

A NOTE ON HARDY-TYPE INEQUALITIES

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ABSTRACT. We use a theorem of Cartlidge and the technique of Redheffer's "recurrent inequalities" to give some results on inequalities related to Hardy's inequality.

1. INTRODUCTION

Suppose throughout that $p \neq 0$, $\frac{1}{p} + \frac{1}{q} = 1$. Let l^p be the Banach space of all complex sequences $\mathbf{a} = (a_n)_{n \geq 1}$ with norm

$$\|\mathbf{a}\| := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} < \infty.$$

The celebrated Hardy's inequality ([11], Theorem 326) asserts that for $p > 1$,

$$(1.1) \quad \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^n a_k \right|^p \leq \left(\frac{p}{p-1} \right)^p \sum_{k=1}^{\infty} |a_k|^p.$$

Among the many papers that have appeared providing new proofs, generalizations and sharpenings of (1.1), we refer the reader to the work of G. Bennett [2]-[6] for his study of factorable matrices.

Hardy's inequality can be regarded as a special case of the following inequality:

$$\sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} c_{j,k} a_k \right|^p \leq U \sum_{k=1}^{\infty} |a_k|^p,$$

in which $C = (c_{j,k})$ and the parameter p are assumed fixed ($p > 1$), and the estimate is to hold for all real sequences \mathbf{a} . The l^p operator norm of C is then defined as the p -th root of the smallest value of the constant U :

$$\|C\|_{p,p} = U^{\frac{1}{p}}.$$

Hardy's inequality thus asserts that the Cesàro matrix operator C , given by $c_{j,k} = 1/j$, $k \leq j$, and 0 otherwise, is bounded on l^p and has norm $\leq p/(p-1)$. (The norm is in fact $p/(p-1)$.)

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We say a matrix A is a summability matrix if its entries satisfy $a_{j,k} \geq 0$, $a_{j,k} = 0$ for $k > j$ and $\sum_{k=1}^j a_{j,k} = 1$. We say a summability matrix A is a weighted mean matrix if its entries satisfy

$$(1.2) \quad a_{j,k} = \lambda_k / \Lambda_j, \quad 1 \leq k \leq j; \quad \Lambda_j = \sum_{i=1}^j \lambda_i, \quad \lambda_i \geq 0, \quad \lambda_1 > 0.$$

We refer to the n -tuple $(a_{n,1}, a_{n,2}, \dots, a_{n,n})$ as the n -th row of a summability matrix A and then have the following result of Bennett ([6], Theorem 1.14) for the l^p operator norm of A .

Theorem 1.1. *Let $p > 1$ be fixed and suppose A is a summability matrix. If the rows of A are decreasing, then $\|A\|_{p,p} \geq p/(p-1)$. If the rows of A are increasing, then $\|A\|_{p,p} \leq p/(p-1)$.*

The above theorem, when applied to weighted mean matrixes, gives the following inequality ([6], Corollary 4.10).

Theorem 1.2. *If $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and $0 < p < 1$, then*

$$\sum_{n=1}^{\infty} \left| \frac{\sum_{i=1}^n \lambda_i a_i^p}{\sum_{i=1}^n \lambda_i} \right|^{1/p} \leq \left(\frac{1}{1-p} \right)^{1/p} \sum_{n=1}^{\infty} |a_n|.$$

Even though the constant in the above theorem is best possible, some improvement may be possible with specific choices of the λ_i 's. For example, the following two inequalities were claimed to hold by Bennett ([5], pages 40-41; see also [6], page 407):

$$(1.3) \quad \sum_{n=1}^{\infty} \left| \frac{1}{n^\alpha} \sum_{i=1}^n (i^\alpha - (i-1)^\alpha) a_i \right|^p \leq \left(\frac{\alpha p}{\alpha p - 1} \right)^p \sum_{n=1}^{\infty} |a_n|^p,$$

$$(1.4) \quad \sum_{n=1}^{\infty} \left| \frac{1}{\sum_{i=1}^n i^{\alpha-1}} \sum_{i=1}^n i^{\alpha-1} a_i \right|^p \leq \left(\frac{\alpha p}{\alpha p - 1} \right)^p \sum_{n=1}^{\infty} |a_n|^p,$$

whenever $\alpha > 0, p > 1, \alpha p > 1$.

We have no access to the proofs of Bennett and, as far as we know, his proofs are not yet published. However, we find that the following unpublished result of J. Carlidge [8] is very useful to treat the above two inequalities. Here we quote his theorem according to [2] (Theorem C, p. 416):

Theorem 1.3. *Let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by (1.2). If*

$$(1.5) \quad L = \sup_n \left(\frac{\Lambda_{n+1}}{\lambda_{n+1}} - \frac{\Lambda_n}{\lambda_n} \right) < p,$$

then $\|A\|_{p,p} \leq p/(p-L)$.

We will apply the above theorem to prove (1.3) for $\alpha \geq 1, p > 1, \alpha p > 1$ and (1.4) for $p > 1, \alpha \geq 2, \alpha p > 1$ and $p > 1, 0 < \alpha \leq 1, \alpha p > 1$ in section 3.

Suppose $a_n \geq 0$. By a change of variables $a_n \rightarrow a_n^{1/p}$ and letting $p \rightarrow \infty$, (1.1) gives the well-known Carleman's inequality:

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k \right)^{\frac{1}{n}} \leq e \sum_{n=1}^{\infty} a_n.$$

We refer the reader to the survey article [14] and the references therein for an account of Carleman’s inequality. Among the various generalizations of Carleman’s inequality, we mention a result of E. Love [13], who proved that for $\alpha > 0, \lambda_i = i^\alpha - (i - 1)^\alpha$,

$$(1.6) \quad \sum_{n=1}^{\infty} \left(\prod_{i=1}^n a_i^{i^\alpha - (i-1)^\alpha} \right)^{1/n^\alpha} \leq e^{\frac{1}{\alpha}} \sum_{n=1}^{\infty} a_n,$$

and the constant $e^{\frac{1}{\alpha}}$ is best possible. We note here that after a change of variables $a_n \rightarrow a_n^{1/p}$, (1.6) corresponds to the limiting case $p \rightarrow \infty$ of (1.3).

R. Redheffer gave a remarkable proof of Hardy’s inequality in [15] by developing the method of “recurrent inequalities”. His method also works for Carleman’s inequality. Another proof of Carleman’s inequality was given by him in [16], and his result has been generalized by H. Alzer [1] and most recently by J. Pečarić and K. Stolarsky [14], who proved for $b_n > 0, N \geq 1, G_n = (\prod_{i=1}^n a_i)^{1/n}$,

$$\sum_{n=1}^N \Lambda_n (b_n - 1) G_n + \Lambda_N G_N \leq \sum_{n=1}^N \lambda_n G_n b_n^{\Lambda_n / \lambda_n}.$$

In this paper, we will use Redheffer’s method to give a weighted version of his treatment of Hardy’s and Carleman’s inequalities. As we shall see, our result for $1 < p < \infty$ is less satisfactory than that of Cartlidge’s and the result is almost the same as his for the limiting case.

From now on we will assume $a_n \geq 0$ for $n \geq 1$ and any infinite sum converges.

2. LEMMAS

Lemma 2.1. *If $s \geq 1, n \geq 1$, then*

$$(2.1) \quad \frac{(n + 1)^s}{(n + 1)^s - n^s} - \frac{n^s}{n^s - (n - 1)^s} \leq \frac{1}{s},$$

$$(2.2) \quad \frac{s}{s + 1} \frac{n^s (n + 1)^s}{(n + 1)^s - n^s} \leq \sum_{i=1}^n i^s.$$

Inequality (2.2) reverses when $-1 < s \leq 1$.

Proof. This is a result of V. Levin and S. Stečkin, see Lemma 2 on page 18 in [12]. Inequality (2.1) is equivalent to inequality (2) in their proof of Lemma 2. They only proved the case $s \geq 0$ for (2.2), but one checks easily that their proof extends to the case $s > -1$. □

Lemma 2.2 ([6, Lemma 2.11]). *Fix $p > 1$. Suppose that A, B are matrices with non-negative entries, and that the rows of B are decreasing, i.e.*

$$(2.3) \quad b_{j,k} \geq b_{j,k+1} \quad (j, k = 1, 2, \dots).$$

If

$$(2.4) \quad \sum_{k=1}^n a_{j,k} \geq \sum_{k=1}^n b_{j,k} \quad (j, n = 1, 2, \dots),$$

then $\|A\|_{p,p} \geq \|B\|_{p,p}$.

Lemma 2.3 ([7, Proposition 5]). *For $n \geq 1, 0 \leq \alpha < 1$,*

$$\left(\frac{\sum_{i=1}^n i^{\alpha-1}/n}{\sum_{i=1}^{n+1} i^{\alpha-1}/(n+1)}\right)^{\frac{1}{\alpha-1}} \geq \frac{n+1}{n+2} \geq \frac{n}{n+1}.$$

Lemma 2.4. *Let $\Lambda_k = \sum_{i=1}^k \lambda_i, \lambda_i > 0$ and $S_n = \sum_{i=1}^n \lambda_i a_i$. Let $0 \neq p < 1$ be fixed and let $(\mu_i)_{i \geq 1}, (\eta_i)_{i \geq 1}$ be two sequences of real numbers such that $\mu_i \leq \eta_i$ for $0 < p < 1$ and $\mu_i \geq \eta_i$ for $p < 0$. Then for $n \geq 2$,*

$$(2.5) \quad \sum_{i=2}^{n-1} [\mu_i - (\mu_{i+1}^q - \eta_{i+1}^q)^{1/q}] S_i^{1/p} + \mu_n S_n^{1/p} \leq (\mu_2^q - \eta_2^q)^{1/q} \lambda_1^{1/p} a_1^{1/p} + \sum_{i=2}^n \eta_i \lambda_i^{1/p} a_i^{1/p}.$$

Proof. This is essentially due to R. Redheffer [15]. We note for $k \geq 2$,

$$(2.6) \quad \mu_k S_k^{1/p} - \eta_k \lambda_k^{1/p} a_k^{1/p} = S_{k-1}^{1/p} (\mu_k (1+t)^{1/p} - \eta_k t^{1/p}) \leq (\mu_k^q - \eta_k^q)^{1/q} S_{k-1}^{1/p},$$

with $t = \lambda_k a_k / S_{k-1}$ (compare this with the one on page 688 of [15]). The lemma then follows by summing (2.6) for $2 \leq k \leq n$. \square

Lemma 2.5. *Let $\Lambda_k = \sum_{i=1}^k \lambda_i, \lambda_i > 0$ and $G_k = (\prod_{i=1}^k a_i^{\lambda_i})^{1/\Lambda_k}$. Then for $\mu_i > 0, n \geq 2$,*

$$G_1 + \sum_{i=2}^{n-1} \left(\frac{\Lambda_i \mu_i}{\lambda_i} - \frac{\Lambda_i}{\lambda_{i+1}}\right) G_i + \frac{\Lambda_n \mu_n}{\lambda_n} G_n \leq \left(1 + \frac{\Lambda_1}{\lambda_2}\right) a_1 + \sum_{i=2}^n \mu_i \frac{\Lambda_i}{\lambda_i} a_i.$$

Proof. This is essentially due to R. Redheffer [15]. We note for $k \geq 2, \mu > 0, \eta > 0$,

$$\mu G_k - \eta a_k = G_{k-1} (\mu t - \eta t^{\frac{\Lambda_k}{\lambda_k}}) \leq G_{k-1} \left(\frac{\Lambda_{k-1}}{\lambda_k}\right) \eta^{\frac{-\lambda_k}{\Lambda_{k-1}}} \left(\frac{\mu \lambda_k}{\Lambda_k}\right)^{\frac{\Lambda_k}{\Lambda_{k-1}}},$$

where $t^{\frac{\Lambda_k}{\lambda_k}} = a_k / G_{k-1}$ (compare this with the one on page 686 of [15]). By setting $\mu_k \Lambda_k / \lambda_k = \mu, \eta = \mu_k^{\frac{\Lambda_k}{\lambda_k}}$, we get

$$(2.7) \quad \frac{\Lambda_k \mu_k}{\lambda_k} G_k - a_k \mu_k^{\frac{\Lambda_k}{\lambda_k}} \leq \frac{\Lambda_{k-1}}{\lambda_k} G_{k-1}.$$

The lemma then follows by summing (2.7) for $2 \leq k \leq n$ and adding $G_1 = a_1$. \square

3. APPLICATIONS OF CARTLIDGE'S THEOREM

We say a weighted mean matrix A given by (1.2) is generated by a logarithmico-exponential function if for all sufficiently large $n, \lambda_n := l(n)$, where $l(x)$ is a positive logarithmico-exponential function, and a logarithmico-exponential function on $[x_0, \infty]$ is defined by Hardy [10] as a real-valued function defined by a finite combination of ordinary algebraic symbols (viz, $+, -, \times, \div, \sqrt{}$) and the functional symbols $\log(\cdot)$ and $e^{(\cdot)}$, operating on the real variable x and on real constants.

We note first the following theorem of F. Cass and W. Kratz [9]:

Theorem 3.1. *Let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by (1.2), with A generated by a logarithmico-exponential function. Suppose $\lim_{n \rightarrow \infty} \Lambda_n / n \lambda_n = L < p$. Then $\|A\|_{p,p} < \infty$ if and only if $L < p$, in which case $p/(p-L) \leq \|A\|_{p,p}$.*

It is easy to see $\lim_{n \rightarrow \infty} n^{\alpha-1}/(n^\alpha - (n-1)^\alpha) = 1/\alpha$ and the simplest Euler-Maclaurin formulae give

$$\sum_{i=1}^n f(i) = \int_1^n f(x)dx + f(1) + \int_1^n (x - [x])f'(x)dx,$$

for f having continuous derivative f' , where $[x]$ denotes the largest integer not exceeding x . It then follows that

$$\sum_{i=1}^n i^{\alpha-1} = n^\alpha/\alpha + o(n^\alpha).$$

Thus for any fixed $p > 1$, thanks to Theorem 3.1, we know (1.3)-(1.4) can only hold for $\alpha > 0, \alpha p > 1$, and if they hold for some $\alpha > 0, \alpha p > 1$, then the constant $(\alpha p/(\alpha p - 1))^p$ is best possible.

Now we apply Cartlidge's Theorem to get

Corollary 3.1. *Inequality (1.3) holds for $p > 1, \alpha \geq 1, \alpha p > 1$ and the constant there is best possible.*

Proof. Apply Theorem 1.3 with $\lambda_i = i^\alpha - (i-1)^\alpha$. It follows from Lemma 2.1 that $\Lambda_{n+1}/\lambda_{n+1} - \Lambda_n/\lambda_n \leq 1/\alpha$, for $n \geq 1, p > 1, \alpha \geq 1, \alpha p > 1$, and this completes the proof. \square

We note Corollary 3.1 implies (1.6) for $\alpha \geq 1$. Now if we apply Theorem 1.3 to (1.4), we need to show

$$\sum_{i=1}^{n+1} i^{\alpha-1}/(n+1)^{\alpha-1} - \sum_{i=1}^n i^{\alpha-1}/n^{\alpha-1} = 1 + \left(\frac{1}{(n+1)^{\alpha-1}} - \frac{1}{n^{\alpha-1}}\right) \sum_{i=1}^n i^{\alpha-1} \leq 1/\alpha.$$

By Lemma 2.1 we get

Corollary 3.2. *Inequality (1.4) holds for $p > 1, \alpha \geq 2, \alpha p > 1$ and $p > 1, 0 < \alpha \leq 1, \alpha p > 1$ and the constant there is best possible.*

We note here (1.5) fails to hold when $0 < \alpha < 1$ in (1.3) or $1 < \alpha < 2$ in (1.4) by checking the case $n = 1$. We also note if (1.3) holds for $p > 1, 0 < \alpha \leq 1, \alpha p > 1$, then (1.4) holds for $p > 1, 0 < \alpha \leq 1, \alpha p > 1$ since we can apply Lemma 2.2, here with A the matrix given by (1.3) and B the matrix given by (1.4). It is easy to check that condition (2.3) is satisfied by B and (2.4) follows from Lemma 2.3.

4. GENERALIZATIONS OF REDHEFFER'S RESULTS

Theorem 4.1. *Assume the same conditions in Lemma 2.4 and let $0 < p < 1$ be fixed. Suppose there exists a positive constant c such that $c^{-1} + 1 \leq c^{-1/p}$ and*

$$(4.1) \quad c \leq 1 - p + (1 - p)(\lambda_i^{-q} - \lambda_{i-1}^{-q})\Lambda_{i-1}\lambda_i^{q/p}, i \geq 2.$$

Then for $0 < p < 1$,

$$(4.2) \quad \sum_{i=1}^{\infty} (S_i/\Lambda_i)^{1/p} \leq c^{-1/p} \sum_{i=1}^{\infty} a_i^{1/p}.$$

Proof. It suffices to prove the theorem for any integer $n \geq 1$. We note first that condition (4.1) is equivalent to

$$(4.3) \quad q^{-1}(1 - c^{-1} + c^{-1}\Lambda_{i-1}\lambda_i^{q/p}(\lambda_{i-1}^{-q} - \lambda_i^{-q})) \geq 1, i \geq 2.$$

By setting $\eta_i = \lambda_i^{-1/p}, \mu_i^q = \lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q$ in (2.5), we can rewrite the left-hand side of (2.5) as

$$\sum_{i=2}^{n-1} [(\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q)^{1/q} - (\Lambda_i/c\lambda_i^q)^{1/q}] S_i^{1/p} + \mu_n S_n^{1/p}.$$

By the mean value theorem,

$$\begin{aligned} & (\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q)^{1/q} - (\Lambda_i/c\lambda_i^q)^{1/q} \\ & \geq q^{-1}(\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q - \Lambda_i/c\lambda_i^q)(\Lambda_i/c\lambda_i^q)^{-1/p} \\ & = q^{-1}(1 - c^{-1} + c^{-1}\Lambda_{i-1}\lambda_i^{q/p}(\lambda_{i-1}^{-q} - \lambda_i^{-q}))(\Lambda_i/c)^{-1/p} \\ & \geq (\Lambda_i/c)^{-1/p}. \end{aligned}$$

Here the last inequality follows from (4.3). Thus (2.5) becomes

$$\sum_{i=1}^n (S_i/\Lambda_i)^{1/p} \leq (c^{-1} + 1)a_1^{1/p} + c^{-1/p} \sum_{i=2}^n a_i^{1/p} \leq c^{-1/p} \sum_{i=1}^n a_i^{1/p}.$$

This completes the proof. □

We note here if $0 < \lambda_1 \leq \lambda_2 \leq \dots$, we can take $c = 1 - p$ in (4.1) and one checks easily for $0 < p < 1$, $(1 - p)^{-1} + 1 < (1 - p)^{-1/p}$. Theorem 4.1 then implies Theorem 1.2.

We also note the constant given by the above theorem may be less satisfactory. For example the case $\alpha = 2, p = 2$ in (1.3) corresponds to the case $\lambda_i = 2i - 1, p = 1/2, c = 3/4$ in (4.2). However, direct calculation shows (4.1) is not satisfied in this case. Of course one may try to prove directly

$$(\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q)^{1/q} - (\Lambda_i/c\lambda_i^q)^{1/q} \geq (\Lambda_i/c)^{-1/p}.$$

But one checks this fails for $i = 2$.

Similarly, the case $\alpha = 2, p = 2$ in (1.4) corresponds to the case $\lambda_i = i, p = 1/2, c = 3/4$ in (4.2). One checks in this case that (4.1) holds for $i \geq 2$. However, $c^{-1} + 1 = 7/3 > 16/9 = c^{-2}$, so the coefficient of a_1 is slightly larger.

Now we focus our attention on Carleman-type inequalities.

Theorem 4.2. *Assume the same conditions in Lemma 2.5, and let $f(x)$ be a real-valued function defined for $x \geq 2$ such that $f(n) = \Lambda_n/\lambda_n$ for $n \geq 2$ and $0 \leq f(x+1) - f(x) \leq 1/\alpha$ for some $\alpha > 0$. If $(1 + \frac{\Lambda_1}{\lambda_2}) \leq e^{1/\alpha}$ for the same α , then*

$$\sum_{n=1}^{\infty} \left(\prod_{i=1}^n a_i^{\lambda_i} \right)^{1/\Lambda_n} \leq \left(1 + \frac{\Lambda_1}{\lambda_2} \right) a_1 + \sum_{i=2}^n a_i \left(1 + \frac{f(i+1) - f(i)}{f(i)} \right)^{f(i)} \leq e^{1/\alpha} \sum_{n=1}^{\infty} a_n.$$

Proof. It suffices to prove the theorem for any integer $n \geq 2$. Setting $\mu_i = f(i+1)/f(i)$ in Lemma 2.5 we get

$$\begin{aligned} \sum_{i=1}^n G_i &\leq \sum_{i=1}^{n-1} G_i + f(n+1)G_N \leq \left(1 + \frac{\Lambda_1}{\lambda_2}\right)a_1 + \sum_{i=2}^n a_i \left(1 + \frac{f(i+1) - f(i)}{f(i)}\right)^{f(i)} \\ &\leq e^{1/\alpha} \sum_{n=1}^{\infty} a_n \end{aligned}$$

by the conditions of the theorem, and this completes the proof. \square

Apply Theorem 4.2 to $\lambda_1 = 1, \lambda_i = \alpha^{i-1} - \alpha^{i-2}, i \geq 2$ for some $\alpha > 1$. Then $f(x) = \alpha/(\alpha - 1)$ and we get

Corollary 4.1. For $\alpha > 1$,

$$\sum_{n=1}^{\infty} \left(a_1 \prod_{k=2}^n a_k^{\alpha^{k-1} - \alpha^{k-2}}\right)^{1/\alpha^{n-1}} \leq \left(1 + \frac{1}{\alpha - 1}\right)a_1 + \sum_{n=2}^{\infty} a_n.$$

Apply Theorem 4.2 to $\lambda_i = \alpha^i, i \geq 1$ for some $\alpha > 0$. Then $f(i+1) - f(i) = \alpha^{-i}$ and we get

Corollary 4.2. For $\alpha > 0$,

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{\alpha^{k-1}}\right)^{(\alpha^n - 1)/(\alpha - 1)} \leq \left(1 + \frac{1}{\alpha}\right)a_1 + \sum_{n=2}^{\infty} e^{1/\alpha^n} a_n \leq \sum_{n=1}^{\infty} e^{1/\alpha^n} a_n.$$

We end the paper by noting that if we take $\lambda_i = (i(i+1))^{-1}$ in Theorem 4.2, then $f(x) = x^2$ and we get back a result of Redheffer (see [15], page 693):

Corollary 4.3.

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k(k+1)}\right)^{(n+1)/n} \leq \sum_{n=1}^{\infty} e^{2/n} a_n.$$

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