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A NOTE ON ADJOINT LINEAR SYSTEMS

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Abstract. In this note, we give a weak estimate on the separation of tangent directions of the conjecture of Fujita for adjoint linear systems on smooth varieties.

1. Introduction

Let X be a smooth projective variety of dimension n. For a Cartier divisor A and a point $p \in X$, we define

$$\mathrm{Bv}(A,p) = \bigcap_{D \in |A-p|} \mathbb{P}(T_p D),$$

where $|A-p|$ is the sublinear system of $|A|$ whose members are divisors in $|A|$ passing through p and $\mathbb{P}(T_p D)$ is the projectivised tangent space of D at p as a subspace of the projectivised tangent space $\mathbb{P}(T_p X)$ of X at p. The conjecture of Fujita [3] asserts that, for every ample divisor H on X and every point $p \in X$,

$$\mathrm{Bv}(K_X + tH, p) = \emptyset \tag{1.1}$$

if $t \geq n+2$. This has been known for curves, surfaces [6], and recently it was proved for Fano threefolds in [5] using the classification of Fano threefolds.

The purpose of this note is to prove the following weak estimate for varieties of arbitrary dimensions.

Theorem 1.1. *Let X be a smooth projective variety of dimension n and H an ample Cartier divisor on X. Then for every point p of X,*

$$\dim \mathrm{Bv}(K_X + tH, p) \leq n-2$$

if $t \geq n^2+n+1$. Equivalently, general members of $|K_X + tH - p|$ are smooth at p.

Let us give a sketch of the proof. Let p be a point in X. Our aim is to construct a $\mathbb{Q}$-divisor G on X satisfying the following properties. We refer to the next section for the notation used here.

1. $tH - G$ is an ample $\mathbb{Q}$-divisor on X.
2. The connected component at p of the multiplier ideal scheme for G, noted by $Z(G)$, is supported only at p and its length is at least 2.

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Then a standard application of the Nadel vanishing theorem implies that a general element of $|K_X + tH - p|$ is smooth at p. However note that we cannot construct G so that Z contains a preassigned tangent direction. In the proof, we will need the following induction step.

Suppose we have created a $\mathbb{Q}$-divisor G so that $Z(G)$ contains a length two subscheme supported at p and that G satisfies the following properties: For any rational $0 < \epsilon << 1$,

(1) the multiplier ideal scheme, $Z((1-\epsilon)G)$, for $(1-\epsilon)G$ does not contain any subscheme of length two supported at p;
(2) the "difference" of multiplier ideals of G and $(1-\epsilon)G$ is given by a positive-dimensional integral subscheme Z through p.

In this case, we need to consider two separate cases. First, suppose that $Z((1-\epsilon)G)$ contains p. We consider a family of $\mathbb{Q}$-divisors $\{B_t\}$ on Z with a multiplicity at x_t no less than the dimension of Z for a general t. Let D_t be a local $\mathbb{Q}$-divisor on X near x_t such that $D_t|_Z = B_t$ and $ord_{x_t} D_t \geq ord_{x_t} B_t$. Then by considering the coefficient of the exceptional divisor of the blowing up of X at x_t, one can deduce that $Z((1-\epsilon)G + D_t)$ contains x_t. Next we take a global lifting D'_t of B_t in X. Theorem 2.7 will guarantee that $Z((1-\epsilon)G + D'_t)$ still contains x_t. Since p is always contained in this new multiplier ideal by our assumption, it contains a length two subscheme. Now by specializing D'_t to D' at p, we can assure that the connected component of $Z((1-\epsilon)G + D)$ at p contains a length two subscheme. If this connected component has dimension zero, then we are done. Otherwise we will have to repeat this process. When $Z((1-\epsilon)G)$ does not contains p, the argument is similar, except that in this case we need to make sure that $Z((1-\epsilon)G + D_t)$ contains a length two subscheme supported at x_t by choosing B_t with a higher multiplicity at x_t.

Notation. We will work over the field of complex numbers.

- $\underset{\mathbb{Q}}{\sim}$ $\mathbb{Q}$-linear equivalence.
- $\lceil \cdot \rceil$ round-up.

2. Preliminaries

Here we collect some properties of multiplier ideals which are needed later in the proof. We refer to [1] and [4] for the details.

Definition 2.1. Let X be a complete variety and let W a finite subscheme of X. Let H be a Cartier divisor on X. We say that $|H|$ separates W if the following natural restriction is surjective:

$$H^0(X, \mathcal{O}_X(H)) \longrightarrow H^0(W, \mathcal{O}_W \otimes \mathcal{O}_X(H)) \longrightarrow 0.$$

Definition 2.2. Let X be a variety. Let G be an effective $\mathbb{Q}$-Cartier divisor on X.

(1) Let p be a smooth point of X. Let $f : Y \longrightarrow X$ be the blowing up of X at p and let E be the exceptional divisor over p. We define the order of G at p, $\operatorname{ord}_p G$, to be the coefficient of E in f^*G.
(2) Let W be a length two subscheme of X consisting of two distinct smooth points p and q in X. We define the order of G at W, $\operatorname{ord}_W G$, to be the minimum between $\operatorname{ord}_p G$ and $\operatorname{ord}_q G$.

(3) Let W be a non-reduced length two subscheme of X supported at a smooth point p of X, i.e. $W = \{p, v\}$ for some $v \in \mathbb{P}(T_pX)$. Let $f : U \longrightarrow X$ be the blowing up of X at p with the exceptional divisor E and $v \in E$. We define the order of G at W, $\operatorname{ord}_W G$, to be $\frac{1}{2} \operatorname{ord}_v f^*G$.

Lemma 2.3 (Lemma 2.4 in [4])**.** *Let X be a complete variety of dimension n, let L be a nef and big $\mathbb{Q}$-Cartier divisor on X, and let W be a finite subscheme of $l := \operatorname{length} W \leq 2$ supported on the smooth locus of X. For every $\epsilon > 0$, there exists an effective $\mathbb{Q}$-divisor D such that $D \underset{\mathbb{Q}}{\sim} L$ and*

$$\operatorname{ord}_W \geq \sqrt[n]{\frac{1}{l} L^n} - \epsilon.$$

Multiplier ideals. Let X be a smooth variety, W a 0-dimensional closed subscheme of X, G an effective $\mathbb{Q}$-Cartier divisor on X, and $f : Y \to X$ a log resolution for (X, G). Since $f_*\mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X$,

$$f_*\mathcal{O}_Y(\lceil K_Y - f^*(K_X + G)\rceil) \subset \mathcal{O}_X.$$

We call this ideal sheaf the multiplier ideal for G. Let $Z(G)$ be the scheme defined by this ideal and note the multiplier ideal by $\mathcal{I}_{Z(G)}$.

Remark 2.4. By a standard method, one can easily check that $\mathcal{I}_{Z(G)}$ is independent of the choice of log resolution.

We say that G is pseudo-critical at W if $W \subset Z(G)$, but $W \not\subset Z(\lambda G)$ for any $\lambda < 1$. Furthermore, we say the pair (G, f), or simply G, is critical at W if G is pseudo-critical at W, and there is a unique prime divisor F on Y such that

$$\lceil K_Y - f^*(K_X + G)\rceil = \lceil K_Y - f^*(K_X + (1-\epsilon)G)\rceil - F$$

for all sufficiently small $0 < \epsilon << 1$. We call F the critical component of G and $f(F)$ the critical variety of G at W.

Remark 2.5 (Remark 3.4 in [4]). Suppose G is ample and pseudo-critical at W. Then, using the so-called tie-braking technique, one can perturb G a little bit so that the new divisor (together with the same log resolution) is critical at W.

Theorem 2.6 (Theorem 3.7 in [4])**.** *We assume that X is complete. Let H be a Cartier divisor on X such that $H - K_X - G$ is nef and big. Let $\mathcal{I}$ be an ideal sheaf of $\mathcal{O}_X$ satisfying $\mathcal{I}_{Z(G)} \subset \mathcal{I}$ and $\dim \operatorname{Supp} \mathcal{I}/\mathcal{I}_{Z(G)} = 0$. Suppose that $\mathcal{I}_Z(G) \subset \mathcal{I}_W$ but $\mathcal{I} \not\subset \mathcal{I}_W$. Then there is $D \in |H|$ such that $\operatorname{length} W = \operatorname{length}(D \cap W) + 1$.*

Theorem 2.7 (Theorem 3.9 in [4])**.** *Suppose G is critical at W with the critical variety Z. Let B be a non-zero effective $\mathbb{Q}$-Cartier divisor on Z. Let D_1 and D_2 be two liftings of B, i.e. $B = D_i|_Z$ for $i = 1, 2$. If*

$$W \subset Z((1-s)D_1 + (1-t)G)$$

for all sufficiently small s and t, then

$$W \subset Z((1-s)D_2 + (1-t)G)$$

for all sufficiently small s and t.

Definition 2.8. Suppose G is critical at W with the critical variety Z. Let B be an effective $\mathbb{Q}$-Cartier divisor on Z. An effective $\mathbb{Q}$-Cartier divisor D is said to be a nice lifting of B with respect to G if

$$D|_Z = B \qquad \text{and} \qquad \operatorname{Supp} \mathcal{J} \subsetneqq Z$$

where $\mathcal{J}$ is defined by

$$0 \to \mathcal{I}_{Z((1-t)G+D)} \to \mathcal{I}_{Z((1-t)G)} \to \mathcal{J} \to 0$$

for $0 < t << 1$.

Proposition 2.9 (Proposition 3.10 in [4])**.** *Suppose G is critical at W with the critical variety Z. Let B be an effective $\mathbb{Q}$-Cartier divisor on Z. Let A be a $\mathbb{Q}$-ample Cartier divisor on X such that $B \underset{\mathbb{Q}}{\sim} A|_Z$. Then there is a nice lifting D of B with respect to G such that $D \underset{\mathbb{Q}}{\sim} A$.*

Proposition 2.10 (Proposition 3.12 in [4])**.** *Suppose that G is critical at $\{p\}$ with the critical variety Z of dimension $d > 0$. If D is a $\mathbb{Q}$-Cartier divisor on X with $\operatorname{ord}_p D > d$, then $p \in Z((1-\epsilon)G + D)$ for all $0 < \epsilon << 1$.*

3. Proof of Theorem 1.1

In the spirit of [2] we will prove below a result which is a bit stronger. Theorem 1.1 follows from it easily.

Theorem 3.1. *Let X be a smooth projective variety of dimension n and let H be a Cartier divisor on X. Then for every point p of X,*

$$\dim \mathrm{Bv}(K_X + H, p) \leq n - 2$$

if $H^d \cdot Z \geq \{n^2 + n + 1\}^d$ for every subvariety Z of dimension d. Equivalently, general members of $|K_X + H - p|$ are smooth at p.

The proof of Theorem 3.1 uses the following two lemmas on the behavior of multiplier ideals in a family.

Lemma 3.2 (Proposition 2.7 in [1])**.** *Let X be a variety and let p be a smooth point of X. Let T be the normalization of an irreducible affine curve containing p. Let q be a preimage of p in T. Let $\{D_t\}_{t \in T}$ be an algebraic family of $\mathbb{Q}$-Cartier divisors on X. Suppose $t \in Z(D_t)$ for general t. Then $p \in Z(D_q)$.*

Lemma 3.3. *Let X be a variety and let p be a smooth point of X. Let T be the normalization of an irreducible affine curve containing p. Let q be a preimage of p in T and let W be the image in X of the length two subscheme of T supported at q. Let $\{D_t\}_{t \in T}$ be an algebraic family of $\mathbb{Q}$-Cartier divisors on X. If $\{t, p\} \subset Z(D_t)$ for general t, then $W \subset Z(D_q)$.*

Proof. Let $\phi : Y \longrightarrow X \times T$ be a log resolution of $\mathcal{D} := \bigcup_t D_t$. Then

$$\phi^{-1}(X \times \{t\}) \longrightarrow X \times \{t\}$$

is a log resolution of D_t and $Z(\mathcal{D}) \cap X \times \{t\} = Z(D_t)$ for general t. Thus $W \subset Z(\mathcal{D}) \cap X \times \{p\}$, and by Proposition 2.1 in [1], $W \subset Z(D_p)$. □

Proof of Theorem 3.1. We fix a positive rational number $0 < \delta < \frac{1}{2n}$ and set $A := \frac{1}{n^2+n+1} H$. From Lemma 2.3, we have a $\mathbb{Q}$-divisor $D \underset{\mathbb{Q}}{\sim} (2n+\delta)A$ such that $\operatorname{ord}_p D > 2n \geq n + 1$. Then $Z(\eta D) \subset \mathcal{I}_p^2$ for some $\eta \leq \frac{n+1}{\operatorname{ord}_p D} < 1$. Let η_1 be the positive rational number such that $\eta_1 D$ becomes pseudo-critical at p. We have $\eta_1 \leq \eta < 1$. By Remark 2.5 we may assume that $\eta_1 D$ is critical at p with the critical variety Z_1. There are two possibilities.

If $\dim Z_1 > 0$, then we move to 3.1 with $G = \eta_1 D$, $\lambda = \eta_1(2n + \delta)$, and $Z = Z_1$.

Suppose that $\dim Z_1 = 0$. In this case, we need to increase η_1. Let η_2 be the smallest rational number such that $Z(\eta_2 D)$ contains a length two subscheme of X. Note that $\eta_2 \le \eta < 1$. Again by Remark 2.5, we may assume that there is at most one positive-dimensional irreducible component of $Z(\eta_2 D)$ through p. We need to consider two separate cases.

First assume that $Z(\eta_2 D)$ has a unique positive-dimensional component, say Z_2, through p. Then $Z(c\eta_2 D) = \{p\}$ for all $0 << c < 1$ near p and $\eta_2 D$ is critical at every length two subscheme W in Z_2 with $\operatorname{Supp} W = p$. Now we proceed to 3.2 with $G = \eta_2 D$, $\lambda = \eta_2(2n+\delta)$, and $Z = Z_2$.

If $Z(\eta_2 D)$ does not have a positive-dimensional irreducible component through p, then we go to 3.3 with $G = \eta_2 D$ and $\lambda = \eta_2(2n+\delta) < 2n+1$.

3.1. **Induction step: case 1.** Let G be a $\mathbb{Q}$-divisor on X such that

(1) $G \underset{\mathbb{Q}}{\approx} \lambda A$ for a rational number $0 < \lambda$,
(2) G is critical at p,
(3) Z, the critical variety of G, has a positive dimension $d > 0$.

Let T be the normalization of an irreducible affine curve containing p. Let q be a preimage of p in T and let W be the image in X of the length two subscheme of T supported at q. By applying Lemma 2.3 over the function field of T, we obtain an algebraic family of $\mathbb{Q}$-Cartier divisors $\{B_t\}_{t\in T}$ on Z such that $B_t \underset{\mathbb{Q}}{\approx} (d+\delta)A|_Z$ and, for a general t, $\operatorname{ord}_t B_t > d$. By Proposition 2.9, there exists a nice lifting $\widetilde{D}_q$ of $2B_q$ with respect to G. We put $D_q = \frac{1}{2}\widetilde{D}_q$. We can lift D_q to an algebraic family $\{D_t\}$ of $\mathbb{Q}$-divisors so that $D_t|_Z = B_t$ and $D_t \underset{\mathbb{Q}}{\approx} (d+\delta)A$. For a general t we choose a local lifting D'_t of B_t with $\operatorname{ord}_t D'_t \ge \operatorname{ord}_t B_t$. From Proposition 2.10, we have $t \in Z(cG + D'_t)$ for a general t and a positive rational number $0 << c < 1$. Since G is critical at a general point t of T, $t \in Z(cG + D_t)$ by Theorem 2.7. Thus $p \in Z(cG+D_q)$ by Lemma 3.2. Let η_1 be the constant such that $cG+\eta_1 D_q$ becomes pseudo-critical at p. Then $\eta_1 \le 1$. By Remark 2.5 we may assume that $cG + \eta_1 D_q$ is critical at p with the critical variety Z_1. By construction, we know that Z_1 is a proper subvariety of Z.

Suppose that $\dim Z_1 > 0$. Then we go back to the beginning of 3.1 with $G = cG + \eta_1 D_q$, $\lambda = c\lambda + \eta_1(d+\delta)$, and $Z = Z_1$.

Suppose that $\dim Z_1 = 0$. Since $\{p,t\} \subset Z(cG+\eta_1 D_q + D_t)$ by Proposition 2.10, $W \subset Z(cG+\eta_1 D_q + D_q)$ by Lemma 3.3. Let η_2 be the smallest rational number such that $Z(cG + \eta_1 D_q + \eta_2 D_q)$ contains a length two subscheme of X. Then $\eta_2 \le 1$. By Remark 2.5 we may assume that there is at most one positive-dimensional irreducible component of $Z(cG + \eta_1 D_q + \eta_2 D_q)$ supported at p. If there exists such a component, say Z_2, then we go to 3.2 with $G = cG + \eta_1 D_q + \eta_2 D_q$, $\lambda = c\lambda + (\eta_1+\eta_2)(d+\delta)$, and $Z = Z_2$. Otherwise, we go to 3.3 with $G = cG+\eta_1 D_q+\eta_2 D_q$ and $\lambda = c\lambda + (\eta_1+\eta_2)(d+\delta) < \lambda + 2(d+\delta) < \lambda + 2d + \frac{1}{n}$.

3.2. **Induction step: case 2.** Let G be a $\mathbb{Q}$-divisor on X such that

(1) $G \underset{\mathbb{Q}}{\approx} \lambda A$ for a rational number $0 < \lambda$,
(2) $Z(cG) = \{p\}$ near p for all rational numbers $0 << c < 1$,
(3) $Z(G)$ has a unique positive-dimensional irreducible component, say Z, through p.

Let T be the normalization of an irreducible affine curve containing p. Let q be a preimage of p in T and let W be the image in X of the length two subscheme of T supported at q. By Lemma 2.3, we have an algebraic family of $\mathbb{Q}$-Cartier divisors

$\{B_t\}_{t\in T}$ on Z such that $B_t \underset{\mathbb{Q}}{\sim} (d+\delta)A|_Z$ and, for a general t, $\operatorname{ord}_t B_t > d$. By Proposition 2.9 there exists a nice lifting D_q of B_q with respect to G. We can lift D_q to an algebraic family $\{D_t\}$ of $\mathbb{Q}$-divisors so that $D_t|_Z = B_t$ and $D_t \underset{\mathbb{Q}}{\sim} (d+\delta)A$. For a general t, we choose a local lifting D'_t of B_t with $\operatorname{ord}_t D'_t \geq \operatorname{ord}_t B_t$. By Proposition 2.10, $t \in Z(cG+D'_t)$ for a general t. Since G is critical at t for a general t, $\{p,t\} \subset Z(cG+D_t)$ by Theorem 2.7. Then $W \subset Z(cG+D_q)$ by Proposition 3.3. The support of $Z(cG+D_q)$ is properly contained in the support of $Z(G)$ near p by construction. Let η be the smallest positive rational number such that $Z(cG+\eta D_q)$ contains a length two subscheme supported at p. Clearly $\eta \leq 1$. By Remark 2.5, we may assume that there exists at most one positive-dimensional irreducible component passing through p. If there exists such a component, say Z', then we go back to the beginning of 3.2 with $G = cG+\eta D_q$, $\lambda = c\lambda + \eta(d+\delta)$, and $Z = Z'$. Otherwise we proceed to 3.3 with $G = cG+\eta D_p$ and $\lambda = c\lambda+\eta(d+\delta) < \lambda + d + \frac{1}{2n}$.

3.3. **Final step:** By our choice of δ and the construction,

(1) $G \underset{\mathbb{Q}}{\sim} \lambda A = \frac{\lambda}{n^2+n+1}H$ for some $0 < \lambda < n^2+n+1$,
(2) $Z(G)$ has an isolated support at p, and
(3) $Z(G)$ contains a length two subscheme W supported at p.

Since $H-G$ is nef and big by (1), Theorem 2.6 and (2) imply that $|K_X+H|$ separates the connected component of $Z(G)$ at p. Then $|K_X+H|$ separates every length two subscheme W (which exists by the condition (3)) of $Z(G)$ with $\operatorname{Supp} W = p$. Thus general elements of $|K_X+H-p|$ are smooth at p. □

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