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BOUNDEDNESS OF THE FIRST EIGENVALUE OF THE p-LAPLACIAN

ANA-MARIA MATEI

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ABSTRACT. We prove that for any $p > 1$, any compact manifold of three or more dimensions carries Riemannian metrics of volume one with the first eigenvalue of the p-Laplacian arbitrarily large.

INTRODUCTION

The p-Laplace operator has been extensively studied in recent years, especially in the context of a bounded domain in $\mathbb{R}^n$. Recently, there has been an increasing interest in the study of this operator – and in particular of its first eigenvalue – in the more general setting of Riemannian manifolds.

Let M be a compact connected manifold. The p-Laplace operator ($p > 1$) associated to a Riemannian metric g on M is given by

$$\Delta_p f := \delta(|df|^{p-2} df)\,,$$

where $\delta = -div_g$ is the adjoint of d for the L^2-norm induced by g on the space of differential forms. This operator can be viewed as a natural generalization of the well-known Laplace-Beltrami operator which corresponds to $p = 2$. When $\partial M = \emptyset$, the nonlinear partial differential equation

$$\Delta_p f = \lambda |f|^{p-2} f \tag{0.1}$$

characterizes the critical points of the p-energy functional $E_p(f) = \int_M |df|^p \, \nu_g$ under the constraint $\int_M |f|^p \, \nu_g = 1$ (ν_g denotes the Riemannian volume element induced by g).

The real numbers λ for which this equation has nontrivial solutions are called *eigenvalues* of Δ_p. They represent the critical energy levels. The associated solutions, i.e. the critical functions, are called *eigenfunctions*.

Obviously, zero is an eigenvalue of Δ_p, the associated eigenfunctions being the constant functions. The set $\sigma_p(M, g)$ of the nonzero eigenvalues is a nonempty, unbounded subset of $(0, \infty)$ [6]. Its infimum $\lambda_{1,p}(M, g) = \inf \sigma_p(M, g)$ is itself a positive eigenvalue called the *first eigenvalue* of Δ_p and has the following variational characterization [14] :

$$\lambda_{1,p}(M, g) = \inf \left\{ \frac{\int_M |df|^p \ \nu_g}{\int_M |f|^p \, \nu_g} \mid f \in W^{1,p}(M) \setminus \{0\},\ \int_M |f|^{p-2} f \, \nu_g = 0 \right\}. \tag{0.2}$$

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In the present work we consider $\lambda_{1,p}$ as a functional on the space of Riemannian metrics on M. Since $\lambda_{1,p}$ is noninvariant under dilatations, we will study the restriction of $\lambda_{1,p}$ to $\mathcal{M}(M)$, the space of Riemannian metrics of volume one on M. The question we ask is whether $\lambda_{1,p}$ is bounded on $\mathcal{M}(M)$?

A classical elementary argument [3] shows that the infimum of $\lambda_{1,2}$ on this space is zero. The same argument shows that in fact for any $p > 1$ the infimum of $\lambda_{1,p}$ on $\mathcal{M}(M)$ is equally zero.

A series of authors (Bérard Bergery and Bourguignon [2], Muto [10], Tanno [11], Urakawa [13]) have showed in various degrees of generality that if M is a sphere of dimension $m \geq 3$, then there is no upper bound for $\lambda_{1,2}$ on $\mathcal{M}(S^m)$. Colbois and Dodziuk [4] (and independently Xu [15]) have extended this result to any compact manifold of dimension $m \geq 3$.

In this paper we prove that this phenomenon is still true for the functional $\lambda_{1,p}$, for any $p > 1$.

1. Preliminaries

We review here some basic properties of the first eigenvalue for the Dirichlet and the Neumann problems for Δ_p.

Let Ω be a domain in M and consider the Dirichlet problem:

$$\begin{cases} \Delta_p f & = \mu \, |f|^{p-2} f \quad \text{in } \Omega, \\ f & = 0 \qquad \text{on } \partial\Omega \, . \end{cases}$$

Denote by $\mu_{1,p}(\Omega, g)$ the infimum of the set of eigenvalues for this problem. Here too, $\mu_{1,p}(\Omega, g)$ is a positive eigenvalue with the variational characterization

$$\mu_{1,p}(\Omega, g) = \inf \left\{ \frac{\int_\Omega |df|^p \nu_g}{\int_\Omega |f|^p \nu_g} \mid f \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}.$$

The eigenfunctions associated to $\mu_{1,p}(\Omega, g)$ are essentially unique; they form a one-dimensional space. Moreover, they have no zeros in Ω (see for instance [7]).

In [14], Veron proved the following result:

$$\lambda_{1,p}(M, g) = \min_{(\omega_1, \omega_2)} \max \left\{ \mu_{1,p}(\omega_1, g), \mu_{1,p}(\omega_2, g) \right\}, \tag{1.1}$$

where (ω_1, ω_2) runs over the set of couples of nonempty disjoint open subsets of M.

Consider now the Neumann problem for Δ_p on Ω:

$$\begin{cases} \Delta_p f & = \lambda |f|^{p-2} f \quad \text{in } \Omega, \\ df(\eta) & = 0 \qquad \text{on } \partial\Omega \, , \end{cases}$$

where η denotes the exterior normal unit vector field to $\partial\Omega$.

Denote by $\lambda^N_{1,p}(\Omega, g)$ the infimum of the set of nonzero eigenvalues for this problem. By mimicking the proof of the closed case [14] we obtain that $\lambda^N_{1,p}(\Omega, g)$ is itself a positive eigenvalue with the variational characterization

$$\lambda^N_{1,p}(\Omega, g) = \inf \left\{ \frac{\int_\Omega |df|^p \, \nu_g}{\int_\Omega |f|^p \, \nu_g} \mid f \in W^{1,p}(\Omega) \setminus \{0\} \, , \, \int_\Omega |f|^{p-2} f \, \nu_g = 0 \right\}.$$

A more general regularity result [12] says that the eigenfunctions for these problems are C^1, with gradient locally Hölder continuous.

2. Riemannian metrics with arbitrarily large $\lambda_{1,p}$

Let M be a compact connected manifold and denote by $\mathcal{M}(M)$ the space of Riemannian metrics of volume one on M.

It is known that we may diffeomorphically transform a local disc of M into a "mushroom" attached to M by a cylinder C. We can choose a Riemannian metric $g_s \in \mathcal{M}(M)$ such that some subset $C_0 \subset C$ is the Riemannian product $[0,l] \times S^{m-1}(s)$ (l, s are positive constants and $S^{m-1}(s)$ denotes the $(m-1)$-dimensional Euclidean sphere of radius s). Let M_1 and M_2 be the connected components of $M \setminus C_0$. We may choose g_s such that $\mathrm{Vol}(M_1, g_s) = \mathrm{Vol}(M_2, g_s)$.

Let us consider a C^∞ function f equal to 1 on M_1, to -1 on M_2 and linear along C_0 (constant on the cross sections of C_0). Then f satisfies the orthogonality condition $\int_M |f|^{p-2} f \nu_{g_s} = 0$ and

$$\lim_{s\to 0} \frac{\int_M |df|^p \nu_{g_s}}{\int_M |f|^p \nu_{g_s}} = 0.$$

The variational characterization for $\lambda_{1,p}(M, g_s)$ implies $\lim_{s\to 0} \lambda_{1,p}(M, g_s) = 0$, i.e.:

Proposition 2.1. *For any $p > 1$,*

$$\inf_{g\in\mathcal{M}(M)} \lambda_{1,p}(M, g) = 0\,.$$

Our main result is addressing the nonexistence of universal upper bounds for $\lambda_{1,p}$ on $\mathcal{M}(M)$:

Theorem 2.2. *For any $p > 1$, every compact manifold M of dimension $m \geq 3$ carries Riemannian metrics of volume one with $\lambda_{1,p}$ arbitrarily large.*

In the case $p = 2$ this Theorem is due to Colbois and Dodziuk [4].

The proof of Theorem 2.2 is divided into two parts: for $1 < p < 2$ the path we follow is reminiscent of the case $p = 2$; however, new ideas are required since the arguments from the linear case do not apply when $p \neq 2$. In the case $p > 2$, the proof follows directly from a monotonicity result.

Proof. *I. The case $1 < p \leq 2$*

The main part of the proof consists in constructing a domain $D \subset S^m$ (S^m denotes the m-dimensional unit Euclidean sphere) and a family of Riemannian metrics $(g_n)_{n\in\mathbb{N}}$ on S^m such that

$$\lim_{n\to\infty} \lambda^N_{1,p}(D, g_n) > 0\,, \ \lim_{n\to\infty} \lambda_{1,p}(S^m, g_n) > 0 \ \text{ and } \ \lim_{n\to\infty} \mathrm{Vol}(D, g_n) = \infty\,. \tag{2.1}$$

We do this first on S^3 and then by following a path similar to [5] we show how to construct such domains and Riemannian metrics inductively on S^m, $m > 3$ (can and $\widetilde{can}$ denote the canonical metrics on S^3, respectively $C\mathbb{P}^1$). For any $x \in S^3$, denote by $V_x = Ker(d\pi_x)$ and by H_x the space orthogonal to V_x in T_xS^3 w.r.t. can_x. Let $\{g_t\}_{t>0}$ be the *Berger metrics* defined by

$$g_{t\,|V_x\times H_x} = 0, \quad g_{t\,|V_x\times V_x} = t^2 can_{|V_x\times V_x}, \quad g_{t\,|H_x\times H_x} = can_{|H_x\times H_x}\,. \tag{2.2}$$

The map π is a Riemannian submersion from (S^3, g_t) to $(C\mathbb{P}^1, \widetilde{can})$ with totally geodesic fibers $F_x = \pi^{-1}(\pi(x))$ isometric to $(S^1, t^2 can)$.

Lemma 2.3. *Let* $A = \{(x_1, x_2, x_3, x_4) \in S^3 \mid x_3^2 + x_4^2 < \frac{1}{4}\}$*. Then for any* $1 < p \leq 2$*,*

$$\lambda_{1,p}^N(A, g_t) \geq \lambda_{1,p}(S^3, g_t) \geq 2^{-p}\mu_{1,p}(A, g_t).$$

Proof of Lemma 2.3. Fix t and let $a = \{x \in S^3 \mid x = (x_1, x_2, 0, 0)\}$, $b = \{x \in S^3 \mid x = (0, 0, x_3, x_4)\}$ and $\mathcal{S} = S^3 \setminus \{a \cup b\}$. Since a and b have codimension 2 in S^3, [1, Lemma 1] implies that $C_0^\infty(\mathcal{S})$ is dense in $W^{1,2}(S^3)$ and therefore in $W^{1,p}(S^3)$ for $1 < p \leq 2$. We can then replace $W^{1,p}(S^3)$ by $C_0^\infty(\mathcal{S})$ in the variational characterization of $\lambda_{1,p}(S^3, g_t)$.

Parameterize $\mathcal{S}$ by

$$\Phi : (0,1) \times [0, 2\pi) \times [0, 2\pi) \to \mathcal{S},$$

$$\Phi(r, \theta_1, \theta_2) = (\sqrt{1-r^2}\cos\theta_1, \sqrt{1-r^2}\sin\theta_1, r\cos\theta_2, r\sin\theta_2)\,.$$

It is immediate that $\Phi(r, \theta_1, \theta_2) \in A \setminus a \;\Leftrightarrow\; r \in (0, \frac{1}{2})$. Consider the diffeomorphism $\tau : A \setminus a \to \mathcal{S}$ defined by

$$\tau(\Phi(r, \theta_1, \theta_2)) = \Phi(2r, \theta_1, \theta_2)\,. \tag{2.3}$$

Let $f \in C_0^\infty(\mathcal{S}) \setminus \{0\}$ with $\int_S |f|^{p-2} f\, \nu_{g_t} = 0$. The function $f \circ \tau \in W_0^{1,p}(A \setminus a) \subset W_0^{1,p}(A)$ and the variational characterization of $\mu_{1,p}(A, g_t)$ yields

$$\mu_{1,p}(A, g_t) \leq \frac{\int_{A\setminus a} |d(f\circ\tau)|_{g_t}^p\, \nu_{g_t}}{\int_{A\setminus a} |f\circ\tau|^p\, \nu_{g_t}}\,, \tag{2.4}$$

where $|\cdot|_{g_t}$ denotes the Hilbert-Schmidt norm w.r.t. g_t. To evaluate the right side of inequality (2.4) note that for any $x \in \mathcal{S}$, $x = \Phi(r, \theta_1, \theta_2)$, the vectors

$$v = d\Phi\,(0,1,1) = \left(-\sqrt{1-r^2}\sin\theta_1, \sqrt{1-r^2}\cos\theta_1, -r\sin\theta_2, r\cos\theta_2\right),$$

$$u_1 = d\Phi(0, -\tfrac{r}{\sqrt{1-r^2}}, \tfrac{\sqrt{1-r^2}}{r}) = \left(r\sin\theta_1, -r\cos\theta_1, -\sqrt{1-r^2}\sin\theta_2, \sqrt{1-r^2}\cos\theta_2\right),$$

$$u_2 = d\Phi(\sqrt{1-r^2}, 0, 0) = \left(-r\cos\theta_1, -r\sin\theta_1, \sqrt{1-r^2}\cos\theta_2, \sqrt{1-r^2}\sin\theta_2\right)$$

form an orthonormal basis of T_xS^3 w.r.t. can_x. Moreover $v = \frac{d\mathbf{c}}{ds}\,|_{s=0}$ where $\mathbf{c} : [0, 2\pi) \to F_x$, $\mathbf{c}(s) = \Phi(r, \theta_1 + s, \theta_2 + s)$. It is easy to see that $\pi \circ \mathbf{c}(s) = \pi(x)$ for all s and therefore $v \in Ker(d\pi_x) = V_x$. Hence, $\{t^{-1}v, u_1, u_2\}$ is an orthonormal basis of T_xS^3 w.r.t. g_t, for any $t > 0$. On the other hand, (2.3) implies

$$d\tau(v) = v\,, \quad d\tau(u_1) = u_1\,, \quad d\tau(u_2) = 2u_2. \tag{2.5}$$

We have

$$\begin{aligned} |d(f\circ\tau)|_{g_t}^p &= \left(|d(f\circ\tau)(t^{-1}v)|^2 + |d(f\circ\tau)(u_1)|^2 + |d(f\circ\tau)(u_2)|^2\right)^{\frac{p}{2}} \\ &= \left(|df(t^{-1}v)|^2 + |df(u_1)|^2 + |df(2u_2)|^2\right)^{\frac{p}{2}} \circ \tau \\ &\leq 2^p |df|_{g_t}^p \circ \tau\,. \end{aligned}$$

Integrating over $A \setminus a$ we obtain

$$\begin{aligned}\int_{A\setminus a} |d(f\circ\tau)|^p_{g_t}\nu_{g_t} &\leq 2^p \int_{A\setminus a} |df|^p_{g_t} \circ \tau\, \nu_{g_t} \\ &= 2^{p-1} \int_{A\setminus a} |df|^p_{g_t} \circ \tau\, \nu_{\tau^* g_t} \\ &= 2^{p-1} \int_{\mathcal{S}} |df|^p_{g_t}\, \nu_{g_t}\end{aligned} \tag{2.6}$$

(where in the first equality we have used $\nu_{\tau^* g_t} = 2\nu_{g_t}$, which follows immediately from (2.5)). Similarly

$$\int_{A\setminus a} |f\circ\tau|^p \nu_{g_t} = \frac{1}{2}\int_{\mathcal{S}} |f|^p\, \nu_{g_t}\,. \tag{2.7}$$

Equations (2.4), (2.6) and (2.7) imply that

$$\mu_{1,p}(A, g_t) \leq 2^p \frac{\int_{\mathcal{S}} |df|^p\, \nu_{g_t}}{\int_{\mathcal{S}} |f|^p\, \nu_{g_t}}\,.$$

Taking now the infimum over all $f \in C_0^\infty(\mathcal{S}) \setminus \{0\}$ with $\int_{\mathcal{S}} |f|^{p-2} f \nu_{g_t} = 0$ we obtain

$$\mu_{1,p}(A, g_t) \leq 2^p \lambda_{1,p}(S^3, g_t)\,.$$

To prove the left inequality in Lemma 2.3 note that since $C^\infty(\overline{A})$ is dense in $W^{1,p}(A)$, by the same argument as above we obtain that $C_0^\infty(\overline{A} \setminus a)$ is dense in $W^{1,p}(A)$. We can thus replace $W^{1,p}(A)$ by $C_0^\infty(\overline{A}\setminus a)$ in the variational characterization for $\lambda^N_{1,p}(A, g_t)$. Let $f \in C_0^\infty(\overline{A}\setminus a) \setminus \{0\}$ with $\int_{A\setminus a} |f|^{p-2} f\, \nu_{g_t} = 0$. Let $B = \{(x_1, x_2, x_3, x_4) \in S^3 \mid x_3^2 + x_4^2 > \frac{1}{4}\}$ and consider the diffeomorphism $\Psi : \mathcal{S} \to \mathcal{S}$ given by

$$\Psi(\Phi(r, \theta_1, \theta_2)) = \Phi(1 - r, \theta_1, \theta_2). \tag{2.8}$$

Extend f into a function $F \in W_0^{1,p}(\mathcal{S})$ defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in \overline{A} \setminus a, \\ f \circ \Psi(x) & \text{if } x \in B \setminus b. \end{cases}$$

Equation (2.8) implies $d\Psi(v) = v$, $d\Psi(u_1) = u_1$ and $d\Psi(u_2) = -u_2$. Hence, Ψ is an isometry of $(\mathcal{S}, can)$; in fact, since v is invariant by $d\Psi$, Ψ is an isometry of $(\mathcal{S}, g_t)$, for any $t > 0$. In particular, given that $\Psi(A \setminus a) = B \setminus b$, we have

$$\int_{\mathcal{S}} |dF|^p_{g_t}\, \nu_{g_t} = 2\int_{A\setminus a} |df|^p_{g_t}\, \nu_{g_t}, \qquad \int_{\mathcal{S}} |F|^p\, \nu_{g_t} = 2\int_{A\setminus a} |f|^p\, \nu_{g_t},$$
$$\int_{\mathcal{S}} |F|^{p-2} F\, \nu_{g_t} = 2\int_{A\setminus a} |f|^{p-2} f\, \nu_{g_t} = 0.$$

The variational characterization for $\lambda_{1,p}(S^3, g_t)$ yields

$$\lambda_{1,p}(S^3, g_t) \leq \frac{\int_{\mathcal{S}} |dF|^p_{g_t} \nu_{g_t}}{\int_{\mathcal{S}} |F|^p \nu_{g_t}} = \frac{\int_{A\setminus a} |df|^p_{g_t} \nu_{g_t}}{\int_{A\setminus a} |f|^p \nu_{g_t}}\,.$$

Taking now the infimum over all $f \in C_0^\infty(\overline{A}\setminus a) \setminus \{0\}$ with $\int_A |f|^{p-2} f\, \nu_{g_t} = 0$ we obtain $\lambda_{1,p}(S^3, g_t) \leq \lambda^N_{1,p}(A, g_t)$. This ends the proof of Lemma 2.3. $\square$

Lemma 2.4. *Let A be as in Lemma* 2.3. *Then for any* $1 < p \leq 2$,

$$\mu_{1,p}(A, g_t) \geq 2^{\frac{p}{2}-1} \lambda_{1,p}(C\mathbb{P}^1, \widetilde{can}) .$$

Proof of Lemma 2.4. Fix t and let $f \in W_0^{1,p}(A)$ be an eigenfunction for $\mu_{1,p}(A, g_t)$. Then f has no zeros in A, and we may assume that $f > 0$ in A. It is easy to see that for any $x \in A$, $F_x \subset A$. We can construct therefore a positive function $\tilde{f} : \pi(A) \subset C\mathbb{P}^1 \to \mathbb{R}$ by setting

$$\tilde{f}(\pi(x))^p = \int_{F_x} f^p \nu_{g_t} . \tag{2.9}$$

Let u_1, u_2 be two vector fields such that for all $x \in A$, $\{u_1, u_2\}$ is an orthonormal basis of H_x w.r.t. g_t. Then $\{\tilde{u}_1 = d\pi(u_1), \tilde{u}_2 = d\pi(u_2)\}$ is an orthonormal basis of $T_{\pi(x)}C\mathbb{P}^1$ w.r.t. $\widetilde{can}$. Given that the fibers F_x are totally geodesic, we can apply Lemma 3.14 in [2] to the function f^p and the vector field u_1 and obtain

$$p\tilde{f}^{p-1} d\tilde{f}(\tilde{u}_1) = p \int_{F_x} f^{p-1} df(u_1)\, \nu_{g_t} .$$

The Hölder inequality yields

$$\begin{aligned}
\tilde{f}^{p-1} |d\tilde{f}(\tilde{u}_1)| &\leq \textstyle\int_{F_x} f^{p-1} |df(u_1)| \nu_{g_t} \\
&\leq \left(\textstyle\int_{F_x} f^p \nu_{g_t} \right)^{\frac{p-1}{p}} \left(\textstyle\int_{F_x} |df(u_1)|^p \nu_{g_t} \right)^{\frac{1}{p}} \\
&= \tilde{f}^{p-1} \left(\textstyle\int_{F_x} |df(u_1)|^p \nu_{g_t} \right)^{\frac{1}{p}} .
\end{aligned}$$

Since $\tilde{f} > 0$, it follows that $|d\tilde{f}(\tilde{u}_1)|^p \leq \int_{F_x} |df(u_1)|^p \nu_{g_t}$. In the same way we obtain $|d\tilde{f}(\tilde{u}_2)|^p \leq \int_{F_x} |df(u_2)|^p \nu_{g_t}$. Hence

$$\begin{aligned}
|d\tilde{f}|^p_{\widetilde{can}} &= \left(|d\tilde{f}(\tilde{u}_1)|^2 + |d\tilde{f}(\tilde{u}_2)|^2 \right)^{\frac{p}{2}} \leq |d\tilde{f}(\tilde{u}_1)|^p + |d\tilde{f}(\tilde{u}_2)|^p \\
&\leq \textstyle\int_{F_x} |df(u_1)|^p + |df(u_2)|^p \nu_{can} \\
&\leq 2^{1-\frac{p}{2}} \textstyle\int_{F_x} \left(|df(u_1)|^2 + |df(u_2)|^2 \right)^{\frac{p}{2}} \nu_{can} \\
&\leq 2^{1-\frac{p}{2}} \textstyle\int_{F_x} |df|^p_{g_t} \nu_{can}
\end{aligned} \tag{2.10}$$

(where we have used the inequality: $(a+b)^{\frac{p}{2}} \leq a^{\frac{p}{2}} + b^{\frac{p}{2}} \leq 2^{1-\frac{p}{2}}(a+b)^{\frac{p}{2}}$ for any $a, b > 0$ and $1 < p \leq 2$). Integrating (2.9) and (2.10) over $\pi(A)$ we obtain

$$\int_{\pi(A)} |\tilde{f}|^p \nu_{\widetilde{can}} = \int_A |f|^p \nu_{g_t} \text{ and } \int_{\pi(A)} |d\tilde{f}|^p \nu_{\widetilde{can}} \leq 2^{1-\frac{p}{2}} \int_A |df|^p_{g_t} \nu_{g_t} . \tag{2.11}$$

In particular, (2.11) implies that $\tilde{f} \in W^{1,p}(\pi(A))$. In fact, since $f \in W_0^{1,p}(A)$ and $F_x \subset \partial A$ for any $x \in \partial A$, it is immediate from (2.9) that $\tilde{f} \in W_0^{1,p}(\pi(A))$. The variational characterization for $\mu_{1,p}(\pi(A), \widetilde{can})$ and (2.11) imply then

$$\mu_{1,p}(\pi(A), \widetilde{can}) \leq \frac{\int_{\pi(A)} |d\tilde{f}|^p \nu_{\widetilde{can}}}{\int_{\pi(A)} |\tilde{f}|^p \nu_{\widetilde{can}}} \leq 2^{1-\frac{p}{2}} \frac{\int_A |df|^p_{g_t} \nu_{g_t}}{\int_A |f|^p \nu_{g_t}} = 2^{1-\frac{p}{2}} \mu_{1,p}(A, g_t) . \tag{2.12}$$

In the same way we can prove that $\mu_{1,p}(\pi(B), \widetilde{can}) \leq 2^{1-\frac{p}{2}}\mu_{1,p}(B, g_t)$, where B is as in Lemma 2.3. The map Ψ is an isometry between $A \setminus a$ and $B \setminus b$ and therefore it is straightforward that $\mu_{1,p}(A, g_t) = \mu_{1,p}(B, g_t)$. Since the open sets $\pi(A)$ and $\pi(B)$ are disjoint, (1.1) and (2.12) imply

$$\lambda_{1,p}(C\mathbb{P}^1, \widetilde{can}) \leq \max\{\mu_{1,p}(\pi(A), \widetilde{can}), \mu_{1,p}(\pi(B), \widetilde{can}\} \leq 2^{1-\frac{p}{2}}\mu_{1,p}(A, g_t)\,.$$

□

Combining Lemma 2.3 and Lemma 2.4 we obtain

$$\lambda^N_{1,p}(A, g_t) \geq \lambda_{1,p}(S^3, g_t) \geq 2^{-\frac{p}{2}-1}\lambda_{1,p}(C\mathbb{P}^1, \widetilde{can})\,.$$

Finally, since $\nu_{g_t} = t\nu_g$ we have $\mathrm{Vol}(A, g_t) \to \infty$ as $t \to \infty$. In particular, we can find a sequence $t_n \to \infty$ such that the metrics $g_n := g_{t_n}$ and the domain A satisfy (2.1) on S^3.

We will now construct a domain $D \subset S^{m+1}$ and a family of metrics on S^{m+1} with the property (2.1), given such domain and metrics on S^m. To do this we follow the general idea from [5].

Let $D \subset S^m$ and $(g_n)_{n\in\mathbb{N}}$ with property (2.1). (In fact we need only the last two assumptions in (2.1) for the construction on S^{m+1}.) Let $C = S^m \times (0,1)$ endowed with the metric $h_n = g_n \times dr^2$ where dr^2 denotes the Euclidean metric on $(0,1)$. Obviously $\mathrm{Vol}(C, h_n) = \mathrm{Vol}(S^m, g_n) > \mathrm{Vol}(D, g_n)$. Let f_n be an eigenfunction for $\lambda^N_{1,p}(C, h_n)$ and denote by f_n^+, f_n^- its positive, respectively, its negative part. In the same way as in [8, Lemma 3], it is straightforward that

$$\lambda^N_{1,p}(C, h_n) = \frac{\int_C |df_n|^p_{h_n}\,\nu_{h_n}}{\int_C |f_n|^p\,\nu_{h_n}} = \frac{\int_C |df_n^+|^p_{h_n}\,\nu_{h_n}}{\int_C |f_n^+|^p\,\nu_{h_n}} = \frac{\int_C |df_n^-|^p_{h_n}\,\nu_{h_n}}{\int_C |f_n^-|^p\,\nu_{h_n}}.$$

After multiplying f_n^+ and f_n^- by constants we may assume that $\int_C |f_n^+|^p\,\nu_{h_n} = \int_C |f_n^-|^p\,\nu_{h_n} = 1$. Then

$$\lambda^N_{1,p}(C, h_n) = \int_C |df_n^+|^p_{h_n}\,\nu_{h_n} = \int_C |df_n^-|^p_{h_n}\,\nu_{h_n}\,. \tag{2.13}$$

Consider the positive functions $\tilde{f}_n^+(r)$ and $\tilde{f}_n^-(r)$ defined by

$$\tilde{f}_n^+(r)^p = \int_{S^m} |f_n^+|^p(\cdot, r)\,\nu_{g_n} \quad \text{and} \quad \tilde{f}_n^-(r)^p = \int_{S^m} |f_n^-|^p(\cdot, r)\,\nu_{g_n}\,. \tag{2.14}$$

As in the proof of Lemma 2.4, we can apply Lemma 3.14 of [2] to the function $(\tilde{f}_n^+)^p$ and the vector field $\frac{\partial}{\partial r}$ and obtain

$$|(\tilde{f}_n^+)'|^p \leq \int_{S^m} |df_n^+\left(\frac{\partial}{\partial r}\right)|^p \nu_{g_n}\,. \tag{2.15}$$

Integrating (2.14) and (2.15) we get $\int_0^1 |\tilde{f}_n^+|^p\,dr = \int_C |f_n^+|^p\,\nu_{h_n} = 1$ and

$$\int_0^1 |(\tilde{f}_n^+)'|^p\,dr \leq \int_C |df_n^+\left(\frac{\partial}{\partial r}\right)|^p \nu_{h_n} \leq \int_C |df_n^+|^p_{h_n}\nu_{h_n} = \lambda^N_{1,p}(C, h_n)\,. \tag{2.16}$$

Suppose now that $\lim_{n\to\infty} \lambda^N_{1,p}(C, h_n) = 0$. Then (2.16) implies that the sequence $(\tilde{f}_n^+)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(0,1)$. After extracting a subsequence, we may assume that there exists a function $\tilde{f}_0^+ \in W^{1,p}(0,1)$ such that $\tilde{f}_n^+$ converges to $\tilde{f}_0^+$ weakly in $W^{1,p}(0,1)$ and strongly in $L^p(0,1)$. The weak convergence and (2.16) yield

$$\int_0^1 |(\tilde{f}_0^+)'|^p\, dr \le \liminf_{n\to\infty} \int_0^1 |(\tilde{f}_n^+)'|^p\, dr \le \lim_{n\to\infty} \lambda_{1,p}^N(C,h_n) = 0 \quad \Rightarrow \quad \tilde{f}_0^+ \text{ is constant}\,.$$

The strong convergence implies that $\int_0^1 |\tilde{f}_0^+|^p\, dr = \lim_{n\to\infty} \int_0^1 |\tilde{f}_n^+|^p\, dr = 1$. Since $\tilde{f}_0^+$ is constant, we have $\tilde{f}_0^+ \equiv 1$, i.e. $\tilde{f}_n^+ \to 1$ in $L^p(0,1)$. Similarly, after extracting a subsequence again, we obtain $\tilde{f}_n^- \to 1$ in $L^p(0,1)$. In particular,

$$\lim_{n\to\infty} \int_{S^m} |f_n^+|^p(\cdot,r)\,\nu_{g_n} = \lim_{n\to\infty} \int_{S^m} |f_n^-|^p(\cdot,r)\,\nu_{g_n} = 1 \quad \text{for almost all } r\,. \tag{2.17}$$

On the other hand, (2.13) implies

$$\lambda_{1,p}^N(C,h_n) = \int_0^1 \int_{S^m} |df_n^+|_{h_n}^p\,\nu_{g_n}\,dr \ge \int_0^1 \int_{S^m} |df_n^+|_{g_n}^p\,\nu_{g_n}\,dr\,,$$

and a similar inequality for f_n^-. Hence

$$\lim_{n\to\infty} \int_{S^m} |df_n^+|_{g_n}^p(\cdot,r)\,\nu_{g_n} = \int_{S^m} |df_n^-|_{g_n}^p(\cdot,r)\,\nu_{g_n} = 0 \quad \text{for almost all } r\,. \tag{2.18}$$

Combining (2.17) and (2.18) we obtain

$$\lim_{n\to\infty} \frac{\int_{S^m} |df^+|_{g_n}^p(\cdot,r)\,\nu_{g_n}}{\int_{S^m} |f^+|^p(\cdot,r)\,\nu_{g_n}} = \lim_{n\to\infty} \frac{\int_{S^m} |df^-|_{g_n}^p(\cdot,r)\,\nu_{g_n}}{\int_{S^m} |f^-|^p(\cdot,r)\,\nu_{g_n}} = 0 \;\text{ for almost all } r\,. \tag{2.19}$$

Fix r and denote by ω_n^+, ω_n^- the (disjoint) supports in S^m of $f_n^+(\cdot,r)$ and $f_n^-(\cdot,r)$ respectively. The variational characterization for $\mu_{1,p}(\omega_r^+,g_n)$ and $\mu_{1,p}(\omega_r^-,g_n)$ together with (1.1) implies

$$\begin{aligned}
&\max\left\{ \frac{\int_{S^m} |df_n^+|_{g_n}^p(\cdot,r)\,\nu_{g_n}}{\int_{S^m} |f_n^+|^p(\cdot,r)\,\nu_{g_n}}\,,\ \frac{\int_{S^m} |df_n^-|_{g_n}^p(\cdot,r)\,\nu_{g_n}}{\int_{S^m} |f^-|^p(\cdot,r)\,\nu_{g_n}} \right\}\\
&\ge \max\{\mu_{1,p}(\omega_n^+,g_n), \mu_{1,p}(\omega_n^-,g_n)\}\\
&\ge \lambda_{1,p}(S^m,g_n)\,.
\end{aligned}$$

Equation (2.19) implies then that $\lim_{n\to\infty} \lambda_{1,p}(S^m,g_n) = 0$. This contradicts our choice of g_n. It follows that $\lim_{n\to\infty} \lambda_{1,p}^N(C,h_n) \ne 0$ and therefore (after extracting a subsequence) we may assume $\lim_{n\to\infty} \lambda_{1,p}^N(C,h_n) > 0$.

To conclude our construction on S^{m+1} we need the first two assertions of the following technical result [9]:

Lemma 2.5. *Let (M,g) be a compact Riemannian manifold, $p>1$ and $\Omega \subset M$ a domain with Lipschitzian boundary. Then for any $\delta > 0$, there exists a metric $\tilde{g}$ on M such that:*

- $g_{|\Omega} = \tilde{g}_{|\Omega}$,
- $\lambda_{1,p}(M,\tilde{g}) > \lambda_{1,p}^N(\Omega,g) - \delta$,
- $|\mathrm{Vol}(M,\tilde{g}) - \mathrm{Vol}(\Omega,g)| < \delta$.

We can now glue two caps to C to obtain a manifold diffeomorphic to S^{m+1}. Let $2\delta = \lim_{n\to\infty} \lambda_{1,p}^N(C,h_n) > 0$. Lemma 2.5 implies that we can extend h_n into a metric $\tilde{h}_n$ on the whole of S^{m+1} such that $\lambda_{1,p}(S^{m+1},\tilde{h}_n) > \lambda_{1,p}^N(C,h_n) - \delta$. Then

C and (after extracting a subsequence) the family $(\tilde{h}_n)_{n\in\mathbb{N}}$ will satisfy the property (2.1) on S^{m+1}.

Now let M be an arbitrarily compact manifold of dimension $m \geq 3$. Consider a domain $D \subset S^m$ and a family of Riemannian metrics $(g_n)_{n\in\mathbb{N}}$ on S^m with the property (2.1). Let $x_0 \in S^m$ such that $x_0 \notin D$ and let U be a small open set in M diffeomorphic to $S^m \setminus \{x_0\}$ (since $S^m \setminus \{x_0\}$ is diffeomorphic to a ball in $\mathbb{R}^m$, it suffices to take U diffeomorphic to a ball). We can pull back the metric g_n via this diffeomorphism into a metric $\bar{g}_n$ on U. Then (D, g_n) is isometric to $(\Omega, \bar{g}_n)$, where Ω is some subset of U. We have

$$\lambda_{1,p}^N(D, g_n) = \lambda_{1,p}^N(\Omega, \bar{g}_n) \text{ and } \mathrm{Vol}(\Omega, \bar{g}_n) = \mathrm{Vol}(D, g_n) \to \infty \text{ as } n \to \infty\,.$$

Now let $2\delta = \lim_{n\to\infty} \lambda_{1,p}^N(\Omega, \bar{g}_n) > 0$. We can perturb each $\bar{g}_n$ outside Ω as in Lemma 2.5 and obtain a family of metrics $(\tilde{g}_n)_{n\in\mathbb{N}}$ on M such that $\lambda_{1,p}(M, \tilde{g}_n) > \lambda_{1,p}^N(\Omega, \bar{g}_n) - \delta$. Since $\mathrm{Vol}(M, \tilde{g}_n) > \mathrm{Vol}(\Omega, \bar{g}_n)$ we have $\lim_{n\to\infty} \mathrm{Vol}(M, \tilde{g}_n) = \infty$. Now let $\widehat{g}_n = \mathrm{Vol}(M, \tilde{g}_n)^{-\frac{2}{m}} \tilde{g}_n$. Then $\widehat{g}_n \in \mathcal{M}(M)$ and

$$\lambda_{1,p}(M, \widehat{g}_n) = \mathrm{Vol}(M, \tilde{g}_n)^{\frac{p}{m}} \lambda_{1,p}(M, \tilde{g}_n) \to \infty \text{ as } n \to \infty\,.$$

(If $M = S^m$, it suffices to construct the metrics $\widehat{g}_n = \mathrm{Vol}(S^m, g_n)^{-\frac{2}{m}} g_n \in \mathcal{M}(S^m)$ to obtain $\lim_{n\to\infty} \lambda_{1,p}(S^m, \widehat{g}_n) = \infty$.) This ends the proof of the case $1 < p \leq 2$.

II. The case $p > 2$

In this case the proof follows directly from the following monotonicity result:

Proposition 2.6. *Let (M, g) be a compact Riemannian manifold. Then the map*

$$p \mapsto p\,\lambda_{1,p}(M, g)^{\frac{1}{p}}$$

is strictly increasing on $(1, \infty)$.

Proof of Proposition 2.6. Let $1 < p < q$ and A_1, A_2 be the nodal domains of an eigenfunction for $\lambda_{1,q}(M, g)$. Then $\lambda_{1,q}(M, g) = \mu_{1,q}(A_1, g) = \mu_{1,q}(A_2, g)$ [8].

For any bounded domain Ω of M, Lindqvist [7] proved that the map $p \mapsto p\,\mu_{1,p}(\Omega)^{\frac{1}{p}}$ is strictly increasing on $]1, \infty[$. We then have

$$p\,\mu_{1,p}(A_1, g)^{\frac{1}{p}} < q\,\mu_{1,q}(A_1, g)^{\frac{1}{q}} \text{ and } p\,\mu_{1,p}(A_2, g)^{\frac{1}{p}} < q\,\mu_{1,q}(A_2, g)^{\frac{1}{q}}\,.$$

On the other hand, (1.1) implies $\lambda_{1,p}(M, g) \leq \max\{\mu_{1,p}(A_1, g), \mu_{1,p}(A_2, g)\}$. Hence

$$\begin{aligned} p\lambda_{1,p}(M, g)^{\frac{1}{p}} &\leq \max\left\{p\mu_{1,p}(A_1, g)^{\frac{1}{p}}, p\mu_{1,p}(A_2, g)^{\frac{1}{p}}\right\} \\ &< \max\left\{q\mu_{1,q}(A_1, g)^{\frac{1}{q}}, q\mu_{1,q}(A_2, g)^{\frac{1}{q}}\right\} \\ &= q\,\mu_{1,q}(A_1, g)^{\frac{1}{q}} \\ &= q\,\lambda_{1,q}(M, g)^{\frac{1}{q}}\,. \end{aligned}$$

□

Finally, Proposition 2.6 implies that for any $p > 2$,

$$\lambda_{1,p}(M, g) > \left(\frac{2}{p}\right)^p (\lambda_{1,2}(M, g))^{\frac{p}{2}}\,.$$

Since the functional $g \to \lambda_{1,2}(M, g)$ is not bounded on $\mathcal{M}(M)$ it follows that for any $p > 2$ the functional $g \to \lambda_{1,p}(M, g)$ is also not bounded on $\mathcal{M}(M)$. This ends the proof of Theorem 2.2. □

Remark 2.7. Proposition 2.6 provides a simple method for obtaining lower bounds for $\lambda_{1,p}(M,g)$, $p>2$, in terms of other geometric invariants. For example, it is known that $\lambda_{1,2}(M,g) > \frac{h^2_{(M,g)}}{4}$ where $h_{(M,g)}$ is the Cheeger isoperimetric constant of (M,g). We then have

$$\lambda_{1,p}(M,g) > \left(\frac{2}{p}\right)^p (\lambda_{1,2}(M,g))^{\frac{p}{2}} > \left(\frac{h_{(M,g)}}{p}\right)^p \quad \text{for any} \quad p>2.$$

Note that in [8] we proved that $\lambda_{1,p}(M,g) > \left(\frac{h_{(M,g)}}{p}\right)^p$ for any $p>1$.

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Department of Mathematics and Computer Science, Loyola University New Orleans, 6363 St. Charles Avenue, New Orleans, Louisiana 70118

E-mail address: amatei@loyno.edu