\partial x_1}(x) - \frac{\partial \tilde{b}_2}{\partial x_2}(x).$$

2. The proof of I $\Longrightarrow$ II

We begin with the reduction of (4). Set

$$a(x,\xi) = \sum_{|\alpha|=2}\frac{2!}{\alpha!}a_{\sigma(\alpha)}(x)\xi^{\alpha}$$

for short. If

$$\sup_{(T,x,\xi)\in\mathbb{R}\times\mathbb{T}^2\times\mathbb{R}^2}\left|\int_0^T a(x+tp'(\xi),\xi)dt\right| + \infty, \tag{6}$$

then (4) holds since $a(x+tp'(\xi),\xi)$ is a 2π-periodic function in $t\in\mathbb{R}$ for any $(x,\xi)\in\mathbb{T}^2\times\Lambda$. (6) is equivalent to an apparently weaker condition

$$\sup_{(T,x,\xi)\in\mathbb{R}\times\mathbb{T}^2\times\mathbb{Q}^2}\left|\int_0^T a(x+tp'(\xi),\xi)dt\right| + \infty. \tag{7}$$

For $\xi\in\mathbb{Q}^2$, there exist $\alpha\in\mathbb{Z}^2$ and $l\in\mathbb{N}$ such that $\xi = \alpha/l$. Changing the variable by $t = l^2 s$, we have

$$\int_0^T a(x+tp'(\xi),\xi)dt = \int_0^{T/l^2} a(x+sp'(\alpha),\alpha)dt.$$

Then, (7) is equivalent to

$$\sup_{(T,x,\alpha)\in\mathbb{R}\times\mathbb{T}^2\times\mathbb{Z}^2}\left|\int_0^T a(x+tp'(\alpha),\alpha)dt\right| < +\infty. \tag{8}$$

To prove I $\Longrightarrow$ II, we shall show the contraposition of I $\Longrightarrow$ (8).

Suppose that (8) fails to hold; that is, for any $n\in\mathbb{N}$, there exist $T_n\in\mathbb{R}$, $x_n\in\mathbb{T}^2$ and $\alpha\in\mathbb{Z}^2$ such that

$$\left|\int_0^{T_n} a(x_n+tp'(\alpha),\alpha)dt\right| \geqslant 2n.$$

We shall show that the energy inequality (3) fails to hold. First, we consider the case that $T_n >> 0$ and

$$\int_0^{T_n} a(x_n+tp'(\alpha),\alpha)dt \geqslant 2n.$$

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Since $[0,T_n]\times\mathbb{T}^2$ is compact and

$$(s,x)\in[0,T_n]\times\mathbb{T}^2 \longmapsto \int_0^s a(x+tp'(\alpha),\alpha)dt$$

is continuous, there exists $(T,y)\in(0,T_n]\times\mathbb{T}^2$ such that

$$\max_{(s,x)\in[0,T_n]\times\mathbb{T}^2}\int_0^s a(x+tp'(\alpha),\alpha)dt=\int_0^T a(y+tp'(\alpha),\alpha)dt.$$

Pick up $\psi\in C^\infty(\mathbb{T}^2)$ such that $\|\psi\|=1$ and

$$\int_0^T a(x+tp'(\alpha),\alpha)dt \geqslant n \quad \text{in} \quad \operatorname{supp}[\psi].$$

Let u be a complex-valued solution to (1) with a complex-valued given function $f(t,x)$. Then, $L\operatorname{Re}u=\operatorname{Re}f$ and $L\operatorname{Im}u=\operatorname{Im}f$ since all the coefficients in L are real-valued. We construct a sequence of complex-valued asymptotic solutions to $Lu=0$. For $l\in\mathbb{N}$, set

$$u_l(t,x)=e^{itp(l\alpha)+il\alpha\cdot x+\phi_l(t,x)}\psi(x+(t-T/l^2)p'(l\alpha)),$$
$$\phi_l(t,x)=\int_0^{l^2t} a(x+sp'(\alpha),\alpha)ds.$$

Then, $u_l\in C^\infty(\mathbb{R}\times\mathbb{T}^2)$,

$$\|u_l(0)\|=\|\psi\|=1, \tag{9}$$
$$\|u_l(T/l^2)\|=\|\exp(\phi_l(T/l^2,\cdot))\psi(\cdot)\|\geqslant e^n. \tag{10}$$

Next we compute Lu_l. Set $b(x,\xi)=\vec{b}(x)\cdot\xi$ and $v_l(t,x)=e^{-itp(l\alpha)-il\alpha\cdot x}u_l(t,x)$ for short. We deduce

$$\begin{aligned} e^{-itp(l\alpha)-il\alpha\cdot x}Lu_l &= (\partial_t+ip(l\alpha))v_l+p(\partial+il\alpha)v_l \\ &\quad + a(x,\partial+il\alpha)v_l+b(x,\partial+il\alpha)v_l+c(x)v_l, \end{aligned} \tag{11}$$

$$\begin{aligned} (\partial_t+ip(l\alpha))v_l &= ip(l\alpha)v_l+a(x+tp'(l\alpha),l\alpha)v_l \\ &\quad + e^{\phi_l}p'(l\alpha)\cdot\nabla\psi(x+(t-T/l^2)p'(l\alpha)), \end{aligned} \tag{12}$$

$$\begin{aligned} p(\partial+il\alpha)v_l &= (p(\partial)+il\alpha\cdot p'(\partial)-p'(l\alpha)\cdot\nabla-ip(l\alpha))v_l \\ &= (p(\partial)+il\alpha\cdot p'(\partial))v_l-ip(l\alpha)v_l \\ &\quad -\left(\int_0^{l^2t} p'(l\alpha)\cdot\nabla(a(x+sp'(\alpha),\alpha))ds\right)v_l \\ &\quad - e^{\phi_l}p'(l\alpha)\cdot\nabla\psi(x+(t-T/l^2)p'(l\alpha)) \\ &= (p(\partial)+il\alpha\cdot p'(\partial))v_l-ip(l\alpha)v_l \\ &\quad -\left(\int_0^{l^2t} \frac{d}{ds}(a(x+sp'(\alpha),l\alpha))ds\right)v_l \\ &\quad - e^{\phi_l}p'(l\alpha)\cdot\nabla\psi(x+(t-T/l^2)p'(l\alpha)) \\ &= (p(\partial)+il\alpha\cdot p'(\partial))v_l-ip(l\alpha)v_l \\ &\quad - a(x+tp'(l\alpha),l\alpha)v_l+a(x,l\alpha)v_l \\ &\quad - e^{\phi_l}p'(l\alpha)\cdot\nabla\psi(x+(t-T/l^2)p'(l\alpha)), \end{aligned} \tag{13}$$

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$$\begin{aligned} a(x,\partial+il\alpha)v_l &= a(x,\partial)v_l - a(x,l\alpha)v_l \\ &\quad + 2il(\alpha_1 a_1(x)+\alpha_2 a_0(x))\partial_1 v_l \\ &\quad + 2il(\alpha_1 a_0(x)+\alpha_2 a_{-1}(x))\partial_2 v_l, \end{aligned} \tag{14}$$

$$b(x,\partial+il\alpha)v_l = b(x,\partial)v_l - ib(x,l\alpha)v_l. \tag{15}$$

Substituting (12), (13), (14) and (15) into (11), we obtain

$$|Lu_l(t,x)| \leqslant Cl|\alpha| \sum_{|\beta|\leqslant 3} |\partial^\beta v_l(t,x)|.$$

Then, we deduce

$$|Lu_l(t,x)| \leqslant C_0 l|\alpha|(1+T|\alpha|^2)^3 \exp\left(\int_0^T a(y+sp'(\alpha),\alpha)ds\right) \tag{16}$$

for $t \in [0,T/l^2]$. Integrating the $L^2(\mathbb{T}^2)$-norm of (16) over $[0,T/l^2]$, we have

$$\int_0^{T/l^2} \|Lu_l(t)\|dt \leqslant \frac{A_\alpha}{l},$$

$$A_\alpha = 2\pi C_0 T|\alpha|(1+T|\alpha|^2)^3 \exp\left(\int_0^T a(y+sp'(\alpha),\alpha)ds\right).$$

If we take l satisfying $A_\alpha \leqslant l$, then

$$\int_0^{T/l^2} \|Lu_l(t)\|dt \leqslant 1. \tag{17}$$

Combining (9), (10) and (17), we obtain

$$\|u_l(T/l^2)\| \geqslant e^n 2 \geqslant \|u_l(0)\| + \int_0^{T/l^2} \|Lu_l(t)\|dt,$$

which breaks the energy inequality (3).

When $T_n > 0$ and

$$\int_0^{T_n} a(x_n+tp'(\alpha),\alpha)dt \leqslant -2n,$$

we employ a sequence of asymptotic solutions of the form

$$u_l(t,x) = e^{-itp(l\alpha)-il\alpha\cdot x-\phi_l(t,x)}\psi(x+(t-T/l^2)p'(l\alpha)).$$

When $T_n < 0$, the proof above works also in $[T,0]$ for some $T \in [T_n,0]$. The proof of I $\Longrightarrow$ II is finished.

3. The proof of II $\Longrightarrow$ III

To prove II $\Longrightarrow$ III, we need to know the properties of Λ.

Lemma 2. *For any $\alpha \in \mathbb{Z}^2$, there exists $\xi(\alpha) \in \Lambda$ such that $p'(\pm\xi(\alpha)) = \alpha$. Moreover, $\xi(0)=0$, and for $\alpha \neq 0$, $\xi(\alpha) \neq 0$ and*

$$\xi_1(\alpha)\xi_2(\alpha) + \xi_1(-\alpha)\xi_2(-\alpha) \neq 0. \tag{18}$$

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Proof. For the sake of intelligibility, we express two vectors by the entries $(\xi,\eta)\in\Lambda$ and $(\alpha,\beta)\in\mathbb{Z}^2$. We solve a system of quadratic algebraic equations:

$$\eta(2\xi+\eta)=\alpha, \quad \xi(2\eta+\xi)=\beta.$$

Case $(\alpha,\beta)=(0,0)$. Suppose $\eta(2\xi+\eta)=0$ and $\xi(2\eta+\xi)=0$. Then $\eta=0$ or $2\xi+\eta=0$, and $\xi=0$ or $2\eta+\xi=0$. In any case, $(\xi,\eta)=0$ is a unique solution.

Case $\alpha=0$, $\beta\neq 0$. Suppose $\beta\neq 0$, $\eta(2\xi+\eta)=0$ and $\xi(2\eta+\xi)=\beta$. Then, $\eta=0$ or $\eta=-2\xi$, and $\xi(2\eta+\xi)=\beta$. If $\eta=0$, then $\xi^2=\beta$, which implies $\beta>0$ and $(\xi,\eta)=(\pm\sqrt{\beta},0)$. If $\eta=-2\xi$, then $-3\xi^2=\beta$, which implies $\beta<0$ and $(\xi,\eta)=\pm(\sqrt{-\beta/3},-2\sqrt{-\beta/3})$. Then, we have

$$(\xi(0,\beta),\eta(0,\beta))=\begin{cases}\pm(\sqrt{\beta},0) & (\text{if} \quad \beta>0),\\ \pm\left(\sqrt{-\dfrac{\beta}{3}},-2\sqrt{-\dfrac{\beta}{3}}\right) & (\text{if} \quad \beta<0),\end{cases}$$

$$\xi(0,\beta)\eta(0,\beta)+\xi(0,-\beta)\eta(0,-\beta)=-\frac{2}{3}|\beta|\neq 0.$$

Case $\alpha\neq 0$, $\beta=0$. In the same way as the case $\alpha=0$ and $\beta\neq 0$, we have

$$(\xi(\alpha,0),\eta(\alpha,0))=\begin{cases}\pm(0,\sqrt{\alpha}) & (\text{if} \quad \alpha>0),\\ \pm\left(-2\sqrt{-\dfrac{\alpha}{3}},\sqrt{-\dfrac{\alpha}{3}}\right) & (\text{if} \quad \alpha<0),\end{cases}$$

$$\xi(\alpha,0)\eta(\alpha,0)+\xi(\alpha,0)\eta(0,-\alpha)=-\frac{2}{3}|\alpha|\neq 0.$$

Case $\alpha\beta\neq 0$. Suppose $\alpha\beta\neq 0$, $\eta(2\xi+\eta)=\alpha$ and $\xi(2\eta+\xi)=\beta$. $\xi\eta\neq 0$ since $\alpha\beta=\xi\eta(2\eta+\xi)(2\xi+\eta)\neq 0$. Substituting $\eta=-\xi/2+\beta/(2\xi)$ into $\eta(2\xi+\eta)=\beta$, we have $3\xi^4+2(2\alpha-\beta)\xi^2-\beta^2=0$. Then,

$$\xi^2=\frac{\beta-2\alpha\pm\sqrt{(\beta-2\alpha)^2+3\beta^2}}{3}.$$

Since $|\beta-2\alpha|<\sqrt{(\beta-2\alpha)^2+3\beta^2}$ and $\xi^2>0$,

$$\xi^2=\frac{\beta-2\alpha+\sqrt{(\beta-2\alpha)^2+3\beta^2}}{3}.$$

Then, we have

$$\xi(\alpha,\beta)=\pm\sqrt{\frac{\beta-2\alpha+\sqrt{(\beta-2\alpha)^2+3\beta^2}}{3}}.$$

Using $2\xi\eta=\beta-\xi^2=\alpha-\eta^2$, we get

$$2\xi\eta=\frac{2(\alpha+\beta)-\sqrt{(\beta-2\alpha)^2+3\beta^2}}{3}, \tag{19}$$

$$\eta^2=\frac{\alpha-2\beta+\sqrt{(\beta-2\alpha)^2+3\beta^2}}{3}. \tag{20}$$

Here we remark that $\eta^2>0$ is satisfied in (20) since

$$(\sqrt{(\beta-2\alpha)^2+3\beta^2})^2-(\alpha-2\beta)^2=3\alpha^2>0.$$

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Using (19) and (20), we deduce

$$2|\xi\eta| = \chi(\alpha,\beta)\frac{2(\alpha+\beta)-\sqrt{(\beta-2\alpha)^2+3\beta^2}}{3},$$
$$\chi(\alpha,\beta) = \operatorname{sgn}\left(2(\alpha+\beta)-\sqrt{(\beta-2\alpha)^2+3\beta^2}\right).$$

$\chi(\alpha,\beta)$ makes sense for $\alpha\beta \neq 0$ since

$$4(\alpha+\beta)^2 - \left(\sqrt{(\beta-2\alpha)^2+3\beta^2}\right)^2 = 12\alpha\beta \neq 0.$$

Thus, we have

$$\begin{bmatrix}\xi(\alpha,\beta)\\ \eta(\alpha,\beta)\end{bmatrix} = \pm\begin{bmatrix}\sqrt{\dfrac{\beta-2\alpha+\sqrt{(\beta-2\alpha)^2+3\beta^2}}{3}}\\ \chi(\alpha,\beta)\sqrt{\dfrac{\alpha-2\beta+\sqrt{(\beta-2\alpha)^2+3\beta^2}}{3}}\end{bmatrix}.$$

It follows from (19) that

$$\xi(\alpha,\beta)\eta(\alpha,\beta)+\xi(-\alpha,-\beta)\eta(-\alpha,-\beta) = -\frac{1}{3}\sqrt{(\beta-2\alpha)^2+3\beta^2} \neq 0.$$

This completes the proof. □

Finally, we prove II $\Longrightarrow$ III. Express $a_{\sigma(\alpha)}(x)$ by the Fourier series of the form

$$a_{\sigma(\alpha)}(x) = \sum_{\beta\in\mathbb{Z}^2} a_{\sigma(\alpha),\beta}\exp(i\beta\cdot x).$$

Substitute the Fourier series into (4). Then, for $(x,\xi)\in\mathbb{T}^2\times\Lambda$,

$$\begin{aligned} 0 &= \sum_{\beta\in\mathbb{Z}^2}\sum_{|\alpha|=2}\frac{2!}{\alpha!}a_{\sigma(\alpha),\beta}\xi^\alpha e^{i\beta\cdot x}\int_0^{2\pi} e^{it\beta\cdot p'(\xi)}dt \\ &= 2\pi\sum_{\substack{\beta\in\mathbb{Z}^2\\ \beta\cdot p'(\xi)=0}}(a_{1,\beta}\xi_1^2+2a_{0,\beta}\xi_1\xi_2+a_{-1,\beta}\xi_2^2)e^{i\beta\cdot x} \\ &= 2\pi\sum_{\substack{\beta\in\mathbb{Z}^2\\ \beta\cdot p'(\xi)=0}}\{(a_{-1,\beta},a_{1,\beta})\cdot p'(\xi)+2(a_{0,\beta}-a_{1,\beta}-a_{-1,\beta})\xi_1\xi_2\}e^{i\beta\cdot x}. \end{aligned}$$

It follows that if $\beta\cdot p'(\xi)=0$ and $\xi\in\Lambda$, then

$$(a_{-1,\beta},a_{1,\beta})\cdot p'(\xi)+2(a_{0,\beta}-a_{1,\beta}-a_{-1,\beta})\xi_1\xi_2 = 0.$$

In view of Lemma 2, we have

$$(a_{-1,\beta},a_{1,\beta})\cdot\alpha+2(a_{0,\beta}-a_{1,\beta}-a_{-1,\beta})\xi_1(\alpha)\xi_2(\alpha) = 0, \tag{21}$$

$$-(a_{-1,\beta},a_{1,\beta})\cdot\alpha+2(a_{0,\beta}-a_{1,\beta}-a_{-1,\beta})\xi_1(-\alpha)\xi_2(-\alpha) = 0, \tag{22}$$

for $\alpha\in\mathbb{Z}^2$ satisfying $\alpha\cdot\beta=0$. The sum of (21) and (22) is

$$2(a_{0,\beta}-a_{1,\beta}-a_{-1,\beta})(\xi_1(\alpha)\xi_2(\alpha)+\xi_1(-\alpha)\xi_2(-\alpha)) = 0.$$

In view of (18), we get $a_{0,\beta} = a_{1,\beta}+a_{-1,\beta}$ for all $\beta\in\mathbb{Z}^2$. Thus, $a_0(x) = a_1(x) + a_{-1}(x)$, and (21) becomes

$$(a_{-1,\beta},a_{1,\beta})\cdot\alpha = 0 \quad \text{if} \quad \alpha\cdot\beta = 0. \tag{23}$$

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$(a_{-1,0}, a_{1,0}) = 0$ since $\alpha \cdot 0 = 0$ for all $\alpha \in \mathbb{Z}^2$. For $\beta \neq 0$, (23) implies that there exists $\phi_\beta \in \mathbb{C}$ such that $(a_{-1,\beta}, a_{1,\beta}) = i\phi_\beta \beta$. If we set

$$\phi(x) = \sum_{\beta \neq 0} \phi_\beta e^{i\beta \cdot x},$$

then

$$\nabla \phi(x) = \sum_{\beta \neq 0} i\phi_\beta \beta e^{i\beta \cdot x} = \sum_{\beta \neq 0} (a_{-1,\beta}, a_{1,\beta}) e^{i\beta \cdot x} = (a_{-1}(x), a_1(x)),$$

which is desired. The proof of II $\Longrightarrow$ III is finished.

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