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A NOTE ON EXPONENTIAL DECAY PROPERTIES OF GROUND STATES FOR QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. We give an explicit formula for exponential decay properties of ground states for a class of quasilinear elliptic equations in the whole space $\mathbb{R}^N$.

1. Introduction

We consider exponential decay properties of ground states of the quasilinear elliptic equation

$$\Delta_m u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \quad N > m > 1, \tag{1.1}$$

where $\Delta_m u = \operatorname{div}(|Du|^{m-2} Du)$ is the degenerate m-Laplace operator. Here by a ground state we mean a non-negative non-trivial C^1 distribution solution of (1.1) which tends to zero at ∞.

Remark 1. There are some sufficient conditions which guarantee the existence of a positive radial ground state $u(x)$ satisfying $u(x) = u(r)$ with $r = |x|$, $u(0) = \max_{x\epsilon\mathbb{R}^N} u(x)$, $u'(0) = 0$ and $u'(r) < 0$ for $r > 0$, where $u'(r) = \frac{du(r)}{dr}$. See, for example, (H1)—(H3) in reference [6]. A good example for $f(u)$ which satisfies (A1) is:

$$f(u) = -u^{m-1} + u^q \text{ with } m - 1 < q < \frac{N(m-1)+m}{N-m}.$$

From now on we make the following assumption on f:
(A1): $f\colon \mathbb{R}^+ \to \mathbb{R}$ is continuous and there exist positive constants k and δ such that $f(z) + kz^{m-1} = O(z^{m-1+\delta})$ as $z \downarrow 0$.

Before stating our results, let us recollect some facts about ground states for (1.1) with $m = 2$, i.e.,

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^N. \tag{1.2}$$

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Under suitable conditions on f (e.g. [2]) it is well known that the ground state for (1.2) satisfies

$$u(x) = u(|x|) = u(r) > 0,\ u(0) = \max_{x\in\mathbb{R}^N} u(x),$$
$$u'(0) = 0 \quad \text{and} \quad u'(r) < 0 \quad \text{for} \quad r \in (0,\infty).$$

Moreover,

$$\lim_{r\to\infty} u(r) r^{\frac{N-1}{2}} e^r = C_1 \tag{1.3}$$

where $0 < C_1 < \infty$ is a constant, and such precise estimates of asymptotes have been proved to be very useful. For example, for applications of such estimates for $m = 2$, readers can refer to [1], [2], [3], [4], [5], etc. In this note we show that a similar estimate to (1.3) also holds for radial ground states of (1.1) if assumption (A1) is satisfied. Our main result is as follows.

Theorem 1.1. *Let $u(x) = u(r)$ be a positive radial ground state for* (1.1) *and let $f(z)$ satisfy assumptions (A1). Then there exists a sequence of constants $\{c_i\}$ $(i = 1, 2, \cdots)$ such that*

$$\left(-\frac{u'}{u}\right)^{m-1} = \varphi_\infty + \frac{c_1}{r} + \frac{c_2}{r^2} + \cdots + \frac{c_l}{r^l} + \cdots \quad \text{as } r \to \infty$$

where $\{c_i\}$ $(i = 1, 2, \cdots)$ are determined by

$$c_1 = \frac{N-1}{m}\left(\frac{k}{m-1}\right)^{\frac{m-2}{m}}, \quad c_2 = \frac{(N-2)c_1 - \frac{m}{2(m-1)}\varphi_\infty^{\frac{2-m}{m-1}} c_1^2}{m\varphi_\infty^{\frac{1}{m}}},$$

and for $l > 2$, c_l can be uniquely determined by

$$(N-l)\, c_{l-1} - m\varphi_\infty^{\frac{1}{m-1}} c_l = \sum_{i=2}^{l} \frac{F^{(i)}(0)}{i!}\left(\sum_{i_1+\cdots+i_i = l, i_1,\cdots i_i > 0} c_{i_1} c_{i_2} \cdots c_{i_i}\right),$$

where $\varphi_\infty = \left(\frac{k}{m-1}\right)^{\frac{m-1}{m}}$ and $F(\rho) = (m-1)(\varphi_\infty + \rho)^{\frac{m}{m-1}}$. In particular we have

$$\lim_{r\to\infty} u(r) r^{\frac{N-1}{m(m-1)}} e^{\left(\frac{k}{m-1}\right)^{\frac{1}{m}} r} = C_2$$

for some $0 < C_2 < \infty$.

2. Proof of Theorem 1.1

In this section, we will present the proof of Theorem 1.1. First let $u(x) = u(r)$ be a radial ground state as stated in Theorem 1.1. It follows from (1.1) that

$$\left(|u'|^{m-2} u'\right)' + \frac{N-1}{r} |u'|^{m-2} u' + f(u) = 0. \tag{2.1}$$

Let $\varphi(r) = -\frac{|u'|^{m-2}u'}{u^{m-1}}$. Then $|u'|^{m-2} u' = -\varphi u^{m-1}$ and $\frac{u'}{u} = -\varphi^{\frac{1}{m-1}}$. Substituting them into (2.1) yields

$$\varphi' - (m-1)\varphi^{\frac{m}{m-1}} + \frac{N-1}{r}\varphi - \frac{f(u)}{u^{m-1}} = 0. \tag{2.2}$$

Lemma 2.1. $\limsup_{r\to\infty} \varphi(r) < \infty$.

Proof. It follows from assumption (A1) and $\lim_{r\to\infty} u(r) = 0$ that

$$K = \sup_r \frac{|f(u(r))|}{u^{m-1}(r)} < \infty.$$

Noticing from $\varphi(r) \geq 0$ and (2.2) we obtain that as long as $\varphi(r) \geq \left(\frac{4K}{m-1}\right)^{\frac{m-1}{m}}$ and $r \geq \frac{4(N-1)}{m-1}\left(\frac{m-1}{4K}\right)^{\frac{m-1}{m}}$,

$$\varphi' = (m-1)\varphi^{\frac{m}{m-1}} - \frac{N-1}{r}\varphi + \frac{f(u)}{u^{m-1}} \geq \frac{m-1}{2}\varphi^{\frac{m}{m-1}}. \tag{2.3}$$

Now, suppose to the contrary that $\limsup_{r\to\infty} \varphi(r) = \infty$. Let

$$r_1 = \inf\{r \geq \frac{4(N-1)}{m-1}\left(\frac{m-1}{4K}\right)^{\frac{m-1}{m}} : \varphi(r) \geq \left(\frac{4K}{m-1}\right)^{\frac{m-1}{m}}\}.$$

Since $\varphi(0) = 0$ and $\varphi(r) > 0$ for $r > 0$, it follows that $0 < r_1 \leq \infty$. If $r_1 = \infty$ we are done. Next we suppose $r_1 < \infty$. Then $\varphi(r_1) = \left(\frac{4K}{m-1}\right)^{\frac{m-1}{m}}$ and $\varphi'(r) \geq \frac{m-1}{2}\varphi(r)^{\frac{m}{m-1}}$ for all $r \geq r_1$, which blows up before or at $r_2 = r_1 + 2\left(\frac{m-1}{4K}\right)^{\frac{1}{m}}$. This contradicts the well-definedness of φ. The proof of this lemma is completed. □

Lemma 2.2.

$$\lim_{r\to\infty} \varphi = \varphi_\infty = \left(\frac{k}{m-1}\right)^{\frac{m-1}{m}}. \tag{2.4}$$

Proof. Since $\varphi(0) = 0$ and $\varphi(r) > 0$ for $r > 0$ it follows from (2.2) and Lemma 2.1 that

$$0 \leq \liminf_{r\to\infty} \varphi(r) = \alpha < \infty$$

and

$$0 < \limsup_{r\to\infty} \varphi(r) = \beta < \infty.$$

Next we use contradiction arguments to prove (2.4). Suppose

$$\alpha = \liminf_{r\to\infty} \varphi(r) < \limsup_{r\to\infty} \varphi(r) = \beta.$$

Then we may choose two sequences $\{\eta_i\}$ and $\{\zeta_i\}$ going to ∞ as $i \to \infty$ such that

$$\{\eta_i\} \text{ are local minima of } \varphi$$

and

$$\{\zeta_i\} \text{ are local maxima of } \varphi$$

and

$$\eta_i < \zeta_i < \eta_{i+1} \text{ and } \liminf_{r\to\infty} \varphi(r) = \lim_{i\to\infty} \varphi(\eta_i) = \alpha, \limsup_{r\to\infty} \varphi(r) = \lim_{i\to\infty} \varphi(\zeta_i) = \beta.$$

Then at η_i $(i = 1, 2, \cdots)$ we know $u'(\eta_i) = 0$ and thus

$$-(m-1)\varphi(\eta_i)^{\frac{m}{m-1}} + \frac{N-1}{\eta_i}\varphi(\eta_i) - \frac{f(u(\eta_i))}{u(\eta_i)^{m-1}} = 0. \tag{2.5}$$

Similarly at ζ_i $(i = 1, 2, \cdots)$ we get

$$-(m-1)\varphi(\zeta_i)^{\frac{m}{m-1}} + \frac{N-1}{\zeta_i}\varphi(\zeta_i) - \frac{f(u(\zeta_i))}{u(\zeta_i)^{m-1}} = 0, \tag{2.6}$$

$\forall\epsilon > 0$. Since $\lim_{z\downarrow 0} \frac{f(z)}{z^{m-1}} = -k$ and $\lim_{r\to\infty} u(r) = 0$, we can take r_3 sufficiently large such that

$$\left|\frac{f(u(r))}{u(r)^{m-1}} + k\right| < \epsilon \quad \text{for all } r > r_3.$$

Next we take i_0 sufficiently large such that

$$\eta_i > r_3 \text{ for } i > i_0.$$

From (2.5) and (2.6) it follows that for all $i > i_0$:

$$k - \epsilon < (m-1)\varphi(\eta_i)^{\frac{m}{m-1}} - \frac{N-1}{\eta_i}\varphi(\eta_i) < k + \epsilon$$

and

$$k - \epsilon < (m-1)\varphi(\zeta_i)^{\frac{m}{m-1}} - \frac{N-1}{\zeta_i}\varphi(\zeta_i) < k + \epsilon.$$

Letting $i \to \infty$ and noticing the arbitrariness of ϵ, we get

$$(m-1)\alpha^{\frac{m}{m-1}} = k, \qquad (m-1)\beta^{\frac{m}{m-1}} = k,$$

which yields $\alpha = \beta$, contradicting $\alpha < \beta$.

Therefore

$$\lim_{r\to\infty} \varphi(r) = \varphi_\infty = \left(\frac{k}{m-1}\right)^{\frac{m-1}{m}}.$$

The proof of this lemma is complete. □

After presenting the above two lemmas we can now give a proof of our main result.

Proof of Theorem 1.1. We deduce from Lemma 2.2 that

$$\lim_{r\to\infty} \frac{u'}{u} = \lim_{r\to\infty} -\varphi^{\frac{1}{m-1}} = -\left(\frac{k}{m-1}\right)^{\frac{1}{m}}.$$

Therefore $\forall\epsilon > 0$, there exists a constant $0 < c(\epsilon) < \infty$ such that

$$u(r) \le c(\epsilon) e^{-\left(\frac{k-\epsilon}{m-1}\right)^{\frac{1}{m}} r}.$$

In particular taking $\epsilon = \frac{k}{2}$ we have

$$u(r) \le c(k) e^{-\left(\frac{k}{2(m-1)}\right)^{\frac{1}{m}} r}.$$

For convenience, let $\left(\frac{k}{2(m-1)}\right)^{\frac{1}{m}} = \mu$. We know

$$u(r) \le c e^{-\mu r}. \tag{2.7}$$

Next we give a more precise expansion of $\varphi(r)$ at ∞. Let $\varphi = \varphi_\infty + \varphi_1$. We know from (2.2) and Lemma 2.2 that $\lim_{r\to\infty} \varphi_1(r) = 0$ and $\varphi_1(r)$ satisfies

$$\varphi_1' - (m-1)(\varphi_\infty + \varphi_1)^{\frac{m}{m-1}} + \frac{N-1}{r}(\varphi_\infty + \varphi_1) - \frac{f(u)}{u^{m-1}} = 0,$$

i.e.,

$$\begin{aligned} &\varphi_1' - m\varphi_\infty^{\frac{1}{m-1}}\varphi_1 + \frac{N-1}{r}\varphi_1 \\ &= \frac{f(u)}{u^{m-1}} + (m-1)(\varphi_\infty + \varphi_1)^{\frac{m}{m-1}} - m\varphi_\infty^{\frac{1}{m-1}}\varphi_1 - \frac{N-1}{r}\varphi_\infty. \end{aligned} \tag{2.8}$$

Noticing that $\lim_{r\to\infty} u(r) = 0$ and $\frac{f(z)+kz^{m-1}}{z^{m-1}} = O(z^\delta)$ as $z \downarrow 0$, we get for r sufficiently large that

$$\frac{f(u)}{u^{m-1}} = -k + O(u^\delta). \tag{2.9}$$

At the same time for r sufficiently large we have

$$(m-1)(\varphi_\infty + \varphi_1)^{\frac{m}{m-1}} = (m-1)\varphi_\infty^{\frac{m}{m-1}} + m\varphi_\infty^{\frac{1}{m-1}}\varphi_1 + O\left(\varphi_1^2\right). \tag{2.10}$$

Thus from (2.8)–(2.10) it follows that for r sufficiently large,

$$\varphi_1' - m\varphi_\infty^{\frac{1}{m-1}}\varphi_1 + \frac{N-1}{r}\varphi_1 = O(\varphi_1^2) - \frac{N-1}{r}\varphi_\infty + O(u^\delta). \tag{2.11}$$

Multiplying each side of (2.11) by φ_1 and integrating from r to ∞ for r sufficiently large, we get

$$\begin{aligned}\frac{1}{2}\varphi_1^2(r) + \int_r^\infty \left(m\varphi_\infty^{\frac{1}{m-1}} - \frac{N-1}{s} + O(\varphi_1)\right)\varphi_1^2 ds \\ = \int_r^\infty \frac{N-1}{s}\varphi_\infty\varphi_1 ds - \int_r^\infty O(u^\delta)\varphi_1 ds.\end{aligned} \tag{2.12}$$

We can take r sufficiently large such that

$$m\varphi_\infty^{\frac{1}{m-1}} - \frac{N-1}{s} + O(\varphi_1) \geq \frac{1}{2}m\varphi_\infty^{\frac{1}{m-1}} \text{ for } s > r.$$

Therefore for such large r it follows that

$$\varphi_1^2(r) + \int_r^\infty \left(m\varphi_\infty^{\frac{1}{m-1}}\right)\varphi_1^2 ds \leq 2\int_r^\infty \frac{N-1}{s}\varphi_\infty\varphi_1 ds - 2\int_r^\infty O(u^\delta)\varphi_1 ds.$$

Notice that

$$2\int_r^\infty \frac{N-1}{s}\varphi_\infty\varphi_1 ds \leq \frac{1}{4}\left(m\varphi_\infty^{\frac{1}{m-1}}\right)\int_r^\infty \varphi_1^2 ds + \frac{4(N-1)^2\varphi_\infty^2}{m\varphi_\infty^{\frac{1}{m-1}}}\int_r^\infty \frac{1}{s^2}ds$$

and

$$2\int_r^\infty O(u^\delta)\varphi_1 ds \leq \frac{1}{4}\left(m\varphi_\infty^{\frac{1}{m-1}}\right)\int_r^\infty \varphi_1^2 ds + \frac{4C}{m\varphi_\infty^{\frac{1}{m-1}}}\int_r^\infty O(u^{2\delta})ds.$$

By virtue of the above estimates and (2.7) we obtain for r sufficiently large,

$$\begin{aligned}&\varphi_1^2(r) + \frac{1}{2}\left(m\varphi_\infty^{\frac{1}{m-1}}\right)\int_r^\infty \varphi_1^2(s)ds \\ &\leq \frac{4(N-1)^2\varphi_\infty^2}{m\varphi_\infty^{\frac{1}{m-1}}}\frac{1}{r} + \frac{4c}{2\delta m\varphi_\infty^{\frac{1}{m-1}}}e^{-2\delta\mu r} \\ &\leq \frac{8(N-1)^2\varphi_\infty^2}{m\varphi_\infty^{\frac{1}{m-1}}}\frac{1}{r}.\end{aligned} \tag{2.13}$$

Thus we have

$$\varphi_1^2(r) = O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty. \tag{2.14}$$

By this estimate and (2.11) it follows that as $r \to \infty$,

$$\varphi_1' - m\varphi_\infty^{\frac{1}{m-1}}\varphi_1 + \frac{N-1}{r}\varphi_1 = O\left(\frac{1}{r}\right). \tag{2.15}$$

For convenience let $\alpha_0 = m\varphi_\infty^{\frac{1}{m-1}}$. Then we get from (2.15) as $r \to +\infty$,

$$\left(r^{N-1}e^{-\alpha_0 r}\varphi_1\right)' = r^{N-1}e^{-\alpha_0 r}O\left(\frac{1}{r}\right). \tag{2.16}$$

Integrating both sides of (2.16) from r to ∞ yields as $r \to \infty$,

$$\begin{aligned}\varphi_1(r) &= \frac{e^{\alpha_0 r}}{r^{N-1}}\int_r^\infty s^{N-1}O\left(\frac{1}{s}\right)e^{-\alpha_0 s}ds \\ &= O\left(\frac{e^{\alpha_0 r}}{r^{N-1}}\int_r^\infty s^{N-2}e^{-\alpha_0 s}ds\right).\end{aligned} \tag{2.17}$$

Applying integration by parts as many steps as we want we arrive that there exists a sequence of constants $\{a_i\}$ $(i = 1, 2, \cdots)$ such that

$$\begin{aligned}\int_r^\infty s^{N-2}e^{-\alpha_0 s}ds &= a_1 r^{N-2}e^{-\alpha_0 r} + a_2 r^{N-3}e^{-\alpha_0 r} + \cdots \\ &= e^{-\alpha_0 r}\left(a_1 r^{N-2} + a_2 r^{N-3} + \cdots + a_l r^{N-l-1} + \cdots\right)\end{aligned} \tag{2.18}$$

where $a_1 = \frac{1}{\alpha_0}$. Thus it follows from (2.16) that

$$\varphi_1(r) = O(\frac{1}{r}), \text{ as } r \to \infty, \tag{2.19}$$

which is an improvement of (2.14). Using (2.19) and (2.11) we obtain

$$\left(r^{N-1}e^{-\alpha_0 r}\varphi_1\right)' = -r^{N-1}e^{-\alpha_0 r}\left(\frac{N-1}{r}\varphi_\infty + O(\frac{1}{r^2})\right). \tag{2.20}$$

Similar to (2.16)and (2.17), we arrive at

$$\begin{aligned}\varphi_1(r) &= \frac{e^{\alpha_0 r}}{r^{N-1}}\int_r^\infty s^{N-1}e^{-\alpha_0 s}\left(\frac{N-1}{s}\varphi_\infty + O\left(\frac{1}{s^2}\right)\right)ds \\ &= \frac{(N-1)\varphi_\infty a_1}{r} + O\left(\frac{1}{r^2}\right).\end{aligned}$$

Again if we let $\varphi_1 = \frac{(N-1)\varphi_\infty a_1}{r} + \varphi_2$ such that $\varphi_2(r) = O\left(\frac{1}{r^2}\right)$, we obtain from (2.10) and (2.11) that

$$\varphi_2' - \alpha_0\varphi_2 + \frac{N-1}{r}\varphi_2 = \varphi_\infty^{\frac{m-2}{m-1}}\left[\frac{N-1}{2m(m-1)} - \frac{N-2}{m}\right]\frac{N-1}{r^2} + O\left(\frac{1}{r^2}\right).$$

We can then repeat the same process to obtain the expansion as stated in Theorem 1.1 to any polynomial order as we want. Next we need to determine c_i $(i = 1, 2, \cdots)$ in Theorem 1.1. Let $F(\rho) = (m-1)(\varphi_\infty + \rho)^{\frac{m}{m-1}}$. Then the Taylor expansion of $F(\rho)$ at $\rho = 0$ is as follows:

$$\begin{aligned}F(\rho) = {} & (m-1)\varphi_\infty^{\frac{m}{m-1}} + \alpha_0\rho + \frac{m}{2(m-1)}\varphi_\infty^{-\frac{m-2}{m-1}}\rho^2 \\ & - \frac{m(m-2)}{3!(m-1)^2}\varphi_\infty^{-\frac{2m-3}{m-1}}\rho^3 + \cdots + \frac{F^{(n)}(0)}{n!}\rho^n + \cdots\end{aligned} \tag{2.21}$$

where

$$F^{(n)}(0) = \frac{(-1)^n m(m-2)(2m-3)\cdots(lm-l-1)\cdots[(n-2)m-n+1]}{(m-1)^{n-1}}$$

for $n \geq 4$. Thus from (2.8), (2.9) and (2.21) we get

$$\varphi_1' - \alpha_0\varphi_1 + \frac{N-1}{r}\varphi_1 = O(u^\delta) - \frac{N-1}{r}\varphi_\infty + \frac{m}{2(m-1)}\varphi_\infty^{-\frac{m-2}{m-1}}\varphi_1^2 - \frac{m(m-2)}{3!(m-1)^2}\varphi_\infty^{-\frac{2m}{m-1}}\varphi_1^3 + \cdots + \frac{F^{(n)}(0)}{n!}\varphi_1^n + \cdots . \tag{2.22}$$

Substituting

$$\varphi_1(r) = \frac{c_1}{r} + \frac{c_2}{r^2} + \cdots + \frac{c_l}{r^l} + \cdots$$

into (2.22) we get by comparing the coefficients of $\frac{1}{r^n}$ $(n = 1, 2, \cdots)$ that

$$c_1 = \frac{N-1}{m}\left(\frac{k}{m-1}\right)^{\frac{m-2}{m}}$$

and

$$c_2 = \frac{(N-2)c_1 - \frac{m}{2(m-1)}\varphi_\infty^{-\frac{m-2}{m-1}}c_1^2}{\alpha_0}.$$

c_l $(l > 2)$ is determined by

$$(N-l)c_{l-1} - \alpha_0 c_l = \sum_{i=2}^{l}\frac{F^{(i)}(0)}{i!}\left(\sum_{i_1+i_2+\cdots+i_i=l} c_{i_1}c_{i_2}\cdots c_{i_i}\right).$$

Notice that

$$\frac{u'}{u} = -\varphi^{\frac{1}{m-1}}.$$

We know that as $r \to \infty$,

$$\begin{aligned}\frac{u'}{u} &= -\left(\varphi_\infty^{\frac{1}{m-1}} + \frac{1}{m-1}\varphi_\infty^{\frac{2-m}{m-1}}c_1 \cdot \frac{1}{r} + O\left(\frac{1}{r^2}\right)\right)\\ &= -\left(\frac{k}{m-1}\right)^{\frac{1}{m}} - \frac{N-1}{m(m-1)} \cdot \frac{1}{r} + O\left(\frac{1}{r^2}\right),\end{aligned}$$

which yields that

$$u(r) = O\left(r^{-\frac{(N-1)}{m(m-1)}}e^{-\left(\frac{k}{m-1}\right)^{\frac{1}{m}}r}\right) \quad \text{as } r \to \infty,$$

i.e.,

$$\lim_{r\to\infty} u(r)r^{\frac{N-1}{m(m-1)}}e^{\left(\frac{k}{m-1}\right)^{\frac{1}{m}}r} = C_2,$$

for some $0 < C_2 < \infty$. The proof of Theorem 1.1 is complete. $\square$

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