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SPACES OF TYPE BLO FOR NON-DOUBLING MEASURES

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ABSTRACT. The spaces of type BLO for the positive Radon measures satisfying a growth condition on $\mathbb{R}^d$ are introduced. It is shown that some properties which hold for the classical space BLO when μ is a doubling measure remain valid for the spaces of type BLO introduced in this paper, without assuming μ doubling.

Let d, n be some fixed integers with $d \geq 2, 1 \leq n \leq d$, and let μ be a positive Radon measure on $\mathbb{R}^d$ satisfying the growth condition

$$\mu(B(x,r)) \leq C_0 r^n \qquad \text{for all } x \in \mathbb{R}^d, r > 0. \tag{1}$$

We do not assume that μ is doubling.

A kernel $k(\cdot,\cdot) \in L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\})$ is called a Calderón-Zygmund kernel if

1. $|k(x,y)| \leq \dfrac{C}{|x-y|^n}$,
2. there exists $0 < \delta \leq 1$ such that

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}} \quad \text{if} \quad |x-x'| \leq |x-y|/2.$$

The Calderón-Zygmund operator associated to the kernel k and the measure μ is formally defined as

$$Tf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)d\mu(y).$$

We say that T is bounded on $L^p(\mu)$ if the truncated operators

$$T_\epsilon f(x) = \int_{|x-y|>\epsilon} k(x,y)f(y)d\mu(y)$$

are bounded on $L^p(\mu)$ uniformly on $\epsilon > 0$.

The maximal operator T_* associated with the Calderón-Zygmund operator T is defined as

$$T_*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)| \tag{2}$$

for $f \in L^p(\mu), p \in [1,\infty)$.

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When μ is a positive Radon measure satisfying the growth condition (1) and T is bounded on $L^2(\mu)$, the weak type (1,1) and $L^p(\mu)$ estimates for T_* were obtained by F.Nazarov, S.Treil and A.Volberg in [4].

It is well known that if μ is Lebesgue measure and T is bounded on $L^2(\mathbb{R}^d)$, then T_* is bounded from $L^\infty(\mathbb{R}^d)$ into $BMO(\mathbb{R}^d)$ [5], and furthermore, is bounded from $L^\infty(\mathbb{R}^d)$ into $BLO(\mathbb{R}^d)$ [3]. For the positive Radon measure μ satisfying (1), X.Tolsa [6] has introduced the spaces $RBMO(\mu)$, which are the suitable substitutes for the classical spaces $BMO(\mathbb{R}^d)$, and proved that if T is bounded on $L^2(\mu)$, then T is bounded from $L^\infty(\mu)$ into $RBMO(\mu)$.

In this note, for the positive Radon measure μ satisfying (1) we will introduce the spaces $RBLO(\mu)$ as the substitutes for the classical spaces $BLO(\mathbb{R}^d)$ defined by R.R.Coifman and R.Rochberg [2], and will show that if T is bounded on $L^2(\mu)$, then T_* is bounded from $L^\infty(\mu)$ into $RBLO(\mu)$.

We take notation and definitions from [6]. By a cube $Q \subset \mathbb{R}^d$ we mean a closed cube centered at some point $z_Q \in \operatorname{supp}(\mu)$ with sides parallel to the axes. Let $\alpha > 1$ and $\beta > \alpha^n$. A cube with side length $l(Q)$ is said to be (α, β)-doubling if $\mu(\alpha Q) \le \beta\mu(Q)$, where αQ denotes the cube concentric with Q and having side length $\alpha l(Q)$. Given a cube $Q \subset \mathbb{R}^d$, let N be the smallest integer ≥ 0 such that $\alpha^N Q$ is doubling. Denote this doubling cube by $\widetilde{Q}$. If α and β are not specified, by a doubling cube we will mean a $(4\sqrt{d}, (4\sqrt{d})^{n+1})$-doubling cube. Given two cubes $Q \subset R$, we set

$$K_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{l(2^k Q)^n},$$

where $N_{Q,R}$ is the first integer k such that $2^k Q \supset R$.

Let $\rho > 1$ be some fixed constant. Say that $f \in L^1_{\mathrm{loc}}(\mu)$ belongs to $RBMO(\mu)$ if there exists some constant C_1 such that for any cube Q,

$$\frac{1}{\mu(\rho Q)} \int_Q |f - m_{\widetilde{Q}}(f)| d\mu \le C_1 \tag{3}$$

and

$$|m_Q(f) - m_R(f)| \le C_1 K_{Q,R} \qquad \text{for any two doubling cubes } Q \subset R, \tag{4}$$

where $m_Q(f) = \mu(Q)^{-1} \int_Q f d\mu$. The best constant C_1 is the $RBMO(\mu)$ norm of f, which we denote as $\|f\|_*$ and is independent of $\rho > 1$. It is shown by X.Tolsa [6] that one can replace (3) by

$$\frac{1}{\mu(Q)} \int_Q |f - m_Q(f)| d\mu \le C_2 \qquad \text{for any doubling cube } Q, \tag{5}$$

or by

$$\left(\frac{1}{\mu(\rho Q)} \int_Q |f - m_{\widetilde{Q}}(f)|^p d\mu \right)^{1/p} \le C_3 \qquad \text{for } 1 \le p < \infty \text{ and any cube } Q, \tag{6}$$

and the best constants C_2 and C_3 are comparable to $\|f\|_*$.

We are ready to define the spaces of type BLO for the positive Radon measure μ satisfying (1).

Definition 1. We say that $f \in L^1_{\mathrm{loc}}(\mu)$ belongs to $RBLO(\mu)$ if there exists some constant C_4 such that for any doubling cube Q,

$$m_Q(f) - \operatorname*{essinf}_{x\in Q} f(x) \leq C_4 \tag{7}$$

and

$$m_Q(f) - m_R(f) \leq C_4 K_{Q,R} \qquad \text{for any two doubling cubes } Q \subset R. \tag{8}$$

The smallest constant C_4 will be denoted by $\|f\|_{RBLO}$.

Observe that (7) and (8) are equivalent to (7) and (4). It is easy to check that $L^\infty(\mu) \subset RBLO(\mu)$ with $\|f\|_{RBLO} \leq 2\|f\|_{L^\infty(\mu)}$ and $RBLO(\mu) \subset RBMO(\mu)$ with $\|f\|_* \leq 2\|f\|_{RBLO}$.

C.Bennett [1] has obtained a criterion for the classical spaces $BLO(\mathbb{R}^d)$. To give out the $RBLO(\mu)$ criterion we consider the non-centered doubling maximal function

$$Mf(x) = \sup_{\substack{Q\ni x\\ Q:doubling}} m_Q(f) = \sup_{\substack{Q\ni x\\ Q:doubling}} \frac{1}{\mu(Q)}\int_Q f d\mu, \tag{9}$$

where the supremum extends over all doubling cubes Q containing x. By the Lebesgue differentiation theorem, $Mf(x) \geq f(x)$ μ-a.e. $x \in \mathbb{R}^d$. Moreover, $|Mf(x)| \leq M(|f|)(x) := Nf(x)$. The maximal operator N is weak type (1,1) and bounded on $L^p(\mu)$ for $1 < p \leq \infty$ [6], so is M.

Lemma 1. *$f \in RBLO(\mu)$ if and only if $Mf - f \in L^\infty(\mu)$ and f satisfies* (8). *Furthermore,*

$$\|Mf - f\|_{L^\infty(\mu)} \sim \|f\|_{RBLO}. \tag{10}$$

Proof. Suppose first that $f \in RBLO(\mu)$. Then f satisfies (8). Observe that (see [6, Remark 2.3]), by the Lebesgue differentiation theorem, for μ-a.e.$x \in \mathbb{R}^d$ one can find a sequence of doubling cubes $\{Q_k\}_k$ centered at x with $l(Q_k) \to 0$ such that

$$\lim_{k\to\infty} \frac{1}{\mu(Q_k)} \int_{Q_k} f d\mu = f(x).$$

Let x be any such point, and let Q be any doubling cube containing x. Then $f(x) \geq \operatorname*{essinf}_{x\in Q} f(x)$ and so

$$m_Q(f) - f(x) \leq m_Q(f) - \operatorname*{essinf}_{x\in Q} f(x) \leq \|f\|_{RBLO}.$$

Taking the supremum over all doubling cubes containing x, we get

$$Mf(x) - f(x) \leq \|f\|_{RBLO}.$$

Hence $Mf - f \in L^\infty(\mu)$ and $\|Mf - f\|_{L^\infty(\mu)} \leq \|f\|_{RBLO}$.

Conversely, suppose $Mf - f \in L^\infty(\mu)$ and let Q be any doubling cube in $\mathbb{R}^d$. If any $x \in Q$ is such that

$$f(x) < m_Q(f) - \|Mf - f\|_{L^\infty(\mu)},$$

then

$$Mf(x) - f(x) \geq m_Q(f) - f(x) > \|Mf - f\|_{L^\infty(\mu)}.$$

So, for μ-a.e. $x \in Q$,

$$f(x) \geq m_Q(f) - \|Mf - f\|_{L^\infty(\mu)},$$

and consequently,

$$\operatorname*{essinf}_{x\in Q} f(x) \geq m_Q(f) - \|Mf - f\|_{L^\infty(\mu)}.$$

Since f satisfies (8) we get that $f \in RBLO(\mu)$ and

$$\|f\|_{RBLO} \leq C\|Mf - f\|_{L^\infty(\mu)}.$$

□

Our main results are as follows.

Theorem 1. *If the Calderón-Zygmund operator T is bounded on $L^2(\mu)$, then the maximal operator T_* is bounded from $L^\infty(\mu)$ into $RBLO(\mu)$.*

Theorem 2. *If $f \in RBMO(\mu)$ and Mf satisfies (8), then $Mf \in RBLO(\mu)$ and*

$$\|Mf\|_{RBLO} \leq C\|f\|_*.$$

In particular, M is bounded on $RBLO(\mu)$.

Theorem 3. *A locally integrable function f belongs to $RBLO(\mu)$ if and only if there exist $h \in L^\infty(\mu)$ and $F \in RBMO(\mu)$ with MF satisfying (8) such that*

$$f = MF + h. \tag{11}$$

Furthermore,

$$\|f\|_{RBLO} \sim \inf(\|F\|_* + \|h\|_{L^\infty(\mu)}) \tag{12}$$

where the infimum extends over all representations of the form (11).

The proof of Theorem 1. Let $x \in \mathbb{R}^d \cap \operatorname{supp}(\mu)$ and Q be any doubling cube containing x. For each fixed cube Q let B be the smallest ball centered at x which contains Q. Then $2B \subset 4\sqrt{d}Q$. If $f \in L^\infty(\mu) \cap L^{p_0}(\mu)$ for some $p_0 \in [1, \infty)$, by $L^2(\mu)$ boundedness of T_* [4], we have

$$\begin{aligned}
\frac{1}{\mu(Q)}\int_Q T_*(f\chi_{2B})d\mu &\leq \frac{1}{\mu(Q)^{1/2}}\left\{\int_{\mathbb{R}^d}\left[T_*(f\chi_{2B})\right]^2 d\mu\right\}^{1/2}\\
&\leq \frac{C}{\mu(Q)^{1/2}}\left\{\int_{\mathbb{R}^d}|f\chi_{2B}|^2 d\mu\right\}^{1/2}\\
&\leq C\frac{\mu(2B)^{1/2}}{\mu(Q)^{1/2}}\|f\|_{L^\infty(\mu)}\\
&\leq C\frac{\mu(4\sqrt{d}Q)^{1/2}}{\mu(Q)^{1/2}}\|f\|_{L^\infty(\mu)}\\
&\leq C\|f\|_{L^\infty(\mu)}. \qquad (13)
\end{aligned}$$

From (2) it follows that

$$T_*(f\chi_{\mathbb{R}^d\setminus 2B})(x) \leq T_*f(x).$$

By this and the conditions of the Calderón-Zygmund kernel, for all $y \in Q$, we have

$$\begin{aligned}
T_*(f\chi_{\mathbb{R}^d\setminus 2B})(y) &\leq |T_*(f\chi_{\mathbb{R}^d\setminus 2B})(y) - T_*(f\chi_{\mathbb{R}^d\setminus 2B})(x)| + T_*(f\chi_{\mathbb{R}^d\setminus 2B})(x)\\
&\leq C\|f\|_{L^\infty(\mu)} + T_*f(x).
\end{aligned}$$

Therefore,

$$\frac{1}{\mu(Q)}\int_Q T_*(f\chi_{\mathbb{R}^d\setminus 2B})d\mu \leq C\|f\|_{L^\infty(\mu)} + T_*f(x).$$

From this and (13), we get

$$\frac{1}{\mu(Q)}\int_Q T_* f d\mu \leq C\|f\|_{L^\infty(\mu)} + T_* f(x). \tag{14}$$

So,

$$\|M(T_* f) - T_* f\|_{L^\infty(\mu)} \leq C\|f\|_{L^\infty(\mu)}. \tag{15}$$

From this and Lemma 1, it suffices to show that $T_* f$ satisfies (8). We apply the argument analogous to [6, pp. 104-105]. Let $Q \subset R$ be any two doubling cubes. Recall that $N_{Q,R}$ is the first integer k such that $2^k Q \supset R$. We denote $Q_R = 2^{N_{Q,R}+1}Q$. Thus, for $x \in Q$ and $y \in R$, we set

$$\begin{aligned} T_\epsilon f(x) = T_\epsilon(f\chi_{2Q})(x) &+ \sum_{k=1}^{N_{Q,R}} T_\epsilon(f\chi_{2^{k+1}Q\setminus 2^k Q})(x) + T_\epsilon(f\chi_{\mathbb{R}^d\setminus Q_R})(x) \\ &- \Big(T_\epsilon(f\chi_{Q_R})(y) + T_\epsilon(f\chi_{\mathbb{R}^d\setminus Q_R})(y)\Big) + T_\epsilon f(y). \end{aligned}$$

Since

$$|T_\epsilon(f\chi_{\mathbb{R}^d\setminus Q_R})(x) - T_\epsilon(f\chi_{\mathbb{R}^d\setminus Q_R})(y)| \leq C\|f\|_{L^\infty(\mu)},$$

we get

$$\begin{aligned} T_* f(x) \leq T_*(f\chi_{2Q})(x) &+ C\Big(1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^{k+1}Q)}{l(2^{k+1}Q)^n}\Big)\|f\|_{L^\infty(\mu)} \\ &+ T_*(f\chi_{Q_R})(y) + T_* f(y). \end{aligned}$$

Now we take the mean over Q for x, and over R for y. We write

$$m_R\big(T_*(f\chi_{Q_R})\big) \leq m_R\big(T_*(f\chi_{Q_R\cap 2R})\big) + m_R\big(T_*(f\chi_{Q_R\setminus 2R})\big).$$

Similar to the previous estimate (13) we obtain

$$m_Q\big(T_*(f\chi_{2Q})\big) \leq C\|f\|_{L^\infty(\mu)}$$

and

$$m_R\big(T_*(f\chi_{Q_R\cap 2R})\big) \leq C\|f\|_{L^\infty(\mu)}.$$

On the other hand, since $l(Q_R) \approx l(R)$, we have

$$m_R\big(T_*(f\chi_{Q_R\setminus 2R})\big) \leq C\|f\|_{L^\infty(\mu)}.$$

Therefore,

$$m_Q(T_* f) - m_R(T_* f) \leq CK_{Q,R}\|f\|_{L^\infty(\mu)}.$$

If $f \notin L^p(\mu)$ for all $p \in [1,\infty)$, then the integral $\int_{|x-y|>\epsilon} k(x,y)f(y)d\mu(y)$ may not be convergent. The operator T_ϵ can be extended to the whole space $L^\infty(\mu)$ following the standard arguments: Given a cube Q_0 centered at the origin with side length $l(Q_0) > 3\epsilon$, we write $f = f_1 + f_2$, with $f_1 = f\chi_{2Q_0}$. For $x \in Q_0$, we define

$$T_\epsilon f(x) = T_\epsilon f_1(x) + \int_{|x-y|>\epsilon} (k(x,y) - k(0,y))f_2(y)d\mu(y).$$

Now both integrals in this equation are convergent. With arguments similar to the case $f \in L^\infty(\mu) \cap L^{p_0}(\mu)$ we complete the proof. □

To prove Theorem 2 and Theorem 3 we require the following lemma.

Lemma 2. *If $f \in RBMO(\mu)$, then for any doubling cube Q,*

$$\frac{1}{\mu(Q)}\int_Q Mf d\mu \le C\|f\|_* + \operatorname*{essinf}_{x\in Q} Mf(x). \tag{16}$$

Moreover, if Mf is finite μ-a.e., then

$$\frac{1}{\mu(Q)}\int_Q Mf d\mu - \operatorname*{essinf}_{x\in Q} Mf(x) \le C\|f\|_*. \tag{17}$$

Proof. Set $\rho = 4\sqrt{d}/5$ in (3). Fix a doubling cube Q. For $f \in RBMO(\mu)$ we write

$$f = (f - m_Q(f))\chi_{3Q} + (m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d\setminus 3Q}).$$

Since M is bounded on $L^2(\mu)$, we have

$$\begin{aligned}
&\int_Q M\big((f - m_Q(f))\chi_{3Q}\big)d\mu \\
\le\ & \mu(Q)^{1/2}\Big\{\int_{\mathbb{R}^d}|M\big((f - m_Q(f))\chi_{3Q}\big)|^2 d\mu\Big\}^{1/2} \\
\le\ & C\mu(Q)^{1/2}\Big\{\int_{\mathbb{R}^d}|(f - m_Q(f))\chi_{3Q}|^2 d\mu\Big\}^{1/2} \\
\le\ & C\mu(Q)^{1/2}\Big(\Big\{\int_{3Q}|f - m_{\widetilde{3Q}}(f)|^2 d\mu\Big\}^{1/2} + \Big\{\int_{3Q}|m_Q(f) - m_{\widetilde{3Q}}(f)|^2 d\mu\Big\}^{1/2}\Big) \\
\le\ & C\mu(Q)^{1/2}\Big(\mu((12\sqrt{d}/5)Q)^{1/2}\|f\|_* + \mu(3Q)^{1/2}\|f\|_* K_{Q,\widetilde{3Q}}\Big) \\
\le\ & C\mu(Q)^{1/2}\mu(4\sqrt{d}Q)^{1/2}\|f\|_* \le C\mu(Q)\|f\|_*.
\end{aligned}$$

Next, we shall show that

$$\frac{1}{\mu(Q)}\int_Q M\big(m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d\setminus 3Q}\big)d\mu \le C\|f\|_* + \operatorname*{essinf}_{x\in Q} Mf(x). \tag{18}$$

It suffices to show that

$$M\big(m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d\setminus 3Q}\big)(x) \le C\|f\|_* + \operatorname*{essinf}_{x\in Q} Mf(x) \quad \mu\text{-a.e. } x \in Q. \tag{19}$$

For this it will be enough to show that

$$\frac{1}{\mu(R)}\int_R \big(m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d\setminus 3Q}\big)d\mu \le C\|f\|_* + \operatorname*{essinf}_{x\in Q} Mf(x) \tag{20}$$

for any doubling cube $R \ni x$. If $R \subset 3Q$, the result follows immediately:

$$\frac{1}{\mu(R)}\int_R \big(m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d\setminus 3Q}\big)d\mu = m_Q(f) \le \operatorname*{essinf}_{x\in Q} Mf(x).$$

So, suppose $R \cap (\mathbb{R}^d \setminus 3Q) \ne \emptyset$. Then $l(R) > l(Q)$ and $3Q \subset 5R$. Write

$$m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d\setminus 3Q} = \big(m_Q(f) - m_{\widetilde{5R}}(f)\big)\chi_{3Q} + \big(f - m_{\widetilde{5R}}(f)\big)\chi_{\mathbb{R}^d\setminus 3Q} + m_{\widetilde{5R}}(f).$$

Obviously, $m_{\widetilde{5R}}(f) \le \operatorname*{essinf}_{x\in Q} Mf(x)$. Further,

$$\begin{aligned}
&\int_R \Big\{ \big(m_Q(f) - m_{\widetilde{5R}}(f)\big)\chi_{3Q} + \big(f - m_{\widetilde{5R}}(f)\big)\chi_{\mathbb{R}^d\setminus 3Q} \Big\} d\mu \\
\le\ & \mu(3Q)|m_Q(f) - m_{\widetilde{5R}}(f)| + \int_{5R} |f - m_{\widetilde{5R}}(f)|\chi_{\mathbb{R}^d\setminus 3Q} d\mu \\
\le\ & \frac{\mu(4\sqrt{d}Q)}{\mu(Q)} \int_Q |f - m_{\widetilde{5R}}(f)| + \int_{5R\setminus 3Q} |f - m_{\widetilde{5R}}(f)| d\mu \\
\le\ & C \int_{5R} |f - m_{\widetilde{5R}}(f)| d\mu \\
\le\ & C\mu(4\sqrt{d}R)\|f\|_* \le C\mu(R)\|f\|_*.
\end{aligned}$$

This establishes (18) and completes the proof of Lemma 2. □

The proof of Theorem 2. By Lemma 2, it suffices to prove that if $f \in RBLO(\mu)$, then Mf satisfies (8). In fact, applying Lemma 1, for any two doubling cubes $Q \subset R$ we have

$$\begin{aligned}
& m_Q(Mf) - m_R(Mf) \\
\le\ & m_Q(Mf) - m_Q(f) + m_R(Mf) - m_R(f) + |m_Q(f) - m_R(f)| \\
\le\ & 2\|Mf - f\|_{L^\infty(\mu)} + \|f\|_* K_{Q,R} \\
\le\ & 2\|f\|_{RBLO} + \|f\|_* K_{Q,R} \\
\le\ & C\|f\|_{RBLO} K_{Q,R}.
\end{aligned}$$

As we remarked above, this completes the proof of Theorem 2. □

Theorem 3 follows immediately from Lemma 1 and Theorem 2, and we omit the details here.

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