A NOTE ON ADJOINT LINEAR SYSTEMS

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Abstract. In this note, we give a weak estimate on the separation of tangent directions of the conjecture of Fujita for adjoint linear systems on smooth varieties.

1. Introduction

Let $X$ be a smooth projective variety of dimension $n$. For a Cartier divisor $A$ and a point $p \in X$, we define

\[ \text{Bv}(A, p) = \bigcap_{D \in \mid A - p \mid} \mathbb{P}(T_p D), \]

where $|A - p|$ is the sublinear system of $|A|$ whose members are divisors in $|A|$ passing through $p$ and $\mathbb{P}(T_p D)$ is the projectivised tangent space of $D$ at $p$ as a subspace of the projectivised tangent space $\mathbb{P}(T_p X)$ of $X$ at $p$. The conjecture of Fujita [3] asserts that, for every ample divisor $H$ on $X$ and every point $p \in X$,

\[ \text{Bv}(K_X + tH, p) = \emptyset \]

if $t \geq n + 2$. This has been known for curves, surfaces [6], and recently it was proved for Fano threefolds in [5] using the classification of Fano threefolds.

The purpose of this note is to prove the following weak estimate for varieties of arbitrary dimensions.

Theorem 1.1. Let $X$ be a smooth projective variety of dimension $n$ and $H$ an ample Cartier divisor on $X$. Then for every point $p$ of $X$,

\[ \dim \text{Bv}(K_X + tH, p) \leq n - 2 \]

if $t \geq n^2 + n + 1$. Equivalently, general members of $|K_X + tH - p|$ are smooth at $p$.

Let us give a sketch of the proof. Let $p$ be a point in $X$. Our aim is to construct a $\mathbb{Q}$-divisor $G$ on $X$ satisfying the following properties. We refer to the next section for the notation used here.

(1) $tH - G$ is an ample $\mathbb{Q}$-divisor on $X$.

(2) The connected component at $p$ of the multiplier ideal scheme for $G$, noted by $Z(G)$, is supported only at $p$ and its length is at least 2.
Then a standard application of the Nadel vanishing theorem implies that a general element of \([K_X + tH - p]\) is smooth at \(p\). However note that we cannot construct \(G\) so that \(Z\) contains a preassigned tangent direction. In the proof, we will need the following induction step.

Suppose we have created a \(\mathbb{Q}\)-divisor \(G\) so that \(Z(G)\) contains a length two subscheme supported at \(p\) and that \(G\) satisfies the following properties: For any rational \(0 < \epsilon << 1\),

1. the multiplier ideal scheme, \(Z((1 - \epsilon)G)\), for \((1 - \epsilon)G\) does not contain any subscheme of length two supported at \(p\);
2. the “difference” of multiplier ideals of \(G\) and \((1 - \epsilon)G\) is given by a positive-dimensional integral subscheme \(Z\) through \(p\).

In this case, we need to consider two separate cases. First, suppose that \(Z((1 - \epsilon)G)\) contains \(p\). We consider a family of \(\mathbb{Q}\)-divisors \(\{B_t\}\) on \(Z\) with a multiplicity at \(x_t\) no less than the dimension of \(Z\) for a general \(t\). Let \(D_t\) be a local \(\mathbb{Q}\)-divisor on \(X\) near \(x_t\) such that \(D_t|_Z = B_t\) and \(\text{ord}_{x_t}D_t \geq \text{ord}_{x_t}B_t\). Then by considering the coefficient of the exceptional divisor of the blowing up of \(X\) at \(x_t\), one can deduce that \(Z((1 - \epsilon)G + D_t)\) contains \(x_t\). Next we take a global lifting \(D_t'\) of \(B_t\) in \(X\). Theorem 2.7 will guarantee that \(Z((1 - \epsilon)G + D_t')\) still contains \(x_t\). Since \(p\) is always contained in this new multiplier ideal by our assumption, it contains a length two subscheme. Now by specializing \(D_t'\) to \(D'\) at \(p\), we can assure that the connected component of \(Z((1 - \epsilon)G + D)\) at \(p\) contains a length two subscheme. If this connected component has dimension zero, then we are done. Otherwise we will have to repeat this process. When \(Z((1 - \epsilon)G)\) does not contains \(p\), the argument is similar, except that in this case we need to make sure that \(Z((1 - \epsilon)G + D_t)\) contains a length two subscheme supported at \(x_t\) by choosing \(B_t\) with a higher multiplicity at \(x_t\).

Notation. We will work over the field of complex numbers.

- \(\sim\) \(\mathbb{Q}\)-linear equivalence.
- \([\cdot]\) round-up.

2. Preliminaries

Here we collect some properties of multiplier ideals which are needed later in the proof. We refer to [1] and [4] for the details.

Definition 2.1. Let \(X\) be a complete variety and let \(W\) a finite subscheme of \(X\). Let \(H\) be a Cartier divisor on \(X\). We say that \(|H|\) separates \(W\) if the following natural restriction is surjective:

\[
H^0(X, \mathcal{O}_X(H)) \longrightarrow H^0(W, \mathcal{O}_W \otimes \mathcal{O}_X(H)) \longrightarrow 0.
\]

Definition 2.2. Let \(X\) be a variety. Let \(G\) be an effective \(\mathbb{Q}\)-Cartier divisor on \(X\).

1. Let \(p\) be a smooth point of \(X\). Let \(f : Y \longrightarrow X\) be the blowing up of \(X\) at \(p\) and let \(E\) be the exceptional divisor over \(p\). We define the order of \(G\) at \(p\), \(\text{ord}_pG\), to be the coefficient of \(E\) in \(f^*G\).
2. Let \(W\) be a length two subscheme of \(X\) consisting of two distinct smooth points \(p\) and \(q\) in \(X\). We define the order of \(G\) at \(W\), \(\text{ord}_W G\), to be the minimum between \(\text{ord}_pG\) and \(\text{ord}_qG\).
(3) Let $W$ be a non-reduced length two subscheme of $X$ supported at a smooth point $p$ of $X$, i.e. $W = \{p, v\}$ for some $v \in \mathbb{P}(T_p X)$. Let $f : U \rightarrow X$ be the blowing up of $X$ at $p$ with the exceptional divisor $E$ and $v \in E$. We define the order of $G$ at $W$, $\text{ord}_W G$, to be $\frac{1}{2} \text{ord}_v f^* G$.

**Lemma 2.3** (Lemma 2.4 in [4]). Let $X$ be a complete variety of dimension $n$, let $L$ be a nef and big $\mathbb{Q}$-Cartier divisor on $X$, and let $W$ be a finite subscheme of $l := \text{length} W \leq 2$ supported on the smooth locus of $X$. For every $\epsilon > 0$, there exists an effective $\mathbb{Q}$-divisor $D$ such that $D \cong L$ and

$$\text{ord}_W \geq \frac{n}{L} - \epsilon.$$  

**Multiplier ideals.** Let $X$ be a smooth variety, $W$ a 0-dimensional closed subscheme of $X$, $G$ an effective $\mathbb{Q}$-Cartier divisor on $X$, and $f : Y \rightarrow X$ a log resolution for $(X, G)$. Since $f_* \mathcal{O}_Y(K_Y/X) = \mathcal{O}_X$,

$$f_* \mathcal{O}_Y([K_Y - f^* (K_X + G)]) \subset \mathcal{O}_X.$$  

We call this ideal sheaf the multiplier ideal for $G$. Let $Z(G)$ be the scheme defined by this ideal and note the multiplier ideal by $\mathcal{I}_{Z(G)}$.

**Remark 2.4.** By a standard method, one can easily check that $\mathcal{I}_{Z(G)}$ is independent of the choice of log resolution.

We say that $G$ is pseudo-critical at $W$ if $W \subset Z(G)$, but $W \not\subset Z(\lambda G)$ for any $\lambda < 1$. Furthermore, we say the pair $(G, f)$, or simply $G$, is critical at $W$ if $G$ is pseudo-critical at $W$, and there is a unique prime divisor $F$ on $Y$ such that

$$[K_Y - f^*(K_X + G)] = [K_Y - f^*(K_X + (1 - \epsilon)G)] - F$$

for all sufficiently small $0 < \epsilon << 1$. We call $F$ the critical component of $G$ and $f(F)$ the critical variety of $G$ at $W$.

**Remark 2.5** (Remark 3.4 in [4]). Suppose $G$ is ample and pseudo-critical at $W$. Then, using the so-called tie-braking technique, one can perturb $G$ a little bit so that the new divisor (together with the same log resolution) is critical at $W$.

**Theorem 2.6** (Theorem 3.7 in [4]). We assume that $X$ is complete. Let $H$ be a Cartier divisor on $X$ such that $H - K_X - G$ is nef and big. Let $\mathcal{I}$ be an ideal sheaf of $\mathcal{O}_X$ satisfying $\mathcal{I}_{Z(G)} \subset \mathcal{I}$ and $\dim \text{Supp} \mathcal{I}/\mathcal{I}_{Z(G)} = 0$. Suppose that $\mathcal{I}_{Z(G)} \subset \mathcal{I}_W$ but $\mathcal{I} \not\subset \mathcal{I}_W$. Then there is $D \in |H|$ such that $\text{length} W = \text{length}(D \cap W) + 1$.

**Theorem 2.7** (Theorem 3.9 in [4]). Suppose $G$ is critical at $W$ with the critical variety $Z$. Let $B$ be a non-zero effective $\mathbb{Q}$-Cartier divisor on $Z$. Let $D_1$ and $D_2$ be two liftings of $B$, i.e. $B = D_i|_Z$ for $i = 1, 2$. If

$$W \subset Z((1 - s)D_1 + (1 - t)G)$$

for all sufficiently small $s$ and $t$, then

$$W \subset Z((1 - s)D_2 + (1 - t)G)$$

for all sufficiently small $s$ and $t$.

**Definition 2.8.** Suppose $G$ is critical at $W$ with the critical variety $Z$. Let $B$ be an effective $\mathbb{Q}$-Cartier divisor on $Z$. An effective $\mathbb{Q}$-Cartier divisor $D$ is said to be a nice lifting of $B$ with respect to $G$ if

$$D|_Z = B \quad \text{and} \quad \text{Supp} J \subsetneq Z.$$
where $J$ is defined by

$$0 \to \mathcal{I}_{Z((1-t)G+D)} \to \mathcal{I}_{Z((1-t)G)} \to J \to 0$$

for $0 < t << 1$.

**Proposition 2.9** (Proposition 3.10 in [H]). Suppose $G$ is critical at $W$ with the critical variety $Z$. Let $B$ be an effective $\mathbb{Q}$-Cartier divisor on $Z$. Let $A$ be a $\mathbb{Q}$-ample Cartier divisor on $X$ such that $B \equiv A|_Z$. Then there is a nice lifting $D$ of $B$ with respect to $G$ such that $D \not\equiv A$.

**Proposition 2.10** (Proposition 3.12 in [H]). Suppose that $G$ is critical at $\{p\}$ with the critical variety $Z$ of dimension $d > 0$. If $D$ is a $\mathbb{Q}$-Cartier divisor on $X$ with $\text{ord}_p D > d$, then $p \in Z((1-\epsilon)G + D)$ for all $0 < \epsilon << 1$.

### 3. Proof of Theorem 3.1

In the spirit of [2] we will prove below a result which is a bit stronger. Theorem 3.1 follows from it easily.

**Theorem 3.1.** Let $X$ be a smooth projective variety of dimension $n$ and let $H$ be a Cartier divisor on $X$. Then for every point $p$ of $X$,

$$\dim \text{Br}(K_X + H, p) \leq n - 2$$

if $H^d \cdot Z \geq \{n^2 + n - 1\}^d$ for every subvariety $Z$ of dimension $d$. Equivalently, general members of $|K_X + H - p|$ are smooth at $p$.

The proof of Theorem 3.1 uses the following two lemmas on the behavior of multiplier ideals in a family.

**Lemma 3.2** (Proposition 2.7 in [H]). Let $X$ be a variety and let $p$ be a smooth point of $X$. Let $T$ be the normalization of an irreducible affine curve containing $p$. Let $q$ be a preimage of $p$ in $T$. Let $\{D_t\}_{t \in T}$ be an algebraic family of $\mathbb{Q}$-Cartier divisors on $X$. Suppose $t \in Z(D_t)$ for general $t$. Then $p \in Z(D_q)$.

**Lemma 3.3.** Let $X$ be a variety and let $p$ be a smooth point of $X$. Let $T$ be the normalization of an irreducible affine curve containing $p$. Let $q$ be a preimage of $p$ in $T$ and let $W$ be the image in $X$ of the length two subscheme of $T$ supported at $q$. Let $\{D_t\}_{t \in T}$ be an algebraic family of $\mathbb{Q}$-Cartier divisors on $X$. If $\{t, p\} \subset Z(D_t)$ for general $t$, then $W \subset Z(D_q)$.

**Proof.** Let $\phi : Y \longrightarrow X \times T$ be a log resolution of $D := \bigcup_D D_t$. Then

$$\phi^{-1}(X \times \{t\}) \longrightarrow X \times \{t\}$$

is a log resolution of $D_t$ and $Z(D) \cap X \times \{t\} = Z(D_t)$ for general $t$. Thus $W \subset Z(D) \cap X \times \{p\}$, and by Proposition 2.1 in [H], $W \subset Z(D_p)$.\qed

**Proof of Theorem 3.1** We fix a positive rational number $0 < \delta < \frac{1}{4n}$ and set $A := \frac{1}{n+2n+1}H$. From Lemma 2.3, we have a $\mathbb{Q}$-divisor $D \equiv (2n+\delta)A$ such that $\text{ord}_p D > 2n \geq n + 1$. Then $Z(\eta D) \subset T^p_\eta$ for some $\eta \leq \frac{n+1}{\text{ord}_p D} < 1$. Let $\eta_1$ be the positive rational number such that $\eta_1 D$ becomes pseudo-critical at $p$. We have $\eta_1 \leq \eta < 1$. By Remark 2.5, we may assume that $\eta_1 D$ is critical at $p$ with the critical variety $Z_1$. There are two possibilities.

If $\dim Z_1 > 0$, then we move to 3.1 with $G = \eta_1 D$, $\lambda = \eta_1(2n + \delta)$, and $Z = Z_1$. 


Suppose that \( \dim Z_1 = 0 \). In this case, we need to increase \( \eta_1 \). Let \( \eta_2 \) be the smallest rational number such that \( Z(\eta_2 D) \) contains a length two subscheme of \( X \). Note that \( \eta_2 \leq \eta < 1 \). Again by Remark 2.5 we may assume that there is at most one positive-dimensional irreducible component of \( Z(\eta_2 D) \) through \( p \). We need to consider two separate cases.

First assume that \( Z(\eta_2 D) \) has a unique positive-dimensional component, say \( Z_2 \), through \( p \). Then \( Z(c\eta_2 D) = \{p\} \) for all \( 0 < c < 1 \) near \( p \) and \( \eta_2 D \) is critical at every length two subscheme \( W \) in \( Z_2 \) with \( \text{Supp} W = p \). Now we proceed to 3.2 with \( G = \eta_2 D, \lambda = \eta_2(2n + \delta) \), and \( Z = Z_2 \).

If \( Z(\eta_2 D) \) does not have a positive-dimensional irreducible component through \( p \), then we go to 3.3 with \( G = \eta_2 D \) and \( \lambda = \eta_2(2n + \delta) < 2n + 1 \).

3.1. Induction step: case 1. Let \( G \) be a \( \mathbb{Q} \)-divisor on \( X \) such that

1. \( G \subseteq \lambda A \) for a rational number \( 0 < \lambda \),
2. \( G \) is critical at \( p \),
3. \( Z \), the critical variety of \( G \), has a positive dimension \( d > 0 \).

Let \( T \) be the normalization of an irreducible affine curve containing \( p \). Let \( q \) be a preimage of \( p \) in \( T \) and let \( W \) be the image in \( X \) of the length two subscheme of \( T \) supported at \( q \). By applying Lemma 2.3 over the function field of \( T \), we obtain an algebraic family of \( \mathbb{Q} \)-Cartier divisors \( \{B_t\}_{t \in T} \) on \( Z \) such that \( B_t \supseteq (d + \delta)A \) and, for a general \( t \), \( \text{ord}_t B_t > d \). By Proposition 2.9, there exists a nice lifting \( \tilde{D}_q \) of \( 2B_q \) with respect to \( G \). We put \( D_q = \frac{1}{\lambda}\tilde{D}_q \). We can lift \( D_q \) to an algebraic family \( \{D_t\} \) of \( \mathbb{Q} \)-divisors so that \( D_t|_Z = B_t \) and \( D_t \supseteq (d + \delta)A \). For a general \( t \) we choose a local lifting \( D_t \) of \( B_t \) with \( \text{ord}_t D_t \geq \text{ord}_t B_t \). From Proposition 2.10 we have \( t \in Z(cG + D_t) \) for a general \( t \) and a positive rational number \( 0 < c < 1 \). Since \( G \) is critical at a general point \( t \) of \( T \), \( t \in Z(cG + D_t) \) by Theorem 2.4. Thus \( p \in Z(cG + D_q) \) by Lemma 3.2. Let \( \eta_1 \) be the constant such that \( cG + \eta_1 D_q \) becomes pseudo-critical at \( p \). Then \( \eta_1 \leq 1 \). By Remark 2.5 we may assume that \( cG + \eta_1 D_q \) is critical at \( p \) with the critical variety \( Z_1 \). By construction, we know that \( Z_1 \) is a proper subvariety of \( Z \).

Suppose that \( \dim Z_1 > 0 \). Then we go back to the beginning of 3.1 with \( G = cG + \eta_1 D_q, \lambda = c\lambda + \eta_1(d + \delta) \), and \( Z = Z_1 \).

Suppose that \( \dim Z_1 = 0 \). Since \( \{p, t\} \subset Z(cG + \eta_1 D_q + D_t) \) by Proposition 2.10, \( W \subset Z(cG + \eta_1 D_q + D_q) \) by Lemma 3.3. Let \( \eta_2 \) be the smallest rational number such that \( Z(cG + \eta_1 D_q + \eta_2 D_q) \) contains a length two subscheme of \( X \). Then \( \eta_2 \leq 1 \). By Remark 2.3 we may assume that there is at most one positive-dimensional irreducible component of \( Z(cG + \eta_1 D_q + \eta_2 D_q) \) supported at \( p \). If there exists such a component, say \( Z_2 \), then we go to 3.2 with \( G = cG + \eta_1 D_q + \eta_2 D_q, \lambda = c\lambda + (\eta_1 + \eta_2)(d + \delta) \), and \( Z = Z_2 \). Otherwise, we go to 3.3 with \( G = cG + \eta_1 D_q + \eta_2 D_q \) and \( \lambda = c\lambda + (\eta_1 + \eta_2)(d + \delta) < \lambda + 2(d + \delta) < \lambda + 2d + \frac{1}{\lambda} \).

3.2. Induction step: case 2. Let \( G \) be a \( \mathbb{Q} \)-divisor on \( X \) such that

1. \( G \subseteq \lambda A \) for a rational number \( 0 < \lambda \),
2. \( Z(cG) = \{p\} \) near \( p \) for all rational numbers \( 0 < c < 1 \),
3. \( Z(G) \) has a unique positive-dimensional irreducible component, say \( Z \), through \( p \).

Let \( T \) be the normalization of an irreducible affine curve containing \( p \). Let \( q \) be a preimage of \( p \) in \( T \) and let \( W \) be the image in \( X \) of the length two subscheme of \( T \) supported at \( q \). By Lemma 2.3 we have an algebraic family of \( \mathbb{Q} \)-Cartier divisors
\{B_t\}_{t \in T}$ on $Z$ such that $B_t \sim (d + \delta)A|Z$ and, for a general $t$, ord$_t B_t > d$. By Proposition 2.9, there exists a nice lifting $D_q$ of $B_q$ with respect to $G$. We can lift $D_q$ to an algebraic family $\{D_t\}$ of $\mathbb{Q}$-divisors so that $D_t|Z = B_t$ and $D_t \sim (d + \delta)A$. For a general $t$, we choose a local lifting $D'_t$ of $B_t$ with ord$_t D'_t \geq$ ord$_t B_t$. By Proposition 2.10, $t \in Z(cG + D'_t)$ for a general $t$. Since $G$ is critical at $t$ for a general $t$, $\{p, t\} \subset Z(cG + D_t)$ by Theorem 2.7. Then $W \subset Z(cG + D_q)$ by Proposition 3.3. The support of $Z(cG + D_t)$ is properly contained in the support of $Z(G)$ near $p$ by construction. Let $\eta$ be the smallest positive rational number such that $Z(cG + \eta D_q)$ contains a length two subscheme supported at $p$. Clearly $\eta \leq 1$. By Remark 3.3, we may assume that there exists at most one positive-dimensional irreducible component passing through $p$. If there exists such a component, say $Z'$, then we go back to the beginning of 3.2 with $G = cG + \eta D_q$, $\lambda = c\lambda + \eta(d + \delta)$, and $Z = Z'$. Otherwise we proceed to 3.3 with $G = cG + D_p$, and $\lambda = c\lambda + \eta(d + \delta) < \lambda + d + \frac{1}{2n}$.

3.3. **Final step:** By our choice of $\delta$ and the construction,
\begin{enumerate}
  \item $G \sim \lambda A = \frac{\lambda}{n^2 + n + 1}H$ for some $0 < \lambda < n^2 + n + 1$,
  \item $Z(G)$ has an isolated support at $p$, and
  \item $Z(G)$ contains a length two subscheme $W$ supported at $p$.
\end{enumerate}
Since $H - G$ is nef and big by (1), Theorem 2.6 and (2) imply that $|K_X + H|$ separates the connected component of $Z(G)$ at $p$. Then $|K_X + H|$ separates every length two subscheme $W$ (which exists by the condition (3)) of $Z(G)$ with Supp $W = p$. Thus general elements of $|K_X + H - p|$ are smooth at $p$. \hfill $\square$

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**References**


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