PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 7, Pages 2131–2135
S 0002-9939(05)07788-9
Article electronically published on January 31, 2005

THE CONVERSE OF THE FOUR VERTEX THEOREM

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(Communicated by Ronald A. Fintushel)

ABSTRACT. We establish the converse to the four vertex theorem without the positivity condition.

1. Introduction

Let $\boldsymbol{T}$ denote the unit circle of the complex plane $\boldsymbol{C}$ and let $\gamma : \boldsymbol{T} \to \boldsymbol{C}$ be the parametrisation of a smooth, simple and closed curve such that $\gamma' \neq 0$ and such that γ is also regular. Here the smoothness condition means that γ is infinitely many times continuously differentiable. If $\kappa : \boldsymbol{T} \to \boldsymbol{R}$ denotes the curvature function of γ, then the four vertex theorem asserts that if κ is not a constant, then κ has at least four critical points $p_i \in \boldsymbol{T}$, $i = 1, \dots, 4$, ordered counterclockwise such that p_1, p_3 are local maxima and p_2, p_4 are local minima and that furthermore $\kappa(p_1) > 0, \kappa(p_3) > 0$ and

$$\max(\kappa(p_2), \kappa(p_4)) < \min(\kappa(p_1), \kappa(p_3)).$$

This result was apparently first proved for the case of closed convex curves by Mukhopadhyaya [4]. For proofs in the case of simple closed curves see Fog [1], Jackson [3] and Vietoris [5].

The converse of this was studied by Gluck [2], who proved that if κ is a smooth and strictly positive function satisfying the above four vertex property, then κ is the curvature function of a smooth, simple and closed curve. The purpose of this note is to establish the converse to the four vertex theorem without the positivity condition.

Theorem 1.1. *Suppose $\kappa : \boldsymbol{T} \to \boldsymbol{R}$ is not a constant and assume that κ has at least four critical points ordered counterclockwise $p_i \in T$, $i = 1, \dots, 4$, such that p_1, p_3 are local maxima and p_2, p_4 are local minima with $\kappa(p_1) > 0, \kappa(p_3) > 0$ and $\max(\kappa(p_2), \kappa(p_4)) < \min(\kappa(p_1), \kappa(p_3))$. If in addition κ is smooth, then κ is the curvature function of a smooth, simple and closed curve.*

Received by the editors March 10, 2003 and, in revised form, March 29, 2004.
2000 *Mathematics Subject Classification.* Primary 53A04.
The author was supported by a grant from the NFR, Sweden.

Björn Dahlberg died on the 30th of January 1998. The results of this paper appeared in his posthumous work. The final version was prepared by Vilhelm Adolfsson and Peter Kumlin, Department of Mathematics, Chalmers University of Technology and Göteborg University, SE-412 96 Göteborg, Sweden; email: vilhelm@math.chalmers.se.

We remark that a smooth function $K : \boldsymbol{T} \to \boldsymbol{R}$ represents the curvature of a smooth, simple and closed curve parametrised by the arc length if and only if the following three conditions are satisfied:

$$\int_0^{2\pi} K\,ds = 2\pi, \tag{1.1}$$

$$\int_0^{2\pi} e^{i\alpha(s)}\,ds = 0, \tag{1.2}$$

$$\int_t^{\tau} e^{i\alpha(s)}\,ds \neq 0 \text{ if } 0 \leq t < \tau < 2\pi. \tag{1.3}$$

Here $\alpha(s) = \int_0^s K(t)\,dt$ and the parametrisation of the associated curve γ is given by

$$\gamma_K(t) = \int_0^t e^{i\alpha(s)}\,ds.$$

The above three conditions all have a geometric interpretation. The first condition expresses that the curve γ_K has a well-determined tangent at $s = 0$, and the second condition expresses that γ_K is a closed curve. Finally, the third condition expresses that γ_K is simple, that is, without self-intersections.

We will say that κ is a *non–normalised curvature function* if

$$\begin{cases} I = & \int_0^{2\pi} \kappa\,dt \neq 0 \text{ and} \\ K = & \frac{2\pi}{I}\kappa \text{ satisfies (1.2) and (1.3).} \end{cases} \tag{1.4}$$

In order to prove the theorem it is enough to show the existence of a smooth diffeomorphism $\varphi : \boldsymbol{T} \to \boldsymbol{T}$ such that $\kappa \circ \varphi$ is a non–normalised curvature function. For letting ψ denote the inverse of φ and setting $K = \frac{2\pi}{I}\kappa \circ \varphi$, $I = \int_0^{2\pi} \kappa \circ \varphi\,dt$, then the curve Γ parametrised by $\Gamma(t) = \frac{2\pi}{I}\gamma_K(\psi(t))$ has κ as its curvature function.

2. Preliminary results

The construction of the diffeomorphisms required for the proof of Theorem 1.1 will be based on the following observations.

Proposition 2.1. *Let $E_j \subset \boldsymbol{T}$, $j = 1, \ldots, 4$, be non-empty, pairwise disjoint open intervals that are ordered counterclockwise on the unit circle with $\bigcup \overline{E_j} = \boldsymbol{T}$. Let a, b be two positive numbers with $a \neq b$ and define the function k by*

$$k = \begin{cases} a \text{ on } E_1 \cup E_3, \\ b \text{ on } E_2 \cup E_4. \end{cases}$$

Suppose furthermore that $\int_0^{2\pi} k\,ds = 2\pi$. Then k is the curvature function of a closed convex curve parametrised by the arc length if and only if $E_3 = \{-w : w \in E_1\}$ and $E_4 = \{-w : w \in E_2\}$.

Proof. We begin by establishing the necessity. Let γ be a curve of length 2π whose curvature is given by $k = k(s)$. Since k is strictly positive it is well known that γ is convex. Let $\kappa = \kappa(\vartheta)$ represent the curvature of γ as a function of the angle ϑ that the tangent forms with the positive x-axis. Then it is easily seen that

$$\int_0^{2\pi} e^{i\vartheta} \frac{1}{\kappa(\vartheta)}\,d\vartheta = 0. \tag{2.1}$$

Let the function $F = F(\vartheta)$ parametrise γ with respect to the angle ϑ that the tangent forms with the x-axis. Now set $e_j = \{\vartheta : F(\vartheta) \in \gamma(E_j)\},\ j = 1, \dots, 4$. Then the e_j's are non–empty, pairwise disjoint open intervals in $\boldsymbol{T}$ that are ordered counterclockwise and for which $\bigcup \overline{e_j} = \boldsymbol{T}$. Then $\kappa = a$ on $U = e_1 \cup e_3$ and $\kappa = b$ on $V = e_2 \cup e_4$. Since $a \neq b$ and $\int_0^{2\pi} e^{i\vartheta}\, d\vartheta = 0$ it follows from (2.1) that

$$\int_U e^{i\vartheta}\, d\vartheta = \int_V e^{i\vartheta}\, d\vartheta = 0. \tag{2.2}$$

Denote by $2l_j$ the length of the interval e_j and let c_j denote its centre. From (2.2) it follows that $e^{ic_1} \sin l_1 + e^{ic_3} \sin l_3 = 0$ and $e^{ic_2} \sin l_2 + e^{ic_4} \sin l_4 = 0$. Since $l_j > 0$ for $j = 1, \dots, 4$ it is easily seen that $c_3 = c_1 + \pi$, $c_2 = c_4 + \pi$, $l_3 = l_1$ and $l_4 = l_2$, which yields the necessity part of the proposition. The sufficiency part is an easy consequence of (1.1), (1.2) and (1.3). □

Proposition 2.2. *For $\alpha \in \boldsymbol{C}$ with $|\alpha| < 1$, let g_α denote the restriction to $\boldsymbol{T}$ of the Möbius transformation*

$$g_\alpha(w) = \frac{w - \alpha}{1 - \bar{\alpha} w}.$$

Let $E_j \subset \boldsymbol{T},\ j = 1, \dots, 4$, be non-empty, pairwise disjoint open intervals that are ordered counterclockwise on the unit circle with $\bigcup \overline{E_j} = \boldsymbol{T}$. Suppose $E_3 = \{-w : w \in E_1\}$ and $E_4 = \{-w : w \in E_2\}$. If $g_\alpha(E_1) = -g_\alpha(E_3)$ and $g_\alpha(E_2) = -g_\alpha(E_4)$, then $\alpha = 0$.

Proof. The proof will be carried out by a contradiction argument. We assume therefore that $\alpha \neq 0$. Let z, ζ denote the end points of the interval E_1. The assumptions on g_α imply that $g_\alpha(-z) = -g_\alpha(z)$ and $g_\alpha(-\zeta) = -g_\alpha(\zeta)$. A straightforward computation shows that therefore $\bar{\alpha} z^2 = \alpha$ and $\bar{\alpha} \zeta^2 = \alpha$. Under the assumption that $\alpha \neq 0$ it follows therefore that $z^2 = \zeta^2$, which is impossible since E_1 is non-empty with length strictly less than π. This contradiction establishes the proposition. □

We will need an infinitesimal version of the above propositions.

Proposition 2.3. *For $\alpha \in \boldsymbol{C}$ with $|\alpha| < 1$, let g_α denote the restriction to $\boldsymbol{T}$ of the Möbius transformation*

$$g_\alpha(w) = \frac{w - \alpha}{1 - \bar{\alpha} w}.$$

Let $f : \boldsymbol{T} \to \boldsymbol{R}$ denote the function defined by $f(e^{i\theta}) = 1$ whenever $|\theta - \frac{\pi}{2}| < \frac{\pi}{4}$ or $|\theta - \frac{3\pi}{2}| < \frac{\pi}{4}$ and zero elsewhere. Let a and b be two positive numbers such that $a \neq b$ and set $\kappa = a(1 - f) + bf$. Furthermore, let a and b be normalised so that $\int_0^{2\pi} \kappa\, dt = 2\pi$ and define $I_\alpha = \int_0^{2\pi} \kappa \circ g_\alpha$. Let A_α be defined by $A_\alpha(0) = 0$ and $A'_\alpha = \kappa \circ g_\alpha$. Suppose that α depends smoothly on a real parameter t such that $\alpha(0) = 0$. Letting $\dot{\alpha}$ denote the derivative with respect to the t-variable evaluated at $t = 0$ we have that $\dot{I} = 0$ and

$$\int_0^{2\pi} \dot{A}(t) e^{iA_0(t)}\, dt = z_0(\xi z_1 + \eta z_2).$$

Here ξ and η are defined by $\dot{\alpha} = \xi + i\eta$, and

$$z_0 = 2\sqrt{2} i (b - a) e^{i\pi w_0}, \qquad w_0 = (a + 2b)/4,$$

$$z_1 = (e^{-ib\frac{\pi}{2}} - 1)/b \quad \text{and} \quad z_2 = (e^{ia\frac{\pi}{2}} - 1)/a.$$

Furthermore the vectors z_1 and z_2 are linearly independent over $\boldsymbol{R}$.

Proof. We begin by selecting a smooth branch of the argument for points in a neighbourhood of $\{e^{i\theta} : \theta_0 \le \theta \le \theta_1\}$, where $0 < \theta_0 < \theta_1 < 2\pi$. Now let G_α denote the argument of the inverse of g_α. It is easily seen that

$$\dot{A}(t) = \begin{cases} 0 & \text{if } 0 < t < \frac{\pi}{4}, \\ (a-b)\dot{G}(\frac{\pi}{4}) & \text{if } \frac{\pi}{4} < t < \frac{3\pi}{4}, \\ (a-b)(\dot{G}(\frac{\pi}{4}) - \dot{G}(\frac{3\pi}{4})) & \text{if } \frac{3\pi}{4} < t < \frac{5\pi}{4}, \\ (a-b)(\dot{G}(\frac{\pi}{4}) - \dot{G}(\frac{3\pi}{4}) + \dot{G}(\frac{5\pi}{4})) & \text{if } \frac{5\pi}{4} < t < \frac{7\pi}{4}, \\ (a-b)(\dot{G}(\frac{\pi}{4}) - \dot{G}(\frac{3\pi}{4}) + \dot{G}(\frac{5\pi}{4}) - \dot{G}(\frac{7\pi}{4})) & \text{if } \frac{7\pi}{4} < t < 2\pi. \end{cases}$$

Since the inverse of g_α is given by $g_{-\alpha}$ it is also easily seen that $\dot{G}(\theta) = 2Im(\dot{\alpha}e^{-i\theta})$, where $Im(w)$ denotes the imaginary part of w. Hence $\dot{G}(\theta + \pi) = -\dot{G}(t)$, so that

$$\dot{A}(t) = \begin{cases} 0 & \text{if } 0 < t < \frac{\pi}{4} \text{ or } \frac{7\pi}{4} < t < 2\pi\ , \\ (a-b)\dot{G}(\frac{\pi}{4}) & \text{if } \frac{\pi}{4} < t < \frac{3\pi}{4}, \\ (a-b)(\dot{G}(\frac{\pi}{4}) - \dot{G}(\frac{3\pi}{4})) & \text{if } \frac{3\pi}{4} < t < \frac{5\pi}{4}, \\ (b-a)\dot{G}(\frac{3\pi}{4}) & \text{if } \frac{5\pi}{4} < t < \frac{7\pi}{4}. \end{cases}$$

In particular, we see that since $I_\alpha = A_\alpha(2\pi)$ we have that $\dot{I} = 0$. By using that $A_0(t+\pi) = \pi + A_0(t)$ and integrating it is easy to verify that the expression for $\dot{A}(t)$ holds. It remains to verify that the vectors z_1 and z_2 are linearly independent over $\boldsymbol{R}$. We first note that the normalisation $\int_0^{2\pi} \kappa\, dt = 2\pi$ means that $a + b = 2$ so that $a, b \in (0, 2)$. Assuming that z_1 and z_2 are not linearly independent, there would exist real numbers α, β such that $\alpha z_1 + \beta z_2 = 0$ where not both α and β equals zero. In fact, both α and β have to be non-zero in this case since otherwise z_1 or z_2 has to equal zero in which case $1 = e^{iw\frac{\pi}{2}}$ for $w = a$ or $w = -b$, which in turn implies that $w = 4n$ for some $n \in \boldsymbol{Z}$ and this contradicts that $a, b \in (0, 2)$. Thus, since both α and β have to be non-zero we get that there is a real number $c \ne 0$ such that

$$c(1 - e^{-ib\frac{\pi}{2}}) = e^{ia\frac{\pi}{2}} - 1 \Leftrightarrow e^{-ib\frac{\pi}{4}}c(e^{ib\frac{\pi}{4}} - e^{-ib\frac{\pi}{4}}) = e^{ia\frac{\pi}{4}}(e^{ia\frac{\pi}{4}} - e^{-ia\frac{\pi}{4}})$$
$$\Leftrightarrow c\sin(b\frac{\pi}{4}) = e^{i(a+b)\frac{\pi}{4}}\sin(a\frac{\pi}{4}) \Leftrightarrow c\sin(b\frac{\pi}{4}) = e^{i\frac{\pi}{2}}\sin(a\frac{\pi}{4})$$

where we used that $a + b = 2$. Consequently,

$$c\sin(b\frac{\pi}{4}) = Re(c\sin(b\frac{\pi}{4})) = Re(i\sin(a\frac{\pi}{4})) = 0.$$

Since $c \ne 0$ we get $\sin(b\frac{\pi}{4}) = 0$, which implies that $b\frac{\pi}{4} = n\pi$ for some $n \in \boldsymbol{Z}$, which contradicts $b \in (0, 2)$. Thus, the assumption of linear dependence must be wrong and this yields the proposition. □

3. Proof of the plane case

Let $f : \boldsymbol{T} \to \boldsymbol{R}$ denote the function defined by $f(e^{i\theta}) = 1$ whenever $|\theta - \frac{\pi}{2}| < \frac{\pi}{4}$ or $|\theta - \frac{3\pi}{2}| < \frac{\pi}{4}$ and zero elsewhere.

The proof of the theorem will be carried out in two steps. First one chooses a diffeomorphism η of the circle such that for some positive numbers a, b we have that $\kappa^* = \kappa \circ \eta \approx a(1-f) + bf$ on $\boldsymbol{T}$. This step is an easy consequence of the four vertex condition. The second step consists in showing the existence of a complex number β with $|\beta| < 1$ such that if g_β denotes the fractional transformation

$$g_\beta(w) = \frac{w-\beta}{1-\bar{\beta}w},\ w \in \boldsymbol{T},$$

then $K = \kappa^* \circ g_\beta$ satisfies (1.4).

We need some definitions in order to construct the first preliminary diffeomorphism. For $j = 1, \ldots, 4$, let $\theta_j = (j-1)\frac{\pi}{2}$ and set $q_j = e^{i\theta_j}$, $A_j = \{e^{i\theta} : |\theta - \theta_j| < \frac{\pi}{4} - \epsilon\}$ for a small positive number ϵ.

By continuity there are points $r_j \in \boldsymbol{T},\ j = 1, \ldots, 4$, also ordered counterclockwise and positive numbers a, b such that $0 < a < b$ and $\kappa(r_1) = \kappa(r_3) = a,\ \kappa(r_2) = \kappa(r_4) = b$. Pick intervals $B_j \subset \boldsymbol{T}, j = 1, \ldots, 4$, such that $r_j \in B_j$ and $|\kappa(w) - \kappa(r_j)| < \epsilon$ for all $w \in B_j$. Let η be any smooth orientation preserving C^∞–diffeomorphism of the circle such that $\eta(A_j) \subset B_j$ for $j = 1, \ldots, 4$. It is now easily seen that $\kappa \circ \eta = a(1-f) + bf + e$, where e is bounded with $\int_0^{2\pi} |e| dt < C\epsilon$ and C is independent of ϵ.

We now claim that if ϵ has been chosen small enough, then there is a $\beta \in \boldsymbol{C}$, $|\beta| \leq \frac{1}{2}$ such that $\kappa^* \circ g_\beta$ is a non–normalised curvature function. To see this set $k^* = \kappa^* \circ g_\beta$, $k = a(1 - f \circ g_\beta) + bf \circ g_\beta$ and let

$$I = \int_0^{2\pi} k\, dt, \qquad I^* = \int_0^{2\pi} k^*\, dt,$$

$K^* = \frac{2\pi}{I^*} k^*$ and $K = \frac{2\pi}{I} k$.

If ϵ has been chosen small enough, then clearly $I^* \neq 0$ whenever $|\beta| \leq \frac{1}{2}$. Set $F^*(\beta, \epsilon) = \int_0^{2\pi} e^{i\alpha^*(s)}\, ds$ and $F(\beta) = \int_0^{2\pi} e^{i\alpha(s)}\, ds$, where α, α^* are defined by $\alpha(0) = \alpha^*(0) = 0$ and $\alpha' = K,\ \alpha^{*\prime} = K^*$. It follows from Propositions 2.1 and 2.2 that $F(\beta) \neq 0$ whenever $\beta \neq 0, |\beta| < 1$. Noticing that $F(0) = 0$ we see from Proposition 2.3 that the restriction of F to a sufficiently small circle centred at 0 has non-vanishing winding number around 0. By a standard topology argument and the fact that $\lim_{\epsilon \downarrow 0} F^*(\beta, \epsilon) = F(\beta)$ it follows that if ϵ_0 has been chosen sufficiently small, then for all $\epsilon \in (0, \epsilon_0)$ there is a $\beta,\ |\beta| < \frac{1}{2}$ such that $F^*(\beta, \epsilon) = 0$. Since the curve $\gamma(t) = \int_0^t e^{i\alpha(u)}\, du$ is simple and closed and $|\alpha - \alpha^*| \leq C\epsilon$, it follows that $\int_t^\tau e^{i\alpha^*(s)}\, ds \neq 0$ whenever $0 < t < \tau < 2\pi$, which completes the proof of the theorem.

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