THE CONVERSE OF THE FOUR VERTEX THEOREM

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(Communicated by Ronald A. Fintushel)

Abstract. We establish the converse to the four vertex theorem without the positivity condition.

1. Introduction

Let $T$ denote the unit circle of the complex plane $C$ and let $\gamma : T \to C$ be the parametrisation of a smooth, simple and closed curve such that $\gamma' \neq 0$ and such that $\gamma$ is also regular. Here the smoothness condition means that $\gamma$ is infinitely many times continuously differentiable. If $\kappa : T \to R$ denotes the curvature function of $\gamma$, then the four vertex theorem asserts that if $\kappa$ is not a constant, then $\kappa$ has at least four critical points $p_i \in T$, $i = 1, \ldots, 4$, ordered counterclockwise such that $p_1, p_3$ are local maxima and $p_2, p_4$ are local minima and that furthermore $\kappa(p_1) > 0, \kappa(p_3) > 0$ and

$$\max(\kappa(p_2), \kappa(p_4)) < \min(\kappa(p_1), \kappa(p_3)).$$

This result was apparently first proved for the case of closed convex curves by Mukhopadhyaya [4]. For proofs in the case of simple closed curves see Fog [1], Jackson [3] and Vietoris [5].

The converse of this was studied by Gluck [2], who proved that if $\kappa$ is a smooth and strictly positive function satisfying the above four vertex property, then $\kappa$ is the curvature function of a smooth, simple and closed curve. The purpose of this note is to establish the converse to the four vertex theorem without the positivity condition.

Theorem 1.1. Suppose $\kappa : T \to R$ is not a constant and assume that $\kappa$ has at least four critical points ordered counterclockwise $p_i \in T$, $i = 1, \ldots, 4$, such that $p_1, p_3$ are local maxima and $p_2, p_4$ are local minima with $\kappa(p_1) > 0, \kappa(p_3) > 0$ and $\max(\kappa(p_2), \kappa(p_4)) < \min(\kappa(p_1), \kappa(p_3))$. If in addition $\kappa$ is smooth, then $\kappa$ is the curvature function of a smooth, simple and closed curve.
We remark that a smooth function $K: T \to \mathbb{R}$ represents the curvature of a smooth, simple and closed curve parametrised by the arc length if and only if the following three conditions are satisfied:

\begin{align}
(1.1) \quad & \int_0^{2\pi} K \, ds = 2\pi, \\
(1.2) \quad & \int_0^{2\pi} e^{i\alpha(s)} \, ds = 0, \\
(1.3) \quad & \int_0^\tau e^{i\alpha(s)} \, ds \neq 0 \quad \text{if} \quad 0 \leq \tau < 2\pi.
\end{align}

Here $\alpha(s) = \int_0^s K(t) \, dt$ and the parametrisation of the associated curve $\gamma$ is given by

$$\gamma_K(t) = \int_0^t e^{i\alpha(s)} \, ds.$$

The above three conditions all have a geometric interpretation. The first condition expresses that the curve $\gamma_K$ has a well-determined tangent at $s = 0$, and the second condition expresses that $\gamma_K$ is a closed curve. Finally, the third condition expresses that $\gamma_K$ is simple, that is, without self-intersections.

We will say that $\kappa$ is a non–normalised curvature function if

\begin{align}
(1.4) \quad & \left\{ \begin{array}{l}
I = \int_0^{2\pi} \kappa \, dt \neq 0 \\
K = \frac{2\pi}{I} \kappa \text{ satisfies (1.2) and (1.3)}.
\end{array} \right.
\end{align}

In order to prove the theorem it is enough to show the existence of a smooth diffeomorphism $\varphi: T \to T$ such that $\kappa \circ \varphi$ is a non–normalised curvature function. For letting $\psi$ denote the inverse of $\varphi$ and setting $K = \frac{2\pi}{I} \kappa \circ \varphi$, $I = \int_0^{2\pi} \kappa \circ \varphi \, dt$, then the curve $\Gamma$ parametrised by $\Gamma(t) = \frac{2\pi}{I} \gamma_K(\psi(t))$ has $\kappa$ as its curvature function.

2. PRELIMINARY RESULTS

The construction of the diffeomorphisms required for the proof of Theorem 1.1 will be based on the following observations.

**Proposition 2.1.** Let $E_j \subset T$, $j = 1, \ldots, 4$, be non-empty, pairwise disjoint open intervals that are ordered counterclockwise on the unit circle with $\bigcup E_j = T$. Let $a, b$ be two positive numbers with $a \neq b$ and define the function $k$ by

$$k = \left\{ \begin{array}{cl}
a & \text{on } E_1 \cup E_3, \\
b & \text{on } E_2 \cup E_4.
\end{array} \right.$$

Suppose furthermore that $\int_0^{2\pi} k \, ds = 2\pi$. Then $k$ is the curvature function of a closed convex curve parametrised by the arc length if and only if $E_3 = \{-w : w \in E_1\}$ and $E_4 = \{-w : w \in E_2\}$.

**Proof.** We begin by establishing the necessity. Let $\gamma$ be a curve of length $2\pi$ whose curvature is given by $k = k(s)$. Since $k$ is strictly positive it is well known that $\gamma$ is convex. Let $\kappa = \kappa(\vartheta)$ represent the curvature of $\gamma$ as a function of the angle $\vartheta$ that the tangent forms with the positive $x$-axis. Then it is easily seen that

\begin{align}
(2.1) \quad & \int_0^{2\pi} e^{i\vartheta} \frac{1}{\kappa(\vartheta)} \, d\vartheta = 0.
\end{align}
Let the function $F = F(\vartheta)$ parametrise $\gamma$ with respect to the angle $\vartheta$ that the tangent forms with the $x$-axis. Now set $e_j = \{ \vartheta : F(\vartheta) \in \gamma(E_j) \}$, $j = 1, \ldots, 4$. Then the $e_j$’s are non-empty, pairwise disjoint open intervals in $T$ that are ordered counterclockwise and for which $\bigcup e_j = T$. Then $\kappa = a$ on $U = e_1 \cup e_3$ and $\kappa = b$ on $V = e_2 \cup e_4$. Since $a \neq b$ and $\int_0^{2\pi} e^{i \vartheta} \, d\vartheta = 0$ it follows from (2.1) that

\begin{equation}
\int_U e^{i \vartheta} \, d\vartheta = \int_V e^{i \vartheta} \, d\vartheta = 0.
\end{equation}

Denote by $2l_j$ the length of the interval $e_j$ and let $c_j$ denote its centre. From (2.2) it follows that $e^{ic_1} \sin l_1 + e^{ic_3} \sin l_3 = 0$ and $e^{ic_2} \sin l_2 + e^{ic_4} \sin l_4 = 0$. Since $l_j > 0$ for $j = 1, \ldots, 4$ it is easily seen that $c_3 = c_1 + \pi$, $c_2 = c_4 + \pi$, $l_3 = l_1$ and $l_4 = l_2$, which yields the necessity part of the proposition. The sufficiency part is an easy consequence of (1.1), (1.2) and (1.3).

**Proposition 2.2.** For $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, let $g_\alpha$ denote the restriction to $T$ of the Möbius transformation

\[ g_\alpha(w) = \frac{w - \alpha}{1 - \bar{\alpha}w}. \]

Let $E_j \subset T$, $j = 1, \ldots, 4$, be non-empty, pairwise disjoint open intervals that are ordered counterclockwise on the unit circle with $\bigcup E_j = T$. Suppose $E_3 = \{ -w : w \in E_1 \}$ and $E_4 = \{ -w : w \in E_2 \}$. If $g_\alpha(E_1) = -g_\alpha(E_3)$ and $g_\alpha(E_2) = -g_\alpha(E_4)$, then $\alpha = 0$.

**Proof.** The proof will be carried out by a contradiction argument. We assume therefore that $\alpha \neq 0$. Let $z, \zeta$ denote the end points of the interval $E_1$. The assumptions on $g_\alpha$ imply that $g_\alpha(-z) = -g_\alpha(z)$ and $g_\alpha(-\zeta) = -g_\alpha(\zeta)$. A straightforward computation shows that therefore $\bar{\alpha} z^2 = \alpha$ and $\bar{\alpha} \zeta^2 = \alpha$. Under the assumption that $\alpha \neq 0$ it follows therefore that $z^2 = \zeta^2$, which is impossible since $E_1$ is non-empty with length strictly less than $\pi$. This contradiction establishes the proposition.

We will need an infinitesimal version of the above propositions.

**Proposition 2.3.** For $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, let $g_\alpha$ denote the restriction to $T$ of the Möbius transformation

\[ g_\alpha(w) = \frac{w - \alpha}{1 - \bar{\alpha}w}. \]

Let $f : T \to \mathbb{R}$ denote the function defined by $f(e^{i \vartheta}) = 1$ whenever $|\vartheta - \frac{\pi}{2}| < \frac{\pi}{4}$ or $|\vartheta - \frac{3\pi}{4}| < \frac{\pi}{4}$ and zero elsewhere. Let $a$ and $b$ be two positive numbers such that $a \neq b$ and set $\kappa = a(1 - f) + bf$. Furthermore, let $a$ and $b$ be normalised so that $\int_0^{2\pi} \kappa \, dt = 2\pi$ and define $I_\alpha = \int_0^{2\pi} \kappa \circ g_\alpha$. Let $A_\alpha$ be defined by $A_\alpha(0) = 0$ and $A_\alpha' = \kappa \circ g_\alpha$. Suppose that $\alpha$ depends smoothly on a real parameter $t$ such that $\alpha(0) = 0$. Letting $\dot{\alpha}$ denote the derivative with respect to the $t$-variable evaluated at $t = 0$ we have that $I = 0$ and

\[ \int_0^{2\pi} \dot{A}(t)e^{i A_\alpha(t)} \, dt = z_0(\xi z_1 + \eta z_2). \]

Here $\xi$ and $\eta$ are defined by $\dot{\alpha} = \xi + i \eta$, and

\[
\begin{align*}
\dot{z}_0 &= 2\sqrt{2}i(b - a)e^{i \pi w_0}, & w_0 &= (a + 2b)/4, \\
\dot{z}_1 &= (e^{-i}\frac{\pi}{4} - 1)/b & \text{and} & & \dot{z}_2 &= (e^{i}\frac{\pi}{4} - 1)/a.
\end{align*}
\]

Furthermore the vectors $z_1$ and $z_2$ are linearly independent over $\mathbb{R}$. 

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Proof. We begin by selecting a smooth branch of the argument for points in a neighbourhood of \( \{e^{i\theta} : \theta_0 \leq \theta \leq \theta_1 \} \), where \( 0 < \theta_0 < \theta_1 < 2\pi \). Now let \( G_\alpha \) denote the argument of the inverse of \( g_\alpha \). It is easily seen that

\[
\dot{A}(t) = \begin{cases} 
0 & \text{if } 0 < t < \frac{\pi}{4}, \\
(a - b)\dot{G}(\frac{\pi}{4}) & \text{if } \frac{\pi}{4} < t < \frac{3\pi}{8}, \cr
(a - b)(\dot{G}(\frac{\pi}{4}) - \dot{G}(\frac{5\pi}{8})) & \text{if } \frac{3\pi}{8} < t < \frac{\pi}{2}, \cr
(a - b)(\dot{G}(\frac{3\pi}{8}) - \dot{G}(\frac{5\pi}{8}) + \dot{G}(\frac{3\pi}{8})) & \text{if } \frac{\pi}{2} < t < \frac{\pi}{4}, \cr
(b - a)\dot{G}(\frac{5\pi}{8}) & \text{if } \frac{\pi}{4} < t < \frac{\pi}{2}. \end{cases}
\]

Since the inverse of \( g_\alpha \) is given by \( g_{-\alpha} \) it is also easily seen that \( \dot{G}(\theta) = 2Im(\alpha e^{-i\theta}) \), where \( Im(w) \) denotes the imaginary part of \( w \). Hence \( \dot{G}(\theta + \pi) = -\dot{G}(t) \), so that

\[
\dot{A}(t) = \begin{cases} 
0 & \text{if } 0 < t < \frac{\pi}{4} \text{ or } \frac{3\pi}{8} < t < 2\pi, \cr
(a - b)\dot{G}(\frac{\pi}{4}) & \text{if } \frac{\pi}{4} < t < \frac{3\pi}{8}, \cr
(a - b)(\dot{G}(\frac{3\pi}{8}) - \dot{G}(\frac{5\pi}{8})) & \text{if } \frac{3\pi}{8} < t < \frac{\pi}{2}, \cr
(b - a)\dot{G}(\frac{5\pi}{8}) & \text{if } \frac{\pi}{2} < t < \frac{\pi}{4}. \end{cases}
\]

In particular, we see that since \( I_\alpha = A_\alpha(2\pi) \) we have that \( \dot{I} = 0 \). By using that \( A_0(t + \pi) = \pi + A_0(t) \) and integrating it is easy to verify that the expression for \( A(t) \) holds. It remains to verify that the vectors \( z_1 \) and \( z_2 \) are linearly independent over \( R \). We first note that the normalisation \( \int_0^{2\pi} \kappa dt = 2\pi \) means that \( a + b = 2 \) so that \( a, b \in (0, 2) \). Assuming that \( z_1 \) and \( z_2 \) are not linearly independent, there would exist real numbers \( \alpha, \beta \) such that \( \alpha z_1 + \beta z_2 = 0 \) where not both \( \alpha \) and \( \beta \) equals zero. In fact, both \( \alpha \) and \( \beta \) have to be non-zero in this case since otherwise \( z_1 \) or \( z_2 \) has to equal zero in which case \( 1 = e^{i\theta b} \) for \( w = a \) or \( w = -b \), which in turn implies that \( w = 4n \) for some \( n \in Z \) and this contradicts that \( a, b \in (0, 2) \). Thus, since both \( \alpha \) and \( \beta \) have to be non-zero we get that there is a real number \( c \neq 0 \) such that

\[
c(1 - e^{-ib\frac{\pi}{4}}) = e^{i(a+b)\frac{\pi}{4}} - 1 \iff e^{-ib\frac{\pi}{4}}c(e^{ib\frac{\pi}{4}} - e^{-ib\frac{\pi}{4}}) = e^{i(a+b)\frac{\pi}{4}}(e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}})
\]

\[
\iff c \sin(b\frac{\pi}{4}) = e^{i(a+b)\frac{\pi}{4}} \sin(a\frac{\pi}{4}) \iff c \sin(b\frac{\pi}{4}) = e^{i\frac{\pi}{4}} \sin(a\frac{\pi}{4})
\]

where we used that \( a + b = 2 \). Consequently,

\[
c \sin(b\frac{\pi}{4}) = \text{Re}(c \sin(b\frac{\pi}{4})) = \text{Re}(i \sin(a\frac{\pi}{4})) = 0.
\]

Since \( c \neq 0 \) we get \( \sin(b\frac{\pi}{4}) = 0 \), which implies that \( b\frac{\pi}{4} = n\pi \) for some \( n \in Z \), which contradicts \( b \in (0, 2) \). Thus, the assumption of linear dependence must be wrong and this yields the proposition.

\[
\square
\]

3. Proof of the plane case

Let \( f : T \to R \) denote the function defined by \( f(e^{i\theta}) = 1 \) whenever \( |\theta - \frac{\pi}{2}| < \frac{\pi}{4} \) or \( |\theta - \frac{3\pi}{2}| < \frac{\pi}{4} \) and zero elsewhere.

The proof of the theorem will be carried out in two steps. First one chooses a diffeomorphism \( \eta \) of the circle such that for some positive numbers \( a, b \) we have that \( \kappa^* = \kappa \circ \eta \approx a(1 - f) + bf \) on \( T \). This step is an easy consequence of the four vertex condition. The second step consists in showing the existence of a complex
number $\beta$ with $|\beta| < 1$ such that if $g_\beta$ denotes the fractional transformation

$$g_\beta(w) = \frac{w - \beta}{1 - \beta w}, \ w \in T,$$

then $K = \kappa^* \circ g_\beta$ satisfies (1.1).

We need some definitions in order to construct the first preliminary diffeomorphism. For $j = 1, \ldots, 4$, let $\theta_j = (j-1)\pi/4$ and set $q_j = e^{i\theta_j}$, $A_j = \{e^{i\theta} : |\theta - \theta_j| < \pi/4 - \epsilon\}$ for a small positive number $\epsilon$.

By continuity there are points $r_j \in T$, $j = 1, \ldots, 4,$ also ordered counterclockwise and positive numbers $a, b$ such that $0 < a < b$ and $\kappa(r_1) = \kappa(r_3) = a$, $\kappa(r_2) = \kappa(r_4) = b$. Pick intervals $B_j \subset T, j = 1, \ldots, 4,$ such that $r_j \in B_j$ and $|\kappa(w) - \kappa(r_j)| < \epsilon$ for all $w \in B_j$. Let $\eta$ be any smooth orientation preserving $C^\infty$–diffeomorphism of the circle such that $\eta(A_j) \subset B_j$ for $j = 1, \ldots, 4$. It is now easily seen that $\kappa \circ \eta = a(1 - f) + bf + e$, where $e$ is bounded with $\int_0^{2\pi} |e| dt < C\epsilon$ and $C$ is independent of $\epsilon$.

We now claim that if $\epsilon$ has been chosen small enough, then there is a $\beta \in C$, $|\beta| \leq 1/2$ such that $\kappa^* \circ g_\beta$ is a non–normalised curvature function. To see this set $k^* = \kappa^* \circ g_\beta$, $k = a(1 - f \circ g_\beta) + bf \circ g_\beta$ and let

$$I = \int_0^{2\pi} k^* dt, \quad I^* = \int_0^{2\pi} k^* dt,$$

$K^* = \frac{2\pi}{T} k^*$ and $K = \frac{2\pi}{T} k$.

If $\epsilon$ has been chosen small enough, then clearly $I^* \neq 0$ whenever $|\beta| \leq 1/2$.

Set $F^*(\beta, \epsilon) = \int_0^{2\pi} e^{i\alpha^*(s)} ds$ and $F(\beta) = \int_0^{2\pi} e^{i\alpha(s)} ds$, where $\alpha, \alpha^*$ are defined by $\alpha(0) = \alpha^*(0) = 0$ and $\alpha, \ k^*$ is bounded with $\int_0^{2\pi} |e| dt < C\epsilon$ and $C$ is independent of $\epsilon$.

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$$I = \int_0^{2\pi} k^* dt, \quad I^* = \int_0^{2\pi} k^* dt,$$

$K^* = \frac{2\pi}{T} k^*$ and $K = \frac{2\pi}{T} k$.

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The References


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