

A SPECIAL CASE OF POSITIVITY (II)

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ABSTRACT. In this note we prove the following special case of Serre’s conjecture on Intersection Multiplicity: Let (R, m) be a regular local ring and let P, Q be two prime ideals such that $\ell(R/(P + Q)) < \infty$, $\dim R/P + \dim R/Q = \dim R$ and dimension of $G_m(R/P) \otimes_{G_m(R)} G_m(R/Q) < 2$. Then $\chi(R/P, R/Q) \geq e_m(R/P)e_m(R/Q)$; here $e_m(T)$ denotes the Hilbert-Samuel multiplicity for any finitely generated module T with respect to m .

Let (R, m) denote a regular local ring of dimension n , essentially of finite type over a field L or a discrete valuation ring V . Let $X = \text{Spec } R$, $W_1 = \text{Spec}(R/P)$, $W_2 = \text{Spec}(R/Q)$; P, Q are prime ideals of R . Assume that $\ell(R/(P+Q)) < \infty$. Let $\pi : \tilde{X} \rightarrow X$ be the blow-up of X at $\{m\}$, E the exceptional divisor and $\eta : E \rightarrow \{m\}$ the induced map, i.e., $\eta = \pi|_{\pi^{-1}\{m\}}$. Since R is a regular local ring of dimension n , $E = \mathbb{P}_K^{n-1}$, where $K = R/m$. Let \tilde{W}_1, \tilde{W}_2 denote the blow-ups of W_1 and W_2 at $\{m\}$. The exceptional divisors for \tilde{W}_1 and \tilde{W}_2 are $\tilde{W}_1 \cap E$ and $\tilde{W}_2 \cap E$, respectively. For any finitely generated R -module T , let $e_m(T)$ denote the Hilbert multiplicity of T and let $G_m(T)$ denote the associated graded module $\bigoplus_{n=0}^{\infty} m^n T / m^{n+1} T$.

In this note we intend to prove the following (with notations as above):

Corollary to the Theorem. *Suppose that $\tilde{W}_1 \cap \tilde{W}_2 \cap E$ is either a finite set of points or empty. Then $\chi(R/P, R/Q) \geq e_m(R/P)e_m(R/Q)$.*

Algebraically, the above condition, i.e. $\tilde{W}_1 \cap \tilde{W}_2 \cap E$ is either a finite set of points or empty, is equivalent to stating that the dimension (henceforth \dim) of $G_m(R/P) \otimes G_m(R/Q)$ is less than 2.

Serre [S] showed that the above inequality holds in the equicharacteristic case for any proper intersection. Tennison [T] proved that equality holds (in the above, “ \geq ”) if $\dim G_m(R/P) \otimes G_m(R/Q) = 0$, i.e. $\tilde{W}_1 \cap \tilde{W}_2 \cap E = \emptyset$. Our approach is completely different from those of Serre and Tennison. We use intersection theory as developed by Fulton and MacPherson [Fu] and Gabber’s theorem of non-negativity [B]. The main point is to understand intersection multiplicity as defined by Serre [S] via the blow-up at $\{m\} \times \{\infty\}$ in $\text{Spec } R \times \mathbb{P}^1$, where (R, m) is a regular local ring of essentially finite type over a discrete valuation ring or a field. We mostly

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utilize deformation to the normal cone, localized Chern characters and Riemann-Roch theorems (chapters 5, 12, 15, 18 and 20) from Fulton's beautiful treatise "Intersection Theory" [Fu].

Brief history of the problem. Let (R, m) be a regular local ring and let M, N be two finitely generated R -modules such that $\ell(M \otimes_R N) < \infty$. Following Serre [S], let us define $\chi^R(M, N) = \sum (-1)^i \ell(\text{Tor}_i^R(M, N))$. We drop "R" from the notation when there is no ambiguity. In "Algèbre Locale Multiplicités" ([S]) Serre made the following conjecture:

Conjecture. *Let R be a regular local ring and let M, N be two finitely generated R -modules such that $\ell(M \otimes_R N) < \infty$. Then $\chi(M, N) \geq 0$, where the sign of equality holds if and only if $\dim M + \dim N < \dim R$.*

Serre proved the conjecture in the equicharacteristic and unramified cases. He also showed that if M, N are any pair of finitely generated modules over a regular local ring R such that $\ell(M \otimes_R N) < \infty$, then $\dim M + \dim N \leq \dim R$,

We divide the conjecture into two parts: the *vanishing part* (or simply *vanishing*) when $\dim M + \dim N < \dim R$ and the *positivity part* (or simply *positivity*) when $\dim M + \dim N = \dim R$.

In the mid-eighties P. Roberts [R1] and H. Gillet and C. Soule [G-S2] independently proved the vanishing part. Their proofs also extend to complete intersections when both modules have finite projective dimension. In the mid-nineties Gabber [B] established the non-negativity of χ . However, the positivity part is still very much open. This is the harder part, due to the fact that positivity or non-negativity implies vanishing [D1].

The reason for putting "(II)" in the title is that in the mid-eighties we proved the following in "A special case of positivity" ([D3]): Suppose that (i) R is of mixed characteristic $p (> 0)$, (ii) M is perfect and (iii) p is a non-zero-divisor on M and $p^t N = 0$ for some $t > 0$. Then $\chi(M, N) > 0$, if M and N intersect properly.

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SECTION 1

Let us first state several results from [Fu] which will be used in the proof of our theorem in this section. Here we assume that all our schemes are of finite type and separated over a regular ring.

For any such scheme X , $K^0(X)$ denotes the Grothendieck group of locally free sheaves (vector bundles) on X , $K_0(X)$ denotes the Grothendieck group of coherent sheaves on X , $\mathbb{A}_i(X)$ denotes the Chow group of i -cycles modulo rational equivalence on X , $\mathbb{A}_*(X) = \bigoplus_{i=0}^{\dim X} \mathbb{A}_i(X)$ and $\mathbb{A}_*(X)_{\mathbb{Q}} = \mathbb{A}_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Result 1 (Part of Riemann-Roch Theorem 18.3, [Fu]). For all schemes X there exist homomorphisms: $\tau_X : K_0(X) \rightarrow \mathbb{A}_*(X)_{\mathbb{Q}}$ such that (i) if $f : X \rightarrow Y$ is proper, $\alpha \in K_0(X)$, then $f_* \tau_X(\alpha) = \tau_Y f_*(\alpha)$, (ii) if $\alpha \in K_0(X)$, $\beta \in K^0(X)$, then

$$\tau_X(\beta \otimes \alpha) = \text{ch}(\beta) \cap \tau_X(\alpha),$$

and (iii) if W is a closed subvariety of X , with $\dim W = n$, then

$$\tau_X(O_W) = [W] + \text{terms of dimension } < n.$$

Result 2 (Example 18.3.12, [Fu]). A Riemann-Roch formula. Let E_\bullet be a complex of locally free sheaves on a scheme Y , which is exact off a closed subscheme X . Let \mathcal{F} be a coherent sheaf on Y . Then $\mathcal{H}_i(E_\bullet \otimes \mathcal{F})$, the homology sheaves of $E_\bullet \otimes \mathcal{F}$, have support in X . We have

$$\tau_X[\sum (-1)^i \mathcal{H}_i(E_\bullet \otimes \mathcal{F})] = \text{ch}_X^Y(E_\bullet) \cap \tau_Y(\mathcal{F}).$$

Result 3 (Example 18.3.13 (b), [Fu]). Let X be a non-singular variety and let W_1, W_2 be closed subvarieties of X . Then

$$[W_1] \cdot [W_2] = \sum (-1)^i Z_t(\text{Tor}_i^X(O_{W_1}, O_{W_2}))$$

in $\mathbb{A}_t(W_1 \cap W_2)_\mathbb{Q}$, where $t = \dim W_1 + \dim W_2 - \dim X$ and for any coherent sheaf \mathcal{F} , $Z_t(\mathcal{F})$ denotes the t -cycle determined by \mathcal{F} in its support with dimension of support $\leq t$, i.e.

$$Z_t(\mathcal{F}) = \sum_{\dim V=t} m_V(\mathcal{F})[V],$$

where $m_V(\mathcal{F})$ is the length of the stalk of \mathcal{F} at the generic point of V over the local ring $O_{V,X}$.

Definition. Following Serre, on any regular scheme X we define $\text{Tor}^X(W_1, W_2)$ in $K_0(W_1 \cap W_2)$ for any two closed subschemes W_1 and W_2 of X by

$$\text{Tor}^X(W_1, W_2) = \sum_{i=0}^{\dim X} (-1)^i [\text{Tor}_i^{O_X}(O_{W_1}, O_{W_2})].$$

For details see [S] and [Fu].

Now we are ready to state and prove our theorem. This is an important special case of 20.4.3 in [Fu], however, no direct proof has been offered there. We suspect that the proof, given below, will also work in the general case. See the Remark for more details.

We continue with the notation in the first paragraph of this article.

Theorem. *Let R be a regular local ring of essentially finite type over a field L or a discrete valuation ring V . Let P and Q be two prime ideals of R such that $\ell(R/(P+Q)) < \infty$ and $\dim R/P + \dim R/Q = \dim R$. Then we have the following:*

$$\chi(R/P, R/Q) = e_m(R/P)e_m(R/Q) + \eta_*[\text{Tor}^{\tilde{X}}(\tilde{W}_1, \tilde{W}_2)].$$

Proof. Let B (resp. B_i) denote the blow-up of $X \times \mathbb{P}^1$ (resp. $W_i \times \mathbb{P}^1$) (the product taken over $\text{Spec } L$ or $\text{Spec } V$ depending on the situation) at $\{m\} \times \infty$. Let X' (respectively W'_i) denote the exceptional divisor and let $j : X' \hookrightarrow B$ (resp. $W'_i \hookrightarrow B_i$) denote the corresponding closed immersion. Then $X' = \text{Proj}(C_{\{m\}}X + 1)$ ($W'_i = \text{Proj}(C_{\{m\}}W_i + 1)$)—here $C_{\{m\}}X$ denotes the corresponding tangent cone. The map $X \times \{\infty\} \hookrightarrow X \times \mathbb{P}^1$ (resp. $W_i \times \infty \hookrightarrow W_i \times \mathbb{P}^1$) induces a closed immersion $k : \tilde{X} \hookrightarrow B$ (resp. $\tilde{W}_i \hookrightarrow B_i$). We also have a map $i : X \hookrightarrow B$ ($W_i \hookrightarrow B_i$) obtained by composing $X \hookrightarrow X \times \mathbb{A}^1(x \rightarrow (x, 0))$ with $X \times \mathbb{A}^1 \hookrightarrow B$ (here $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$).

We thus have the following commutative diagram of closed imbeddings ([Fu]):

$$(1) \quad \begin{array}{ccccc} & & X' & \supset & W'_i \\ & & \downarrow j & & \downarrow \\ W_i & \hookrightarrow & X & \xrightarrow{i} & B & \supset & B_i \\ & & \uparrow k & & \uparrow & & \\ & & \tilde{X} & \supset & \tilde{W}_i & & \end{array}$$

Note that i, j, k are all induced by Cartier divisors on B and $i^*, i_*, j^*, j_*, k^*, k_*$ (resp.) are well defined on $K_0(X), \mathbb{A}_*(X)$ ($K_0(B_i), \mathbb{A}_*(B_i)$, etc.) (see Ch. 1, Ch. 2, Ch. 6, and Ch. 15 of [Fu]). We have

$$i^*[B_i] = [W_i], \quad j^*[B_i] = [W'_i] \quad \text{and} \quad k^*[B_i] = [\tilde{W}_i].$$

$X \times \mathbb{P}^1 = \text{Proj } R[T_1, T_2]$; let $X \times \mathbb{A}^1$ correspond to $\text{Spec } R[T_1/T_2]$ and $\{m\} \times \{\infty\}$ correspond to the ideal $(m, T_2/T_1)$ in $\text{Spec } R[T_2/T_1]$. Let O_B (resp. $O_{B_i}, O_{W'_i}$, $O_{\tilde{W}_i}$) denote the corresponding structure-sheaves for B (resp. B_i, W'_i, \tilde{W}_i).

Let us recall that $i_*[W_2] = j_*[W'_2] + k_*[\tilde{W}_2]$ in $\mathbb{A}_*(B_2)$. This is one of the key points of deformation to the normal cone ([Fu], Example 5.1.1). Hence we have

$$(2) \quad i_*(O_{W_2}) = j_*(O_{W'_2}) + k_*(O_{\tilde{W}_2})$$

in $K_0(B_2)$ ([Fu], Example 15.1.5).

Since B is regular, O_{B_1} has a finite resolution by locally free sheaves (vector bundles) E_\bullet on B .

Now we have the following identities:

$$(3) \quad \begin{aligned} & i_*[\text{Tor}^X(W_1, W_2)] \\ &= i_*[\chi^R(R/P, R/Q)[m]] \\ &= \chi^{R[T_1/T_2]}(R[T_1/T_2]/PR[T_1/T_2], R/Q)[m, T_1/T_2] \\ &= \chi^{R[T_1/T_2]}(i^*(O_{B_1}), i_*(O_{W_2}))[m, T_1/T_2] \\ &= [\tau_B(\text{Tor}^B(B_1, i_*(W_2)))] \quad \text{(iii), Result 1)} \\ &= [\text{ch}_{B_1}^B(E_\bullet) \cap \tau_B(i_*(O_{W_2}))] \quad \text{(Result 2)} \\ &= [\text{ch}_{B_1}^B(E_\bullet) \cap \tau_B(j_*(O_{W'_2}) + k_*(O_{\tilde{W}_2}))] \quad \text{(by (2))} \\ &= [\text{ch}_{B_1}^B(E_\bullet) \cap \tau_B j_*(O_{W'_2}) + \text{ch}_{B_1}^B(E_\bullet) \cap \tau_B k_*(O_{\tilde{W}_2})] \\ &= [\text{ch}_{B_1}^B(E_\bullet) \cap j_* \tau_{X'}(O_{W'_2}) + \text{ch}_{B_1}^B(E_\bullet) \cap k_* \tau_{\tilde{X}}(O_{\tilde{W}_2})] \quad \text{(i), Result 1)} \\ &= [j_* \text{ch}_{W'_1}^{X'}(E'_\bullet) \cap \tau_{X'}(O_{W'_2}) + k_* \text{ch}_{\tilde{W}_1}^{\tilde{X}}(\tilde{E}_\bullet) \cap \tau_{\tilde{X}}(O_{\tilde{W}_2})] \end{aligned}$$

(by commutativity of push-forward of localized Chern characters for proper maps,

$$\begin{aligned} & E'_\bullet = j^*(E_\bullet), \tilde{E}_\bullet = k^*(E_\bullet) \\ &= [j_* \tau_{W'_1}(\Sigma(-1)^i \mathcal{H}_i(E'_\bullet \otimes O_{W'_2})) + k_* \tau_{\tilde{W}_1}(\Sigma(-1)^i \mathcal{H}_i(\tilde{E}_\bullet \otimes O_{\tilde{W}_2}))] \quad \text{(by Result 2)} \\ &= [j_* \tau_{X'} \text{Tor}^{X'}(W'_1, W'_2) + k_* \tau_{\tilde{X}} \text{Tor}^{\tilde{X}}(\tilde{W}_1, \tilde{W}_2)] \quad \text{(by proper push-forward).} \end{aligned}$$

Recall that $X' = \mathbb{P}_K^n$ and $\tau_{X'}[\text{Tor}^{X'}(W'_1, W'_2)] = [W'_1] \cdot [W'_2]$ in $\mathbb{A}_0(W'_1 \cap W'_2)_{\mathbb{Q}}$ by Result 3 (here $0 = \dim W'_1 + \dim W'_2 - \dim X'$). By Bezout's theorem (Theorem 12.3 in [Fu]) $[W'_1] \cdot [W'_2]$ is represented by a 0-cycle of degree $e(W'_1) \cdot e(W'_2) = e_m(R/P)e_m(R/Q)$.

Let σ denote the composite of $B \rightarrow X \times \mathbb{P}^1$ and $X \times \mathbb{P}^1 \rightarrow X$ (projection).

Applying σ_* to both sides of (3), we obtain the required result. □

Corollary. (1) *With the same hypothesis as in the theorem, suppose that $\widetilde{W}_1 \cap \widetilde{W}_2 \cap E$ is either a finite set of points or empty. Then*

$$\chi(R/P, R/Q) \geq e_m(R/P)e_m(R/Q).$$

Proof. Note that the condition on $\widetilde{W}_1 \cap \widetilde{W}_2 \cap E$ is equivalent to stating that $\dim(G_m(R/P) \otimes G_m(R/Q)) \leq 1$. Assume that $\dim(G_m(R/P) \otimes G_m(R/Q)) = 1$. Then $Z_0[\text{Tor}^{\widetilde{X}}(\widetilde{W}_1, \widetilde{W}_2)] = \sum r_i[Q_i]$ where the Q_i 's are a finite number of closed points of \widetilde{X} . Since \widetilde{X} is regular, by Gabber's theorem on non-negativity we have $r_i \geq 0$. Thus $\eta_* \text{Tor}^{\widetilde{X}}(\widetilde{W}_1, \widetilde{W}_2) \geq 0$.

If $\dim(G_m(R/P) \otimes G_m(R/Q)) = 0$, then $\widetilde{W}_1 \cap \widetilde{W}_2 = \varnothing$ and hence $\eta_* \text{Tor}^{\widetilde{X}}(\widetilde{W}_1, \widetilde{W}_2) = 0$.

Thus, the result follows from the above theorem. □

Remark. Though no direct proof was offered in Example (20.4.3), the following remark was made: "The proof of (*) given in 12.5 (should be 12.4!) works equally well with no base field." We were inspired by Theorem 12.4(a) and would like to present it here.

Theorem 12.4(a) ([Fu]). *Let V_1, \dots, V_r be pure-dimensional subschemes of a non-singular variety X of dimension n over a field k with $m = \sum_{i=1}^r \dim(V_i) - (r - 1)n$.*

Let P be an isolated point of $\bigcap_{i=1}^r V_i$, rational over k . Let $\pi : \widetilde{X} \rightarrow X$ be the blow-up of X at P , let E be the exceptional divisor and let $\eta : E \rightarrow P$ be the induced map. Let $\widetilde{V}_i \subset \widetilde{X}$ be the blow-up of V_i at P . Then

$$V_1 \bullet V_2 \bullet \dots \bullet V_r = \prod_{i=1}^r e_P(V_i)[P] + \eta_*(\widetilde{V}_1 \bullet \dots \bullet \widetilde{V}_r)$$

in $A_0(P) = \mathbb{Z}$.

In Example 12.4.4 of [Fu] the above result is extended to the case where P is a component of $\bigcap_{i=1}^r V_i$ such that $\dim P = m$ in the following way:

$$i(P, V_1 \bullet \dots \bullet V_r; X) = \prod e_P(V_i) + q,$$

where q is the coefficient of $[P]$ in $\eta_*(\widetilde{V}_1 \bullet \dots \bullet \widetilde{V}_r)$.

Because of Result 3, our theorem can be viewed as an extension of the above theorem, and the above example to regular schemes of finite type over a DVR.

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