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A TOPOLOGICAL PALEY-WIENER PROPERTY FOR LOCALLY COMPACT GROUPS

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ABSTRACT. We investigate a certain topological Paley-Wiener property and show, for instance, that compact-free nilpotent groups and simply connected solvable groups share this property.

INTRODUCTION

Let f be a bounded and compactly supported measurable function on $\mathbb{R}^n$ ($f \in L_c^\infty(\mathbb{R}^n)$). By the classical Paley-Wiener theorem, the Fourier transform $\widehat{f}$ of f extends to an entire function on $\mathbb{C}^n$. It follows that $f = 0$ whenever $\widehat{f}$ vanishes on a set of positive Lebesgue measure. If, more generally, G is a (second countable) unimodular locally compact group of type I and $\widehat{G}$ denotes the dual space of G, then G is said to satisfy the *weak Paley-Wiener property* if the operator-valued Fourier transform $\pi \to \pi(f)$ of a non-zero function $f \in L_c^\infty(G)$ cannot vanish on a set of positive Plancherel measure on $\widehat{G}$. This weak Paley-Wiener property has been established by several authors for simply connected nilpotent Lie groups [1], [7], [14], [15] and later for completely solvable Lie groups [8].

In this paper we introduce and study another Paley-Wiener type property which can be defined for arbitrary locally compact groups G. We say that G has the *topological Paley-Wiener property* if for every non-zero $f \in L_c^\infty(G)$, the map $\pi \to \pi(f)$ cannot vanish on any non-empty open subset of the reduced dual $\widehat{G}_r$ of G. Our main result is an extension theorem which asserts that if G contains a closed normal subgroup N that has the topological Paley-Wiener property, then so does G, provided that G/N is abelian and compact-free (Theorem 2.2). As a consequence, compact-free nilpotent locally compact groups and simply connected solvable Lie groups share this topological Paley-Wiener property (Corollaries 2.4 and 2.5).

1. PRELIMINARIES AND SOME BASIC RESULTS

Let G be a locally compact group with fixed left Haar measure, and let $C^*(G)$ be the completion of the convolution algebra $L^1(G)$ with respect to the norm $\|f\|_{C^*(G)} = \sup\{\|\pi(f)\|\}$, where the supremum is taken over all $*$-representations π of $L^1(G)$ as an algebra of bounded linear operators in a Hilbert space. Let $P(G)$ be the set of all continuous positive definite functions on G, and let $B(G)$ denote the

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linear span of $P(G)$. Then $B(G)$ can be identified with the dual space $C^*(G)^*$ of $C^*(G)$ by the pairing $\langle f, u\rangle = \int_G f(x)u(x)dx$ for $f \in L^1(G)$ und $u \in B(G)$. With pointwise multiplication and the dual norm, $B(G)$ is a commutative Banach algebra, called the *Fourier-Stieltjes algebra* of G. The *Fourier algebra* $A(G)$ is the norm closure in $B(G)$ of $A_c(G) = B(G) \cap C_c(G)$, the compactly supported functions in $B(G)$. For details regarding $B(G)$ and $A(G)$, see the fundamental paper [5].

As is customary, we shall use the same letter, for example π, for a unitary representation of G and the corresponding $*$-representations of $L^1(G)$ and $C^*(G)$. For any representation π of G, let $A_\pi(G)$ denote the closed linear subspace of $B(G)$ generated by all coefficient functions of π, and $B_\pi(G)$ the w^*-closure of $A_\pi(G)$ in $B(G)$. Then $B_\pi(G)$ can be identified with the dual of the C^*-algebra $\pi(C^*(G))$. Note that, when $\rho = \rho_G$ denotes the regular representation of G on $L^2(G)$, then $A_\rho(G) = A(G)$, and $B_\rho(G)$ equals $B(G)$ if and only if G is amenable.

Moreover, we need to recall some definitions from representation theory. If S and T are sets of unitary representations, then S is weakly contained in T ($S \prec T$) if $\bigcap\{\ker\sigma : \sigma \in S\} \supseteq \bigcap\{\ker\tau : \tau \in T\}$, where $\ker\pi$ denotes the C^*-kernel of a representation π. The sets S and T are said to be weakly equivalent ($S \sim T$) if $S \prec T$ and $T \prec S$. Then, for any two representations σ and τ, $\sigma \prec \tau$ if and only $B_\sigma(G) \subseteq B_\tau(G)$. The support of a representation π, $\operatorname{supp}\pi$, is the closed subset of $\widehat{G}$ consisting of all $\tau \in \widehat{G}$ such that $\tau \prec \pi$. Thus $\pi \sim \widehat{G}$ if and only if $\operatorname{supp}\pi = \widehat{G}$. As general references to dual spaces of locally compact groups and weak containment properties, we mention [4] and [6].

Lemma 1.1. *If G contains a non-trivial compact normal subgroup K, then G cannot have the topological Paley-Wiener property.*

Proof. Let μ_K denote the (normalized) Haar measure of K. Let π be an irreducible representation of G such that $\pi \notin \widehat{G/K}$. If σ is an irreducible subrepresentation of $\pi|_K$, then, since $\sigma \neq 1_K$, by the orthogonality relations $\int_K \langle\sigma(k)\xi, \eta\rangle dk = 0$ for all $\xi, \eta \in \mathcal{H}_\sigma$. It follows that $\pi(\mu_K) = 0$. Now, consider $f = g * \mu_K, g \in C_c(G)$. Then $f \in L_c^\infty(G)$ and $\pi(f) = 0$ for all $\pi \in \widehat{G}_r \setminus \widehat{G/K}$, a non-empty open subset of $\widehat{G}_r$. □

For any $f \in L^1(G)$, let

$$Z_f = \{\pi \in \widehat{G}_r : \pi(f) = 0\},$$

the zero set of f in $\widehat{G}_r$. Note that when studying the topological Paley-Wiener property, we can always assume that the function $f \in L_c^\infty(G)$ actually belongs to $A_c(G)$ since $f * f^*$ is positive definite and $Z_{f*f^*} = Z_f$. We say that $f \in A_c(G)$ generates $B_\rho(G)$ in the w^*-topology if the linear span of all two-sided translates of f is w^*-dense in $B_\rho(G)$.

Lemma 1.2. *Let G be a locally compact group and let $f \in A_c(G)$. Then Z_f has empty interior if and only if f generates $B_\rho(G)$ in the w^*-topology.*

Proof. Let I_f denote the closed ideal of $C_r^*(G)$ generated by $\rho(f)$. If $\overset{\circ}{Z}_f \neq \emptyset$, then the non-zero ideal J of $C_r^*(G)$ defined by $\widehat{J} = \overset{\circ}{Z}_f$ satisfies $J \cap I_f = \{0\}$ since $\widehat{I}_f = \widehat{G}_r \setminus Z_f$. Conversely, if there exists a non-zero ideal J of $C_r^*(G)$ such that $J \cap I_f = \{0\}$, then $\widehat{J} \cap (\widehat{G}_r \setminus Z_f) = \widehat{J} \cap \widehat{I}_f = (J \cap I_f)^\wedge = \emptyset$ and hence $\emptyset \neq \widehat{J} \subseteq Z_f$, so

$\overset{\circ}{Z}_f \neq \emptyset$. Therefore, to prove the lemma it suffices to show that there exists a non-zero ideal J of $C_r^*(G)$ with $J \cap I_f = \{0\}$ if and only if E_f, the w^*-closed subspace of $B_\rho(G)$ generated by all two-sided translates of f, does not coincide with $B_\rho(G)$.

Let Δ denote the modular function of G and for any function h on G, define $\widetilde{h}$ by $\widetilde{h}(y) = \Delta(y^{-1})h(y^{-1})$. Then note that $\langle \rho(h)\rho(g), u\rangle = \langle \rho(g), \widetilde{h} * u\rangle$ for all $g \in C_c(G)$ and $h, u \in A_c(G)$, and hence $\langle \rho(h)T, u\rangle = \langle T, \widetilde{h} * u\rangle$ for all $T \in C_r^*(G)$ and $h, u \in A_c(G)$.

Now let J be an ideal as above. Then $\langle T, \widetilde{f} * u\rangle = 0$ for all $T \in J$ and $u \in A_c(G)$ since $\rho(f)T = 0$. Writing f as $f(\cdot) = \langle \pi(\cdot)\xi, \eta\rangle$ for some representation π of G and $\xi, \eta \in \mathcal{H}_\pi$ and choosing u such that $u \geq 0$ and $\|u\|_1 = 1$, we have

$$\begin{aligned}(f * u)(x) - f(x) &= \int_G u(y^{-1})\langle \pi(x)\pi(y)\xi, \eta\rangle dy - \langle \pi(x)\xi, \eta\rangle \\ &= \left\langle \pi(x)\left(\int_G u(y^{-1})(\pi(y)\xi - \xi)dy\right), \eta\right\rangle.\end{aligned}$$

This implies that

$$\|f * u - f\|_{B_\rho(G)} \leq \|\eta\| \sup\{\|\pi(y)\xi - \xi\| : y^{-1} \in \operatorname{supp} u\}.$$

Since $\langle T, f * u\rangle = 0$, taking for $u \in A_c(G)$ the usual approximate identity for $L^1(G)$, it follows that

$$|\langle T, f\rangle| \leq \|T\| \cdot \|f * u - f\|_{B_\rho(G)} \to 0$$

for each $T \in J$. Since $J^\perp$ is a w^*-closed two-sided translation invariant subspace of $B_\rho(G)$, we conclude that $E_f \subseteq J^\perp$, a proper subspace of $B_\rho(G)$.

Conversely, suppose that $E_f \neq B_\rho(G)$ and let

$$J = \{T \in C_r^*(G) : \langle T, u\rangle = 0 \text{ for all } u \in E_f\}.$$

Then J is a non-zero closed ideal $C_r^*(G)$. We show that $J \cap I_f = \{0\}$. For $T \in C_r^*(G)$, $s, t \in G$ and $u \in A_c(G)$, we have

$$\langle \rho(L_{t^{-1}}R_{s^{-1}}\widetilde{f})T, u\rangle = \Delta(ts)\langle T, (L_sR_tf) * u\rangle.$$

Therefore, it will follow that $J \cap I_f = \{0\}$ once we have verified that if E is a w^*-closed translation invariant subspace of $B_\rho(G)$, then $E * u \subseteq E$ for every $u \in A_c(G)$. This can be seen as follows. If $v(\cdot) = \langle \pi(\cdot)\xi, \eta\rangle \in E$, then

$$(v * u)(\cdot) = \left\langle \pi(\cdot)\left(\int_G u(y^{-1})\pi(y)\xi dy\right), \eta\right\rangle.$$

Given $\epsilon > 0$, there exist $y_1, \ldots, y_n \in G$ and $c_1, \ldots, c_n \geq 0$ such that

$$\left\|\int_G u(y^{-1})\pi(y)\xi dy - \sum_{j=1}^n c_j\pi(y_j)\xi\right\| \leq \epsilon.$$

Then $w(\cdot) = \sum_{j=1}^n c_j\langle \pi(\cdot)\pi(y_j)\xi, \eta\rangle \in E$ and

$$\|v * u - w\|_{B_\rho(G)} \leq \|\eta\| \cdot \left\|\int_G u(y^{-1})\pi(y)\xi dy - \sum_{j=1}^n c_j\pi(y_j)\xi\right\| \leq \epsilon\|\eta\|.$$

This finishes the proof. □

It is worth pointing out that, according to the following lemma, the topological Paley-Wiener property is equivalent to a dichotomy for the subspaces $\overline{B_\pi(G) \cap A_c(G)}$ of $A(G)$, which has been investigated in Section 5 of [13].

Lemma 1.3. *For any locally compact group G, the following two conditions are equivalent:*

(i) *G has the topological Paley-Wiener property.*

(ii) *For any unitary representation π of G, either $B_\pi(G) \cap A_c(G) = \{0\}$ or $\overline{B_\pi(G) \cap A_c(G)} = A(G)$.*

Proof. (i) $\Rightarrow$ (ii) Let π be any representation of G, and suppose there exists $f \in B_\pi(G) \cap A_c(G), f \neq 0$. By (i), Z_f has empty interior and hence f generates $B_\rho(G)$ in the w^*-topology (Lemma 1.2). Thus

$$B_\rho(G) \subseteq \overline{B_\pi(G) \cap A_c(G)}^{w^*} \subseteq B_\pi(G),$$

and this in turn implies that

$$A(G) = \overline{B_\rho(G) \cap A_c(G)} \subseteq \overline{B_\pi(G) \cap A_c(G)},$$

as required.

(ii) $\Rightarrow$ (i) Let f be a non-zero function in $A_c(G)$, and let E denote the closed two-sided invariant subspace of $A(G)$ generated by f. There exists a representation π of G such that $E = \overline{A_\pi(G)}$ [2, Théorème 3.17]. By hypothesis, either $B_\pi(G) \cap A_c(G) = \{0\}$ or $\overline{B_\pi(G) \cap A_c(G)} = A(G)$. However, $f \in B_\pi(G) \cap A_c(G)$. Therefore $\overline{B_\pi(G) \cap A_c(G)} = A(G)$ and hence

$$\overline{E}^{w^*} \subseteq B_\rho(G) = \overline{A(G)}^{w^*} = \overline{B_\pi(G) \cap A_c(G)}^{w^*} \subseteq \overline{E}^{w^*},$$

so that f generates $B_\rho(G)$ in the w^*-topology. By Lemma 1.2, $\mathring{Z}_f \neq \emptyset$. □

An interesting consequence of Lemma 1.3 is that the topological Paley-Wiener property can be interpreted as an approximation property for $A(G)$.

A locally compact group G is called a SIN-group (a group with small invariant neighbourhoods) if there exists a neighbourhood basis of the identity consisting of sets V such that $x^{-1}Vx = V$ for all $x \in G$. SIN-groups provide a large class of locally compact groups comprising all abelian groups, compact groups and discrete groups. For the structure theory of SIN-groups, see [11].

Theorem 1.4. *For a SIN-group G, the following two conditions are equivalent:*

(i) *G contains no non-trivial compact normal subgroup.*

(ii) *G has the topological Paley-Wiener property.*

Proof. This is an immediate consequence of Lemma 1.3 and Theorem 5.6 of [13]. □

We conclude this section with the observation that the topological Paley-Wiener property is in general not inherited by normal subgroups. Indeed, an easy example showing this can be constructed as follows. Let F be a finite abelian group and let $N = \sum_{j=1}^{\infty} F_j$ be the direct sum of copies F_j of F. Let $G = S_\infty \ltimes N$ be the semidirect product, where the infinite symmetric group S_∞ acts on N by permuting the index set $\mathbb{N}$. Obviously, G is a discrete ICC group and hence has the weak Paley-Wiener property (Theorem 1.4), whereas the normal subgroup N is an abelian torsion-group.

2. Extensions and applications

In this section we are going to prove two theorems of the nature that if a locally compact group G contains a closed normal subgroup that has the topological Paley-Wiener property, then under certain additional hypotheses, G also has the topological Paley-Wiener property. Both of these results apply to various (classes of) locally compact groups. A similar, but much less applicable, result has been shown in [13, Proposition 5.3].

Let H be a closed subgroup of G and let τ be a representation of H. The representation of G induced by τ is denoted $\operatorname{ind}_H^G \tau$. In the sequel we shall use that $\pi \otimes \operatorname{ind}_H^G \tau = \operatorname{ind}_H^G(\pi|_H \otimes \tau)$ and that $\pi \prec \operatorname{ind}_H^G(\pi|_H)$ when G is amenable [10, Theorem 5.1].

Lemma 2.1. *Let H be a closed subgroup of the locally compact group G. Let U be a non-empty open subset of $\widehat{G}_r$, and let*

$$V = \{\tau \in \widehat{H}_r : \operatorname{supp}(\operatorname{ind}_H^G \tau) \not\subseteq \widehat{G}_r \setminus U\}.$$

Then V is non-empty and open in $\widehat{H}_r$.

Proof. To show that $\widehat{H}_r \setminus V$ is closed in $\widehat{H}_r$, let $(\tau_\alpha)_\alpha$ be a net in $\widehat{H}_r \setminus V$ such that $\tau_\alpha \to \tau$ for some $\tau \in \widehat{H}_r$. Since inducing is continuous in Fell's subgroup representation topology [6, Theorem 4.2], $\operatorname{ind}_H^G \tau_\alpha \to \operatorname{ind}_H^G \tau$. Since $\operatorname{ind}_H^G \tau_\alpha \prec \widehat{G}_r \setminus U$ for all α and $\widehat{G}_r \setminus U$ is closed in $\widehat{G}_r$, it follows that $\operatorname{supp}(\operatorname{ind}_H^G \tau) \subseteq \widehat{G}_r \setminus U$ and hence $\tau \in \widehat{H}_r \setminus V$.

Assume that $V = \emptyset$, that is, $\operatorname{supp}(\operatorname{ind}_H^G \tau) \subseteq \widehat{G}_r \setminus U$ for all $\tau \in \widehat{H}_r$. Then, since $\rho_G = \operatorname{ind}_H^G \rho_H$, we obtain that

$$\widehat{G}_r \sim \{\operatorname{ind}_H^G \tau : \tau \in \widehat{H}_r\} \sim \bigcup_{\tau \in \widehat{H}_r} \operatorname{supp}(\operatorname{ind}_H^G \tau) \subseteq \widehat{G}_r \setminus U.$$

This contradiction shows that V is non-empty. □

Recall that a locally compact group G is said to be *compact-free* if the identity is the only element of G generating a relatively compact subgroup.

Theorem 2.2. *Let G be a locally compact group containing a closed normal subgroup N such that G/N is abelian and compact-free. If N has the topological Paley-Wiener property, then so does G.*

Proof. Let $f \in C_c(G)$ be such that $\overset{\circ}{Z}_f \neq \emptyset$. We have to show that $L_x f|_N = 0$ for all $x \in G$. Temporarily, fix $\pi \in \overset{\circ}{Z}_f$ and $\xi, \eta \in \mathcal{H}_\pi$, and consider on $\widehat{G/N}$ the function

$$\chi \to \langle (\pi \otimes \chi)(f)(\xi \otimes 1), \eta \otimes 1 \rangle.$$

Define $h \in C_c(G/N)$ by

$$h(xN) = \int_N f(xn)\langle \pi(xn)\xi, \eta \rangle dn,$$

$x \in G$. Then, for $\chi \in \widehat{G/N}$,

$$\begin{aligned} \widehat{h}(\chi) &= \int_{G/N} \chi(xN) \left(\int_N f(xn)\langle \pi(xn)\xi, \eta \rangle \right) d(xN) \\ &= \langle (\pi \otimes \chi)(f)(\xi \otimes 1), \eta \otimes 1 \rangle. \end{aligned}$$

Since G/N is abelian and compact-free, $\widehat{h}$ can vanish only on a set of Haar measure zero in $\widehat{G/N}$ [13, Lemma 5.1]. However, $\widehat{h}$ vanishes on the set

$$X = \{\chi \in \widehat{G/N} : \pi \otimes \chi \in \mathring{Z}_f\},$$

and X is a non-empty open subset of $\widehat{G/N}$ since $\pi \in X$ and the mapping $\chi \to \pi \otimes \chi$ from $\widehat{G/N}$ into $\widehat{G}$ is continuous. We conclude that $h = 0$. Since $\xi, \eta \in \mathcal{H}_\pi$ are arbitrary, it follows that $(\pi \otimes \chi)(f) = 0$ for all $\pi \in \mathring{Z}_f$ and $\chi \in \widehat{G/N}$. Now,

$$\pi \otimes \widehat{G/N} = \{\pi \otimes \chi : \chi \in \widehat{G/N}\} \sim \pi \otimes \operatorname{ind}_N^G 1_N = \operatorname{ind}_N^G(\pi|_N),$$

and since $\pi \prec \rho_G$ implies that $\pi|_N \prec \rho_N$, we get that

$$\operatorname{ind}_N^G(\pi|_N) \prec \operatorname{ind}_N^G \rho_N = \rho_G$$

and therefore $\pi \otimes \widehat{G/N} \subseteq \widehat{G}_r$. Thus we have seen that $\mathring{Z}_f \otimes \widehat{G/N} \subseteq Z_f$. Since $\sigma \to \sigma \otimes \chi$ is a homeomorphism of $\widehat{G}_r$,

$$\mathring{Z}_f \otimes \widehat{G/N} = \bigcup_{\chi \in \widehat{G/N}} \mathring{Z}_f \otimes \{\chi\}$$

is open in $\widehat{G}_r$, whence $\mathring{Z}_f \otimes \widehat{G/N} = \mathring{Z}_f$.

Now, let $x \in G$ and define $g \in C_c(N)$ by $g(n) = f(xn)$. We have to show that $\mathring{Z}_g \neq \emptyset$ for all such functions g. To that end, define a subset V of $\widehat{N}_r$ by

$$V = \{\tau \in \widehat{N}_r : \operatorname{supp}(\operatorname{ind}_N^G \tau) \not\subseteq \widehat{G}_r \setminus \mathring{Z}_f\}.$$

By Lemma 2.1, V is non-empty and open in $\widehat{N}_r$. It remains to verify that $V \subseteq Z_g$. Fix $\tau \in V$ and choose a net $(\pi_\alpha)_\alpha$ in $\widehat{G}_r$ such that $\pi_\alpha|_N \to \tau$ in Fell's topology. We claim that $\pi_\alpha \in \mathring{Z}_f$ eventually. Towards a contradiction, after passing to a subnet if necessary, let us assume that $\pi_\alpha \notin \mathring{Z}_f$ for all α. Since $\mathring{Z}_f = \mathring{Z}_f \otimes \widehat{G/N}$, it follows that

$$\operatorname{ind}(\pi_\alpha|_N) \sim \pi_\alpha \otimes \widehat{G/N} \subseteq \widehat{G}_r \setminus \mathring{Z}_f$$

for all α. Therefore, for each $\sigma \in \operatorname{supp}(\pi_\alpha|_N)$,

$$\operatorname{supp}(\operatorname{ind}_N^G \sigma) \subseteq \operatorname{supp}(\operatorname{ind}_N^G(\pi_\alpha|_N)) \subseteq \widehat{G}_r \setminus \mathring{Z}_f,$$

and hence $\sigma \in \widehat{N}_r \setminus V$. Thus $\pi_\alpha|_N \prec \widehat{N}_r \setminus V$ and so $\tau \in \widehat{N}_r \setminus V$. This contradiction shows that $\pi_\alpha \in \mathring{Z}_f$ eventually. Since

$$\operatorname{ind}_N^G(\pi_\alpha|_N) \sim \pi_\alpha \otimes \widehat{G/N} \subseteq \mathring{Z}_f \otimes \widehat{G/N} \subseteq Z_f,$$

and $\operatorname{ind}(\pi_\alpha|_N) \to \operatorname{ind}_N^G \tau$, it follows that $\operatorname{supp}(\operatorname{ind}_N^G \tau) \subseteq Z_f$. But $\operatorname{ind}_N^G \tau(f) = 0$ implies that $\tau(L_x f|_N) = 0$ (compare [16, Lemma 2.1]), that is, $\tau \in Z_g$. This finishes the proof. □

Corollary 2.3. *Suppose that G possesses a sequence $H_0 = \{e\} \subseteq H_1 \subseteq \ldots \subseteq H_m = G$ of closed subgroups such that H_{j-1} is normal in H_j and H_j/H_{j-1} is compact-free abelian ($1 \leq j \leq m$). Then G has the topological Paley-Wiener property.*

Proof. Using Theorem 2.2, the statement follows immediately by induction on m. □

Corollary 2.4. *Let G be a nilpotent locally compact group. Then G has the topological Paley-Wiener property if and only if G is compact-free.*

Proof. Suppose that G has the topological Paley-Wiener property. Then, by Lemma 1.1, G cannot have a non-trivial compact normal subgroup. It is easy to see that this implies that $G^c = \{e\}$. Indeed, let $\{e\} = Z_0(G) \subseteq Z_1(G) \subseteq \ldots$ denote the ascending central series of G and, assuming that $G^c \neq \{e\}$, let $m \geq 0$ be maximal with the property that $G^c \cap Z_m(G) = \{e\}$. Then pick $a \in G^c \cap Z_{m+1}(G), a \neq e$, and let H be the closed subgroup of G generated by $Z_m(G)$ and a. Then H is normal in G and hence H^c is a closed normal subgroup of G. However, H^c is the closed subgroup generated by a, which is compact.

Conversely, if G is compact-free, then G_0, the connected component of the identity, is simply connected and $D = G/G_0$ is discrete and torsion-free [9, Theorem 8.3]. We now apply Corollary 2.3 as follows. If $G = G_0$, then $Z_m(G)$ and the quotient group $G/Z_m(G)$ are simply connected for all m. Then induction yields that G has the topological Paley-Wiener property. In the general case, note that $D/Z_m(D)$ is torsion-free for all m [3, Corollary 2.11]. Thus, a further induction on the length of nilpotency of D shows that G has the topological Paley-Wiener property. □

Corollary 2.5. *Let G be a simply connected solvable Lie group. Then G has the topological Paley-Wiener property.*

Proof. By the structure theory of simply connected solvable Lie groups, there exists a sequence $H_0 = \{e\} \subseteq H_1 \subseteq \ldots \subseteq H_m = G$ of closed subgroups such that H_{j-1} is normal in H_j and $H_j/H_{j-1} = \mathbb{R}$ (see [12, Satz III.3.30]). Thus G has the topological Paley-Wiener property by Corollary 2.3. □

Theorem 2.6. *Let G be a locally compact group and let N be a closed normal subgroup of G such that G/N is amenable. Suppose there exists a dense subset T of $\widehat{N}_r$ such that for each $\tau \in T$, $\mathrm{ind}_N^G \tau$ is weakly equivalent to some irreducible representation. If N has the topological Paley-Wiener property, then so does G.*

Proof. Let $f \in C_c(G)$ such that $\overset{\circ}{Z}_f \neq \emptyset$. By Lemma 2.1, the set

$$V = \{\tau \in \widehat{N}_r : \mathrm{supp}(\mathrm{ind}_N^G \tau) \not\subseteq \widehat{G}_r \setminus \overset{\circ}{Z}_f\}$$

is non-empty and open in $\widehat{N}_r$. Since T is dense in $\widehat{N}_r$, $T \cap V$ is dense in V. By hypothesis, for every $\tau \in T$ there exists $\pi_\tau \in \widehat{G}$ such that $\pi_\tau \sim \mathrm{ind}_N^G \tau$. Since G/N is amenable, $\pi_\tau \in \widehat{G}_r$. Then $\pi_\tau \in \overset{\circ}{Z}_f$ for each $\tau \in T \cap V$. Indeed, for such τ,

$$\pi_\tau \sim \mathrm{supp}(\mathrm{ind}_N^G \tau) \not\subseteq \widehat{G}_r \setminus \overset{\circ}{Z}_f$$

and hence $\pi_\tau \in \overset{\circ}{Z}_f$. Now, for any $\tau \in V$ there exists a net $(\tau_\alpha)_\alpha$ in $T \cap V$ converging to τ. Then

$$\pi_{\tau_\alpha} \sim \mathrm{ind}_N^G \tau_\alpha \to \mathrm{ind}_N^G \tau,$$

and since $\pi_{\tau_\alpha}(f) = 0$ for all α, it follows that $\mathrm{ind}_N^G \tau(f) = 0$. This in turn implies that $\tau(L_x f|_N) = 0$ for all $x \in G$. Since N has the topological Paley-Wiener property, we conclude that $f = 0$. □

We finish the paper with an example to which Theorem 2.6 applies, but neither Theorem 2.2 nor Proposition 5.3 of [13] applies.

Example 2.7. Let $\mathbb{Z}_2 = \{1, -1\}$, $K = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and $N = \sum_{n=1}^{\infty} \mathbb{Z}$, the restricted direct sum of copies of $\mathbb{Z}$. Let G be the semidirect product $G = K \ltimes N$, where K acts on N by $(\epsilon \cdot x)_n = \epsilon_n x_n$ $(\epsilon = (\epsilon_n)_n \in K, x = (x_n)_n \in N)$. Identifying $\widehat{\mathbb{Z}}$ with $\mathbb{T}$, we have $\widehat{N} = \prod_{n=1}^{\infty} \mathbb{T}$ and, for $z = (z_n)_n \in \widehat{N}$ and $\epsilon = (\epsilon_n)_n \in K$, $\epsilon \cdot z = z$ if and only if $z_n = \overline{z_n}$ for all $n \in \mathbb{N}$ such that $\epsilon_n = -1$. Now, $T = \bigcap_{n=1}^{\infty} \{z \in \widehat{N} : \overline{z_n} \neq z_n\}$ is dense in $\widehat{N}$, and each $z \in T$ has a trivial stability group in K and hence the associated induced representation of G is irreducible. By Theorem 2.6, G has the topological Paley-Wiener property.

In [13, Example 5.4] we have given several other examples of locally compact groups having the topological Paley-Wiener property.

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