

RODRIGUES TYPE FORMULA FOR ORTHOGONAL POLYNOMIALS ON THE UNIT BALL

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ABSTRACT. For a class of weight functions invariant under reflection groups on the unit ball, a family of orthogonal polynomials is defined via a Rodrigues type formula using the Dunkl operators. Their properties and their relation with several other bases are explored.

1. INTRODUCTION

Let $\Pi^d = \mathbb{R}[x_1, \dots, x_d]$ denote the space of polynomials in d variables and Π_n^d denote the subspace of polynomials of degree at most n . We will use the standard multi-index notation. For $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ is a monomial of degree $|\alpha| = \alpha_1 + \dots + \alpha_d$.

The classical orthogonal polynomials on the unit ball $B^d = \{x : \|x\| \leq 1\}$, where $\|x\|$ is the Euclidean norm of x , are orthogonal with respect to the weight function

$$(1.1) \quad W_\mu(x) = (1 - \|x\|^2)^{\mu-1/2}, \quad x \in B^d, \quad \mu > -1/2.$$

The study of these polynomials can be traced back to Hermite; see [1] and [5, Chapt. 12]. Among the first orthogonal bases on B^d being studied is the one given by the Rodrigues type formula

$$(1.2) \quad U_\alpha(x) = (1 - \|x\|^2)^{-\mu+1/2} \partial^\alpha (1 - \|x\|^2)^{n+\mu-1/2}, \quad \alpha \in \mathbb{N}_0^d,$$

where $\partial_i = \partial/\partial x_i$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$, and a family of polynomials that are biorthogonal to U_α .

The purpose of this paper is to consider the analogues of U_α for orthogonal polynomials on B^d with respect to a class of weight functions invariant under a reflection group. We first define the weight function. Let G be a finite reflection group with a fixed positive root system R_+ . Let σ_v denote the reflection along $v \in R_+$, that is, $x\sigma_v = x - 2\langle x, v \rangle/\|v\|^2$ for $x \in \mathbb{R}^d$ with the inner product $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$. Thus, G is a finite orthogonal group generated by the reflections $\{\sigma_v : v \in R_+\}$. Let κ be a multiplicity function defined on R_+ , that is, $\kappa : R_+ \mapsto \mathbb{R}$

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is a G -invariant function. We assume that $\kappa(v) \geq 0$ for all $v \in R_+$. Then the function

$$(1.3) \quad h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa(v)}, \quad x \in \mathbb{R}^d,$$

is a positive homogeneous G -invariant function of order $\gamma := \gamma_\kappa = \sum_{v \in R_+} \kappa_v$. For example, if G is the symmetric group, then $h_\kappa(x) = \prod_{1 \leq i < j \leq d} |x_i - x_j|^\kappa$ and $\gamma = d(d-1)\kappa/2$. In this paper we consider the reflection invariant weight function $W_{\kappa, \mu}$ on the unit ball

$$(1.4) \quad W_{\kappa, \mu}(x) = h_\kappa^2(x)(1 - \|x\|^2)^{\mu-1/2}, \quad x \in B^d, \quad \mu > -1/2$$

where h_κ^2 is as in (1.3). If $\kappa = 0$, then $W_{\kappa, \mu}(x)$ becomes $W_\mu(x)$ in (1.1).

The weight function h_κ was introduced by Dunkl for the purpose of studying the orthogonal structure for polynomials on the unit sphere $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ with respect to the measure $h_\kappa^2(x)d\omega$, where $d\omega$ is the rotation invariant measure on S^{d-1} . The main ingredient in the study is a family of commuting differential-difference operators, \mathcal{D}_i , called Dunkl's operators ([2]),

$$\mathcal{D}_i f(x) = \partial_i f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(x\sigma_v)}{\langle x, v \rangle} \langle e_i, v \rangle, \quad 1 \leq i \leq d,$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i -th component. The set $\{\mathcal{D}_i : 1 \leq i \leq d\}$ generates a commutative algebra of operators containing the h -Laplacian $\Delta_h = \mathcal{D}_1^2 + \dots + \mathcal{D}_d^2$. Let \mathcal{P}_n^d denote the space of homogeneous polynomials of degree n in d variables. If $p \in \mathcal{P}_n^d$, then

$$\int_{S^{d-1}} p(x)q(x)h^2(x)d\omega = 0, \quad \forall q \in \Pi_{n-1}^d,$$

if and only if $\Delta_h p = 0$. The space $\mathcal{H}_n^d(h_\kappa^2) := \mathcal{P}_n^d \cap \ker \Delta_h$ is called the space of h -harmonic polynomials of degree n . If $\kappa(v) = 0$, it agrees with the space of ordinary harmonic polynomials $\mathcal{H}_n^d := \mathcal{P}_n^d \cap \ker \Delta$, where Δ denote the ordinary Laplacian $\Delta = \partial_1^2 + \dots + \partial_d^2$. For the extensive study of h -harmonics and the related work, see [4] and the references therein.

The theory of h -harmonics has also made it possible to study the orthogonal structure of polynomials with respect to $W_{\kappa, \mu}$ on B^d . Initial work in this direction has been carried out in [12, 13], which has led to new results even in the case of the classical weight function W_μ . In the present paper we show that the orthogonal polynomials $U_\alpha(x)$ in (1.2) defined by the Rodrigues type formula can be extended to the reflection invariant weight function $W_{\kappa, \mu}$. That is, we define U_α as follows:

Definition 1.1. For $\alpha \in \mathbb{N}_0^d$, define

$$U_\alpha(x) = (1 - \|x\|^2)^{-\mu+1/2} \mathcal{D}^\alpha (1 - \|x\|^2)^{|\alpha+\mu-1/2|}.$$

We will show that $U_\alpha(x)$ so defined are orthogonal polynomials with respect to $W_{\kappa, \mu}$ on B^d . Properties of these polynomials and their relations with other orthogonal bases will be discussed. Let us mention that recently Dunkl and Dunkl-Cherednik operators have been used to give the Rodrigues type formula for nonsymmetric Hermite and Laguerre polynomials with respect to reflection invariant weight functions ([8, 10]) and for Macdonald's polynomials ([7]).

2. RODRIGUES TYPE FORMULA AND ORTHOGONAL BASES

For $n \in \mathbb{N}_0^d$ let $\mathcal{V}_n^d(W_{\kappa,\mu})$ denote the subspace of polynomials of degree exactly n in Π^d that are orthogonal with respect to $W_{\kappa,\mu}$. A polynomial $p \in \mathcal{V}_n^d(W_{\kappa,\mu})$ if $p \in \Pi_n^d$ and $\int_{B^d} p(x)q(x)W_{\kappa,\mu}(x)dx = 0$ for all $q \in \Pi_{n-1}^d$. We show $U_\alpha \in \mathcal{V}_n^d(W_{\kappa,\mu})$.

The definition of U_α in Definition 1.1 shows that it depends on κ and μ . In the following proposition, we write $U_\alpha^\mu := U_\alpha$ to emphasize its dependence on μ .

Proposition 2.1. *Let $\alpha \in \mathbb{N}_0^d$ and $|\alpha| = n$. Then U_α^μ is a polynomial of degree n , and we have a recursive formula*

$$U_{\alpha+e_i}^\mu(x) = -(2\mu + 1)x_i U_\alpha^{\mu+1}(x) + (1 - \|x\|^2)\mathcal{D}_i U_\alpha^{\mu+1}(x).$$

Proof. Since $(1 - \|x\|^2)^a$ is invariant under the reflection group, a simple computation shows that

$$\begin{aligned} \mathcal{D}_i U_\alpha^{\mu+1}(x) &= \partial_i \left[(1 - \|x\|^2)^{-\mu-1/2} \right] \mathcal{D}^\alpha (1 - \|x\|^2)^{|\alpha|+\mu+1/2} \\ &\quad + (1 - \|x\|^2)^{-\mu-1/2} \mathcal{D}^{\alpha+e_i} (1 - \|x\|^2)^{|\alpha|+\mu+1/2} \\ &= (2\mu + 1)x_i (1 - \|x\|^2)^{-1} U_\alpha^{\mu+1}(x) + (1 - \|x\|^2)^{-1} U_{\alpha+e_i}^\mu(x), \end{aligned}$$

which proves the recursive relation. That U_α^μ is a polynomial of degree $|\alpha|$ is a consequence of this relation (use induction if necessary). □

The following proposition plays a key role in proving that U_α is in fact an orthogonal polynomial with respect to $W_{\kappa,\mu}$ on B^d .

Proposition 2.2. *Assume that p and q are continuous functions and p vanishes on the boundary of B^d . Then*

$$\int_{B^d} \mathcal{D}_i p(x)q(x)h_\kappa^2(x)dx = - \int_{B^d} p(x)\mathcal{D}_i q(x)h_\kappa^2(x)dx.$$

Proof. The proof is similar to the proof of Lemma 3.7 of [3] (Theorem 5.1.18 of [4]). We shall be brief. Assume $\kappa_v \geq 1$. Analytic continuation can be used to extend the range of validity to $\kappa_v \geq 0$. Integration by parts shows

$$\begin{aligned} \int_{B^d} \partial_i p(x)q(x)h_\kappa^2(x)dx &= - \int_{B^d} p(x)\partial_i(q(x)h_\kappa^2(x))dx \\ &= - \int_{B^d} p(x)\partial_i q(x)h_\kappa^2(x)dx - 2 \int_{B^d} p(x)q(x)h_\kappa(x)\partial_i h_\kappa(x)dx. \end{aligned}$$

Let D_i denote the difference part of \mathcal{D}_i , $\mathcal{D}_i = \partial_i + D_i$. For a fixed root v , a simple computation shows that

$$\int_{B^d} D_i p(x)q(x)h_\kappa^2(x)dx = \int_{B^d} p(x) \sum_{v \in R_+} \kappa_i v_i \frac{q(x) + q(x\sigma_v)}{\langle x, v \rangle} h_\kappa^2(x)dx.$$

Adding these two equations and using the fact that

$$h_\kappa(x)\partial_i h_\kappa(x) = \sum_{v \in R_+} \kappa_v v_i \frac{1}{\langle v, x \rangle} h_\kappa^2(x)$$

completes the proof. □

Theorem 2.3. *For $\alpha \in \mathbb{N}_0^d$, the polynomials U_α are elements in $\mathcal{V}_{|\alpha|}^d(W_{\kappa,\mu})$.*

Proof. Let $n = |\alpha|$. For each $\beta \in \mathbb{N}_0^d$ and $|\beta| < n$, we claim that

$$(2.1) \quad \mathcal{D}^\beta(1 - \|x\|^2)^{n+\mu-1/2} = (1 - \|x\|^2)^{n-|\beta|+\mu-1/2}Q_\beta(x)$$

for some $Q_\beta \in \Pi_n^d$. This follows from induction. The case $\beta = 0$ is trivial with $Q_0(x) = 1$. Suppose the equation is true for $|\beta| < n - 1$. Then

$$\begin{aligned} \mathcal{D}^{\beta+e_i}(1 - \|x\|^2)^{n+\mu-1/2} &= \mathcal{D}_i \left[(1 - \|x\|^2)^{n-|\beta|+\mu-1/2}Q_\beta(x) \right] \\ &= \partial_i \left[(1 - \|x\|^2)^{n-|\beta|+\mu-1/2} \right] Q_\beta(x) + (1 - \|x\|^2)^{n-|\beta|+\mu-1/2} \mathcal{D}_i Q_\beta(x) \\ &= (1 - \|x\|^2)^{n-|\beta|+\mu-3/2} \left[-2(n - |\beta| + \mu - 1/2)x_i Q_\beta(x) + (1 - \|x\|^2) \mathcal{D}_i Q_\beta(x) \right] \\ &= (1 - \|x\|^2)^{n-|\beta+e_i|+\mu-1/2} Q_{\beta+e_i}(x), \end{aligned}$$

where $Q_{\beta+e_i}$, as defined above, is clearly a polynomial of degree $|\beta| + 1$, which completes the inductive proof. This formula shows, in particular, that the function $\mathcal{D}^\beta(1 - \|x\|^2)^{n+\mu-1/2}$ vanishes on the boundary of B^d if $|\beta| < n$. For any polynomial $p \in \Pi_{n-1}^d$, using Proposition 2.2 repeatedly gives

$$\begin{aligned} \int_{B^d} U_\alpha(x)p(x)h_\kappa^2(x)(1 - \|x\|^2)^{\mu-1/2}dx &= \int_{B^d} \mathcal{D}^\alpha \left[(1 - \|x\|^2)^{n+\mu-1/2} \right] p(x)h_\kappa^2(x)dx \\ &= (-1)^n \int_{B^d} \mathcal{D}^\alpha p(x)(1 - \|x\|^2)^{n+\mu-1/2}h_\kappa^2(x)dx = 0, \end{aligned}$$

since $\mathcal{D}_i : \mathcal{P}_n^d \mapsto \mathcal{P}_{n-1}^d$, which implies $\mathcal{D}^\alpha p(x) = 0$ as $p \in \Pi_{n-1}^d$. □

By definition, U_α is orthogonal to polynomials of lower degree. However, if $|\alpha| = |\beta|$, U_α and U_β are not necessarily orthogonal. In fact, they are not and there is another family of orthogonal polynomials in $\mathcal{V}_n^d(W_{\kappa,\mu})$ that are biorthogonal to U_α . For its definition, we need the projection operator $\text{proj}_n^B : \mathcal{P}_n^d \mapsto \mathcal{V}_n^d(W_{\kappa,\mu})$. This operator has an explicit formula ([12]). For $p \in \mathcal{P}_n^d$,

$$(2.2) \quad \text{proj}_{B^d} p(x) = \sum_{0 \leq j \leq n/2} \frac{1}{4^j j! (-n - \lambda - \mu + 1)_j} \Delta_h^j p(x),$$

where $\lambda = \gamma_\kappa + (d-1)/2$ and $(a)_j = a(a+1) \dots (a+j-1)$ is the Pochhammer symbol. Let V_κ be the intertwining operator between the family of partial derivatives and Dunkl's operators, which is a linear operator uniquely determined by ([3])

$$V_\kappa 1 = 1, \quad V_\kappa \mathcal{P}_n^d = \mathcal{P}_n^d, \quad \mathcal{D}_i V_\kappa = V_\kappa \partial_i, \quad 1 \leq i \leq d.$$

The explicit formula of V_κ , however, is known only in the case of the group \mathbb{Z}_2^d and S_3 , the symmetric group of three variables. For each $\beta \in \mathbb{N}_0^d$, the function $V_\kappa x^\beta$ is a homogeneous polynomial of degree $|\beta|$. Define

$$R_\beta(x) := \text{proj}_n^B V_\kappa \{x^\beta\} = V_\kappa \left[\sum_{0 \leq j \leq n/2} \frac{1}{4^j j! (-n - \lambda - \mu + 1)_j} \Delta^j x^\beta \right],$$

where the second equality follows from (2.2) and the fact that $\Delta_h V_\kappa = V_\kappa \Delta$. Clearly R_β is an orthogonal polynomial of degree n with respect to $W_{\kappa,\mu}$ and $\{R_\beta : |\beta| = n\}$ is a basis of $\mathcal{V}_n^d(W_{\kappa,\mu})$. It turns out that R_β and U_α are biorthogonal. For $\alpha \in \mathbb{N}_0^d$, write $\alpha! = \alpha_1! \dots \alpha_d!$.

Theorem 2.4. *The two families $\{U_\alpha : |\alpha| = n, \alpha \in \mathbb{N}_0^d\}$ and $\{R_\alpha : |\alpha| = n, \alpha \in \mathbb{N}_0^d\}$ are biorthogonal with respect to $W_{\kappa,\mu}$ on B^d ; more precisely,*

$$a_\kappa \int_{B^d} U_\alpha(x)R_\beta(x)W_{\kappa,\mu}(x)dx = A_\alpha\delta_{\alpha,\beta}, \quad A_\alpha = (-1)^n\alpha! \frac{(\mu + 1/2)_n}{(\lambda + \mu + 1/2)_n}.$$

In particular, $\{U_\alpha : |\alpha| = n\}$ is a basis for $\mathcal{V}_n^d(W_{\kappa,\mu})$.

Proof. Since both U_α and V_β are orthogonal polynomials with respect to $W_{\kappa,\mu}$, we only need to consider the case that they are both polynomials of degree n . Let $\alpha, \beta \in \mathbb{N}_0^d$ and $|\alpha| = |\beta|$. Following the proof of Proposition 2.3, we have

$$\begin{aligned} & a_\kappa \int_{B^d} U_\alpha(x)R_\beta(x)h_\kappa^2(x)(1 - \|x\|^2)^{\mu-1/2}dx \\ &= (-1)^n a_\kappa \int_{B^d} \mathcal{D}^\alpha [R_\beta(x)] (1 - \|x\|^2)^{n+\mu-1/2}h_\kappa^2(x) \\ &= (-1)^n a_\kappa \int_{B^d} \mathcal{D}^\alpha [V_\kappa x^\beta] (1 - \|x\|^2)^{n+\mu-1/2}h_\kappa^2(x) \\ &= \delta_{\alpha,\beta}\alpha!(-1)^n a_\kappa \int_{B^d} (1 - \|x\|^2)^{n+\mu-1/2}h_\kappa^2(x), \end{aligned}$$

since $R_\beta(x) = x^\beta +$ lower degree polynomials and $\mathcal{D}^\alpha V_\kappa x^\beta = V_\kappa \partial^\alpha x^\beta = \alpha! \delta_{\alpha,\beta} V_\kappa 1 = \alpha! \delta_{\alpha,\beta}$. The constant is computed by

$$\begin{aligned} \int_{B^d} (1 - \|x\|^2)^{n+\mu-1/2}h_\kappa^2(x) &= \int_0^1 r^{d-1+2\gamma}(1 - r^2)^{\mu-1/2}dr \int_{S^{d-1}} h_\kappa^2(x')d\omega(x') \\ &= \frac{\Gamma(\gamma + d/2)\Gamma(\mu + 1/2)}{2\Gamma(\gamma + \mu + (d + 1)/2)} \int_{S^{d-1}} h_\kappa^2(x')d\omega(x') \\ &= \frac{(\mu + 1/2)_n}{(\lambda + \mu + 1/2)_n} a_\kappa^{-1}, \end{aligned}$$

since a_κ^{-1} is just the above integral with $n = 0$. Finally, it follows from the biorthogonality that the set $\{U_\alpha : |\alpha| = n\}$ is linearly independent, so that it is a basis of $\mathcal{V}_n^d(W_{\kappa,\mu})$. \square

For our next property of $U_\alpha(x)$, we will need the reproducing kernel $P_n(x, y)$ of $\mathcal{V}_n^d(W_{\kappa,\mu})$. This kernel is characterized by the property that

$$a_\kappa \int_{B^d} P(x, y)q(y)W_{\kappa,\mu}(y)dy = q(x), \quad \forall q \in \mathcal{V}_n^d(W_{\kappa,\mu}).$$

It is shown in [12] that the kernel enjoys a closed formula given by

$$\begin{aligned} (2.3) \quad P_n(x, y) &= \frac{n + \lambda + \mu}{\lambda + \mu} c_\mu \\ &\times V_\kappa \left[\int_{-1}^1 C_n^{\lambda+\mu}(\langle x, \cdot \rangle + t\sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2})(1 - t^2)^{\mu-1}dt \right] (y), \end{aligned}$$

where $c_\mu = 1/\int_{-1}^1 (1 - t^2)^{\mu-1}dt$ and V_κ acts on the variable represented by the dot in the formula. This closed formula for the reproducing kernel has been used to study summability of the Fourier orthogonal expansion on the unit ball. It follows from the biorthogonal property of $\{U_\alpha\}$ and $\{R_\beta\}$ that

$$(2.4) \quad P_n(x, y) = \sum_{|\alpha|=n} A_\alpha^{-1}R_\alpha(x)U_\alpha(y).$$

Indeed, multiplying the right-hand side of the above equation by $R_\beta(y)$ and integrating, the result is $R_\beta(y)$ by the biorthogonality, which shows that the right-hand side satisfies the reproducing property. Since the kernel is unique, it has to equal the left-hand side.

Let $K_n(x, y) = V_\kappa(\langle x, y \rangle^n) / n!$. It is known ([3]) that

$$(2.5) \quad K_n(x, \mathcal{D}^{(y)})p(y) = p(x), \quad \forall p \in \mathcal{P}_n^d,$$

where $\mathcal{D}^{(y)}$ means that \mathcal{D}_i is taken with respect to the variable y and $K_n(x, \mathcal{D}^{(y)})$ is the operator formed by replacing y_i by $\mathcal{D}_i^{(y)}$ in $K_n(x, y)$. A bilinear form $\langle p, q \rangle_h$ is defined on \mathcal{P}_n^d by

$$\langle p, q \rangle_h = p(\mathcal{D})q(x), \quad p, q \in \mathcal{P}_n^d.$$

This form is symmetric in the sense that $\langle p, q \rangle_h = \langle q, p \rangle_h$ ([3]). We have the following lemma that will be useful below.

Lemma 2.5. *Let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = n$ and $q \in \mathcal{P}_j^d$ with $j < n$. Then*

$$(\mathcal{D}^{(y)})^\alpha \left[q(y) V_\kappa^{(y)} \langle x, y \rangle^{n-j} \right] = (n-j)! q(\mathcal{D}) x^\alpha.$$

Proof. Let $\beta \in \mathbb{N}_0^d$ and $|\beta| = j$. As a function in y , $y^\beta V_\kappa \langle x, y \rangle^{n-j}$ is a polynomial in \mathcal{P}_n^d . Using the fact that the bilinear form is symmetric, we get

$$\begin{aligned} (\mathcal{D}^{(y)})^\alpha \left[y^\beta V_\kappa \langle x, y \rangle^{n-j} \right] / (n-j)! &= \langle y^\alpha, y^\beta K_{n-|\beta|}(x, y) \rangle_h \\ &= \langle y^\beta K_{n-|\beta|}(x, y), y^\alpha \rangle_h = K_{n-|\beta|}(x, \mathcal{D}^{(y)}) \mathcal{D}^\beta y^\alpha = \mathcal{D}^\beta x^\alpha, \end{aligned}$$

where the last step follows from the equation (2.5) with $p(y) = \mathcal{D}^\beta y^\alpha$ and n replaced by $n - j$. □

Proposition 2.6. *Let $\alpha \in \mathbb{N}_0^d$ and $|\alpha| = n$. Then*

$$U_\alpha(x) = \sum_{0 \leq 2j \leq n} \frac{2^n (\mu + 1/2)_n}{2^{2j} (\mu + 1/2)_{jj}!} (-1)^{n-j} (1 - \|x\|^2)^j \Delta_h^j x^\alpha.$$

In particular, $U_\alpha(x) = (-1)^n 2^n (\mu + 1/2)_n x^\alpha + (1 - \|x\|^2) Q(x)$, where $Q \in \Pi_{n-2}^d$.

Proof. Since $\mathcal{D}^\alpha R_\beta(y) = \alpha! \delta_{\alpha, \beta}$, applying $(\mathcal{D}^{(y)})^\alpha$ to the equation (2.4) gives

$$\alpha! A_\alpha^{-1} U_\alpha(x) = (\mathcal{D}^{(y)})^\alpha P_n(x, y).$$

We then use the expression of $P_n(x, y)$ in (2.3). Since the leading coefficient of $C_n^\lambda(t)$ is $(\lambda)_n 2^n / n!$, it follows that $P_n(x, y)$, as a function of y , satisfies

$$\begin{aligned} P_n(x, y) &= \frac{n + \lambda + \mu}{\lambda + \mu} \frac{(\lambda + \mu)_n 2^n}{n!} c_\mu \\ &\quad \times V_\kappa \left[\int_{-1}^1 (\langle x, \cdot \rangle + t \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2})^n (1 - t^2)^{\mu-1} dt \right] (y) + \dots \\ &= \frac{(\lambda + \mu + 1)_n 2^n}{n!} c_\mu \sum_{0 \leq 2j \leq n} \binom{n}{2j} b_j (1 - \|x\|^2)^j V_\kappa \left[(1 - \|y\|^2)^j \langle x, \cdot \rangle^{n-2j} \right] (y) \\ &\quad + \text{polynomial of lower degree in } y, \end{aligned}$$

where we have used the binomial formula and the fact that $\int_{-1}^1 t^j(1-t^2)^{\mu-1} dt = 0$ if j is odd, and b_j is given by

$$b_j = c_\mu \int_{-1}^1 t^{2j}(1-t^2)^{\mu-1} dt = \frac{(1/2)_j}{(\mu+1/2)_j}.$$

Consequently, since $\mathcal{D}^\alpha p(x) = 0$ for any $p \in \Pi_m^d$, $m < |\alpha|$, we have

$$\begin{aligned} \alpha! A_\alpha^{-1} U_\alpha(x) &= \frac{(\lambda + \mu + 1)_n 2^{2n}}{n!} c_\mu \sum_{0 \leq j \leq n} \binom{n}{2j} \frac{(1/2)_j}{(\mu + 1/2)_j} (1 - \|x\|^2)^j \\ &\quad \times (\mathcal{D}^{(y)})^\alpha [(1 - \|y\|^2)^j V_\kappa(\langle x, \cdot \rangle^{n-2j})(y)]. \end{aligned}$$

Using Lemma 2.5 with $q(y) = \|y\|^{2j}$, we have

$$\begin{aligned} (\mathcal{D}^{(y)})^\alpha [(1 - \|y\|^2)^j V_\kappa(\langle x, \cdot \rangle^{n-2j})(y)] &= (-1)^j (\mathcal{D}^{(y)})^\alpha [\|y\|^{2j} V_\kappa(\langle x, \cdot \rangle^{n-2j})(y)] \\ &= (-1)^j (n - 2j)! \Delta_h^j x^\alpha. \end{aligned}$$

Putting these equations together we conclude

$$U_\alpha(x) = A_\alpha \frac{(\lambda + \mu + 1)_n 2^{2n}}{\alpha! n!} c_\mu \sum_{0 \leq j \leq n} (-1)^j \frac{n!(1/2)_j}{(2j)!(\mu + 1/2)_j} (1 - \|x\|^2)^j \Delta_h^j x^\alpha.$$

Using the identity $(2j)! = 2^{2j}(1/2)_j j!$ to simplify the constants completes the proof. \square

We note that the expansion is still implicit in a way, since the computation of $\Delta_h^j x^\alpha$ can be difficult. In the case of classical orthogonal polynomials on B^d , Δ_h becomes Δ , and the multinomial formula gives

$$\Delta^j x^\alpha = (\partial_1^2 + \dots + \partial_d^2)^j x^\alpha = \sum_{|\beta|=j} \frac{j!}{\beta!} \partial^{2\beta} x^\alpha = \sum_{|\beta|=j} \frac{j! \alpha!}{\beta! (\alpha - 2\beta)!} x^{\alpha - 2\beta},$$

from which the explicit formula of $U_\alpha(x)$ for W_μ follows. Since Dunkl’s operators commute, the multinomial formula can also be used to expand $\Delta_h x^\alpha$. However, we do not have a nice formula for $\mathcal{D}^\beta x^\alpha$. This is a polynomial of degree $|\alpha| - |\beta|$ in x . For $|\alpha| = |\beta|$ it is equal to $\langle x^\alpha, x^\beta \rangle_h$ and is easily seen to be a polynomial in variable κ . However, we do not know an explicit formula for this quantity. We mention the following recursive formula, which can be used to compute $\mathcal{D}^\beta x^\alpha$ of lower degree.

Proposition 2.7. *Let $\alpha, \beta \in \mathbb{N}_0^d$, $|\beta| \leq |\alpha|$. If $\alpha_i > 0$, then*

$$\mathcal{D}^\beta x^\alpha = x_i \mathcal{D}^\beta x^{\alpha - e_i} + \beta_1 \mathcal{D}^{\beta - e_i} x^{\alpha - e_i} + \sum_{v \in R_+} \kappa_v \langle v, e_i \rangle P_{\beta, v}(\mathcal{D}) x^{\alpha - e_i},$$

where $P_{\beta, v}(y) = (y^\beta - (y\sigma_v)^\beta) / \langle y, \sigma_v \rangle$ is a homogeneous polynomial of degree $|\beta| - 1$.

Proof. We use the product rule of Dunkl’s operators ([4, p. 157])

$$\mathcal{D}_i(fg)(x) = g(x)\mathcal{D}_i f(x) + f(x)\partial_i g(x) + \sum_{v \in R_+} \kappa_v f(x\sigma_v) \langle v, e_i \rangle \frac{g(x) - g(x\sigma_v)}{\langle x, v \rangle}.$$

Using the fact that $\mathcal{D}_i K_n(x, y) = x_i K_{n-1}(x, y)$, the product rule with $g(y) = y^\beta$ and $f(y) = K_{|\alpha|-|\beta|}(x, y)$ shows that

$$\begin{aligned} \mathcal{D}_i^{(y)} [y^\beta K_{|\alpha|-|\beta|}(x, y)] &= y^\beta x_i K_{|\alpha|-|\beta|-1}(x, y) + K_{|\alpha|-|\beta|}(x, y) \beta_i y^{\beta-e_i} \\ &\quad + \sum_{v \in R_+} \kappa_v K_{|\alpha|-|\beta|}(x, y \sigma_v) \langle v, e_i \rangle \frac{y^\beta - (y \sigma_v)^\beta}{\langle x, v \rangle}. \end{aligned}$$

Let $\phi_{\alpha,\beta}(x) = (\mathcal{D}_i^{(y)})^\alpha [y^\beta K_{|\alpha|-|\beta|}(x, y)]$. Since $K_n(xw, yw) = K_n(x, y)$ for all $w \in G$, applying $(\mathcal{D}_i^{(y)})^{\alpha-e_i}$ to the above equation gives

$$\phi_{\alpha,\beta}(x) = x_i \phi_{\alpha-e_i,\beta}(x) + \beta_i \phi_{\alpha-e_i,\beta-e_i}(x) + \sum_{v \in R_+} \kappa_v \langle v, e_i \rangle \sum_{|\tau|=|\beta|-1} c_\tau \phi_{\alpha-e_i,\tau}(x \sigma_v),$$

where $(y^\beta - (y \sigma_v)^\beta) / \langle x, v \rangle = \sum_{|\tau|=|\beta|-1} c_\tau y^\tau$. By Lemma 2.5, $\phi_{\alpha,\beta}(x) = \mathcal{D}^\beta x^\alpha$, which gives the stated formula. \square

3. RELATION TO AN ORTHONORMAL BASIS

The biorthogonality of two bases is useful for finding the orthogonal expansions of a function. For example, for $f \in L^2(W_{\kappa,\mu})$, the Fourier orthogonal expansion of f in terms of the basis $\{U_\alpha\}$ is given by

$$f(x) = \sum_{n=0}^\infty \sum_{|\alpha|=n} b_{\alpha,n} A_\alpha^{-1} U_\alpha(x), \quad b_{\alpha,n} = a_\kappa \int_{B^d} f(x) R_\alpha(x) W_{\kappa,\mu}(x) dx,$$

where the equality holds in the L^2 norm. That the coefficient $b_{\alpha,n}$ is given as above can be easily seen upon integrating the expansion of $f(x) R_\alpha(x)$ over B^d . Thus, the biorthogonality provides an easy way to compute the Fourier coefficient $b_{\alpha,n}$ in the expansion with respect to $U_\alpha(x)$. Note that R_α depends on the intertwining operator V_κ . The following formula proved in [11] is useful for computing integrals of $V_\kappa f$, which works despite the lack of an explicit formula for V_κ .

Lemma 3.1. *For a continuous function f on B^d ,*

$$\int_{S^{d-1}} V_\kappa f(x) h_\kappa^2(x) d\omega(x) = B_\kappa \int_{B^d} f(x) (1 - \|x\|^2)^{\gamma-1} dx$$

where the constant B_κ can be determined by setting $f(x) = 1$.

We use this lemma to work out the expansion in one particular case. Let $C_n^{(\mu,\tau)}(t)$ be polynomials defined by the generating function

$$c_\mu \int_{-1}^1 \frac{1}{(1 - 2xt + t^2)^\mu} (1+t)(1-t^2)^{\tau-1} dt = \sum_{n=0}^\infty C_n^{(\mu,\tau)}(x) t^n.$$

These are orthogonal polynomials with respect to the weight function $w_{\mu,\tau}(t) = |t|^{2\tau} (1-t^2)^{\mu-1/2}$ on $[-1, 1]$ and they are related to the Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$. In particular,

$$(3.1) \quad C_{2n}^{(\mu,\tau)}(t) = \frac{(\mu + \tau)_n}{(\tau + 1/2)_n} P_n^{(\mu-1/2,\tau-1/2)}(2t^2 - 1).$$

It is known that an orthonormal basis of $\mathcal{V}_n^d(W_{\kappa,\mu})$ is given by

$$(3.2) \quad \left\{ f_{j,\beta}(x) := m_j \tilde{C}_{2j}^{(\mu,n-2j+\lambda)}(\|x\|) S_\beta^h(x) : S_\beta^h \in \mathcal{H}_{n-2j}^d(h_\kappa^2), \quad 0 \leq 2j \leq n \right\},$$

in which c_j^{-2} is the $L^2(w_{\lambda,\mu}, [-1, 1])$ norm of $C_{2j}^{(\mu, n-2j+\lambda)}(t)$ and $\{S_\beta^h\}$ is an orthonormal basis of $\mathcal{H}_{n-2j}^d(h_\kappa^2)$. In particular, the polynomial $C_{2n}^{(\mu,\lambda)}(\|x\|)$ is an element of $\mathcal{V}_{2n}^d(W_{\kappa,\mu})$; hence it can be written in terms of $U_\alpha(x)$.

Proposition 3.2. *The following expansion holds:*

$$(3.3) \quad \frac{C_{2n}^{(\mu,\lambda)}(\|x\|)}{C_{2n}^{(\mu,\lambda)}(1)} = \frac{n!}{(\mu + 1/2)_{2n} 2^{2n}} \sum_{|\beta|=n} \frac{1}{\beta!} U_{2\beta}(x).$$

Proof. The fact that $\{U_\alpha : |\alpha| = 2n\}$ is a basis of $\mathcal{V}_{2n}^d(W_{\kappa,\mu})$ shows that we can write

$$C_{2n}^{(\mu,\lambda)}(\|x\|) = \sum_{|\alpha|=2n} b_\alpha A_\alpha^{-1} U_\alpha(x).$$

Using the biorthogonality in Proposition 2.4 and the fact that $R_\kappa(x) = V_\kappa x^\alpha + \dots$, we can compute the coefficient b_α as follows:

$$\begin{aligned} b_\alpha &= a_{\kappa,\mu} \int_{B^d} C_{2n}^{(\mu,\lambda)}(\|x\|) R_\alpha(x) h_\kappa^2(x) (1 - \|x\|^2)^{\mu-1} dx \\ &= a_{\kappa,\mu} \int_{B^d} C_{2n}^{(\mu,\lambda)}(\|x\|) V_\kappa [x^\alpha] h_\kappa^2(x) (1 - \|x\|^2)^{\mu-1} dx \\ &= b_{\kappa,\mu} \int_0^1 r^{2n+d-1+2\gamma} C_{2n}^{(\mu,\lambda)}(r) (1 - r^2)^{\mu-1/2} dr \\ &\quad \cdot c_h \int_{S^{d-1}} V_\kappa(\{\cdot\}^\alpha)(x') h_\kappa^2(x') d\omega(x'), \end{aligned}$$

using the polar coordinates $x = rx'$, $r > 0$ and $x' \in S^{d-1}$, where

$$b_{\kappa,\mu} = 1/ \int_0^1 r^{2\lambda} (1 - r^2)^{\mu-1/2} dr$$

and

$$c_h = 1/ \int_{S^{d-1}} h_\kappa^2(x') d\omega(x').$$

The first integral can be computed using the L^2 norm and the leading coefficient $k_n^{(\mu,\tau)}$ of $C_n^{(\mu,\tau)}(t)$ (cf. [4, p. 27]),

$$\begin{aligned} b_{\kappa,\mu} &\int_0^1 r^{2n+d-1+2\gamma} C_{2n}^{(\mu,\lambda)}(r) (1 - r^2)^{\mu-1/2} dr \\ &= b_{\kappa,\mu} (k^{(\mu,\tau)})^{-1} \int_0^1 [C_{2n}^{(\mu,\lambda)}(r)]^2 (1 - r^2)^{\mu-1/2} dr = \frac{(\mu + 1/2)_n (\lambda + \mu)_n}{(\lambda + \mu + 1)_{2n}}, \end{aligned}$$

The integral of $V_\kappa x^\alpha$ can be computed using Lemma 3.1,

$$c_h \int_{S^{d-1}} V_\kappa(\{\cdot\}^\alpha)(x') h_\kappa^2(x') d\omega = a_\kappa \int_{B^d} x^\alpha (1 - \|x\|^2)^{\gamma-1} dx,$$

where $a_\kappa = 1/ \int_{B^d} (1 - \|x\|^2)^{\gamma-1} dx$, which shows that the integral is zero if one of α_i is odd. If $\alpha = 2\beta$, then the integral over B^d can be evaluated using polar coordinates and the result is

$$c_h \int_{S^{d-1}} V_\kappa(\{\cdot\}^\alpha)(x') h_\kappa^2(x') d\omega = \frac{(1/2)_\beta}{(\lambda + 1/2)_\beta}.$$

Putting these equations together, we have proved that

$$C_{2n}^{(\mu,\lambda)}(\|x\|) = \sum_{|\beta|=n} \frac{(\mu + 1/2)_n(\mu + \lambda)_n}{(\mu + \lambda + 1)_{2n}} \frac{(1/2)_\beta}{(\lambda + 1/2)_\beta} A_{2\beta}^{-1} U_{2\beta}(x).$$

Using the formula of $C_{2n}^{(\mu,\lambda)}(1)$ ([4, p. 27]) and simplifying the coefficients completes the proof. \square

Let us write the expansion explicitly. Using the multinomial formula and the Rodrigues type formula of $U_\beta(x)$, we see that the equation (3.3) gives

$$\begin{aligned} \frac{C_{2n}^{(\mu,\lambda)}(\|x\|)}{C_{2n}^{(\mu,\lambda)}(1)} &= \frac{1}{(\mu + 1/2)_{2n} 2^{2n}} (1 - \|x\|^2)^{-\mu+1/2} \sum_{|\beta|=n} \frac{n!}{\beta!} \mathcal{D}^{2\beta} (1 - \|x\|^2)^{2n+\mu-1/2} \\ &= \frac{1}{(\mu + 1/2)_{2n} 2^{2n}} (1 - \|x\|^2)^{-\mu+1/2} \Delta_h^n (1 - \|x\|^2)^{2n+\mu-1/2}. \end{aligned}$$

In particular, if $\kappa = 0$ and $d = 1$, so that $\lambda = 0$, then $C_{2n}^{(\mu,0)}(\|x\|)$ becomes the Gegenbauer polynomial $C_{2n}^\mu(x)$ and the above formula is precisely the Rodrigues formula for the Gegenbauer polynomials. Furthermore, in polar coordinates, the h -Laplacian can be written as

$$(3.4) \quad \Delta_h = \frac{\partial^2}{\partial r^2} + \frac{2\lambda + 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{h,0},$$

where $\Delta_{h,0}$ is the spherical part of the operator which has h -harmonics as eigenfunctions. More precisely, if $S_n^h \in \mathcal{H}_n^d(h_\kappa^2)$, then

$$(3.5) \quad \Delta_{h,0} S_n^h(x) = -n(n + 2\lambda) S_n^h(x).$$

This leads to the following Rodrigues type formula for $C_{2n}^{(\mu,\lambda)}(t)$.

Proposition 3.3. *For $n \geq 0$,*

$$\frac{C_{2n}^{(\mu,\lambda)}(r)}{C_{2n}^{(\mu,\lambda)}(1)} = \frac{1}{(\mu + 1/2)_{2n} 2^{2n}} (1 - r^2)^{-\mu+1/2} \left(\frac{d^2}{dr^2} + \frac{2\lambda}{r} \frac{d}{dr} \right)^n (1 - r^2)^{2n+\mu-1/2}.$$

By the relation (3.1), this gives a Rodrigues type formula for the Jacobi polynomials, which can be rewritten as

$$P_n^{(\alpha,\beta)}(2r^2 - 1) = \frac{(\alpha + 1)_n}{(\alpha + 1)_{2n} n! 2^{2n}} (1 - r^2)^{-\alpha} \left(\frac{d^2}{dr^2} + \frac{2\beta + 1}{r} \frac{d}{dr} \right)^n (1 - r^2)^{2n+\alpha}.$$

In fact, this formula was discovered by Koornwinder [6] in the course of giving an analytic proof for his addition formula for the Jacobi polynomials.

It is possible to derive a Rodrigues type formula for other elements in the orthonormal basis (3.2). We have the following.

Proposition 3.4. *Let S_{n-2j}^h be an h -spherical harmonic in $\mathcal{H}_n^d(h_\kappa^2)$. Then*

$$\begin{aligned} &\frac{C_{2j}^{(\mu,n-2j+\lambda)}(\|x\|)}{C_{2j}^{(\mu,n-2j+\lambda)}(1)} S_{n-2j}^h(x) \\ &= \frac{1}{(\mu + 1/2)_{2j} 2^{2j}} (1 - \|x\|^2)^{-\mu+1/2} \Delta_h^j \left[(1 - \|x\|^2)^{2j+\mu-1/2} S_{n-2j}^h(x) \right]. \end{aligned}$$

Proof. Let g be a differentiable function defined on $[0, 1]$ and $S_m^h \in \mathcal{H}_m^d(h_\kappa^2)$. We claim that

$$\Delta_h [g(\|x\|)S_m^h(x)] = S_m^h(x) \left(\frac{d^2}{dr^2} + \frac{2\lambda + 2m + 1}{r} \frac{d}{dr} \right) g(r), \quad r = \|x\|.$$

Indeed, since S_m^h is a homogeneous polynomial of degree r , $S_m^h(x) = r^m S_m^h(x')$ under the polar coordinates. Therefore, using (3.4) and (3.5),

$$\begin{aligned} \Delta_h [g(\|x\|)S_m^h(x)] &= \left(\frac{d^2}{dr^2} + \frac{2\lambda + 1}{r} \frac{d}{dr} \right) (r^m g(r)) \cdot S_m^h(x') \\ &\quad - m(m + 2\lambda)g(r)r^{m-2}S_m^h(x') \\ &= \left[\left(\frac{d^2}{dr^2} + \frac{2\lambda + 2m + 1}{r} \frac{d}{dr} \right) g(r) \right] r^m S_m^h(x'), \end{aligned}$$

where the second equality follows from a simple computation. Clearly, we can use the formula repeatedly to get

$$\Delta_h^j [g(\|x\|)S_m^h(x)] = S_m^h(x) \left(\frac{d^2}{dr^2} + \frac{2\lambda + 2m + 1}{r} \frac{d}{dr} \right)^j g(r).$$

Applying the above formula with $g(r) = (1 - r^2)^{2j+\mu-1/2}$ and $m = n - 2j$, the stated formula follows from Proposition 3.3. \square

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