

GAUSSIAN POLYNOMIALS AND INVERTIBILITY

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ABSTRACT. The content of a polynomial f over a commutative ring R is the ideal $c(f)$ of R generated by the coefficients of f . If $c(fg) = c(f)c(g)$ for each polynomial $g \in R[x]$, then f is said to be Gaussian. If $c(f)$ is an invertible ideal of R , then f is Gaussian. An open question has been whether the converse holds for a polynomial whose content is a regular ideal of R . The main theorem shows slightly more than this; namely, if $c(f)$ has no nonzero annihilators, then $c(f)\text{Hom}_R(c(f), R) = R$.

For a pair of commutative rings $R \subseteq S$, the R -content of a polynomial $g \in S[x]$ is the R -submodule, $c(g)$, of S generated by the coefficients of g . If $f \in R[x]$, $c(f)$ is an ideal of R . The Dedekind-Mertens content formula states that for a pair of polynomials f and g over a ring R , there is an integer $m \geq 0$ such that $c(f)^m c(fg) = c(f)^{m+1} c(g)$ (see, for example, [G, Theorem 28.1]). In 1998, Heinzer and Huneke showed that $m + 1$ need not be larger than the number of generators needed to generate $c(g)$ locally [HH2, Theorem 2.1]. Previously, the bound for m was known to be less than or equal to the number of nonzero coefficients of g . If $c(f)$ is an invertible ideal of R , multiplying both sides by the appropriate power of the inverse of $c(f)$ reduces the formula to $c(fg) = c(f)c(g)$ no matter the degree of g . More generally, if $c(f)R_M$ is principal for each maximal ideal M (but perhaps with a nonzero annihilator for some or all M), then Nakayama's Lemma can be used to show that $c(f)c(g)R_M = c(fg)R_M$ for each M (or simply apply the aforementioned [HH2, Theorem 2.1] with the roles of f and g reversed). The conclusion is then reached by noting that an ideal is completely characterized by its localizations at maximal ideals (see, for example, [A], or [AK, Theorem 1.1]). A polynomial $f \in R[x]$ with the property that $c(fg) = c(f)c(g)$ for each $g \in R[x]$ (equivalently, each $g \in T(R)[x]$) is said to be Gaussian. A problem often associated with Kaplansky is to determine whether or not every Gaussian polynomial with regular content has invertible content. In her dissertation, Kaplansky's student Hwa Tsang proved several results about Gaussian polynomials [T1]. The conjecture that a Gaussian polynomial over an integral domain has invertible content appears there. She later published a portion of her work under the name Hwa Tsang Tang [T2]. In the late 1990s, Heinzer and Huneke [HH1] and Glaz and Vasconcelos [GV] answered the implied question in the affirmative in various Noetherian and Noetherian-like settings. The article by Corso and Glaz [CG] gives a good account of what was

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known about the problem prior to the year 2000 or so. At the October 2003 meeting of the American Mathematical Society in Chapel Hill, North Carolina, Alan Loper presented a proof that every Gaussian polynomial over an integral domain has invertible content (in fact, for any ring that is locally an integral domain). The basis of the proof presented in this paper is highly dependent on the work of Loper and his coauthor Moshe Roitman [LR]. Leading up to their proof that a Gaussian polynomial over a commutative ring has invertible content when the ring is locally an integral domain, Loper and Roitman established several interesting properties of Gaussian polynomials for integral domains. Most of these properties can be easily extended to the general setting of commutative rings, frequently with no substantial change in the proof. Also important to the proof presented here is work of Eakin and Sathaye involving stable and prestable ideals [ES] (also employed in both [GV] and [LR]) and a theorem due to Gilmer and Hoffman involving the saturation in $S[x]$ of the set of unit content polynomials of a ring R when S is integral over R [GH, Theorem 3].

We assume throughout that all rings are commutative with $1 \neq 0$. For a ring R , we let $T(R)$ denote the total quotient ring of R and $Q_0(R)$ denote the ring of finite fractions over R . The ring of finite fractions over R can be viewed either as the subring of $T(R[x])$ (the total quotient ring of the polynomial ring $R[x]$) consisting of those elements $b(x)/a(x)$ for which $a(x) = \sum a_i x^i$, $b(x) = \sum b_i x^i \in R[x]$ with $a(x)$ regular (i.e., not a zero divisor) and $a_i b_j = a_j b_i$ for all i and j , or as the equivalence classes of R -module homomorphisms on semiregular ideals with two such homomorphisms declared to be equivalent if they agree on some semiregular ideal. An ideal is semiregular if it contains a finitely generated ideal that has no nonzero annihilators. The latter method of construction is modeled after the construction Lambek presents for the complete ring of quotients [La, Section 2.3]. As with the complete ring of quotients, there is a natural embedding of R and $T(R)$ into $Q_0(R)$ obtained by identifying an element a/b in $T(R)$ (where $a, b \in R$ with b regular) with the R -module homomorphism from the regular ideal bR into R defined by multiplication by a/b . For more details on these constructions see, for example, [Lu2].

In our main theorem we show that if $c(f)$ is a semiregular ideal of R (equivalently, f is a regular element of $R[x]$), then f is Gaussian if and only if $c(f)$ is Q_0 -invertible; i.e., there is an R -submodule J of $Q_0(R)$ such that $c(f)J = R$. The latter is equivalent to $c(f)$ being (semiregular and) locally principal [Lu2, Lemma 6]. According to [AAM], a ring R is strongly Prüfer if each finitely generated semiregular ideal is locally principal. Theorem 8 of [Lu2] shows that R is strongly Prüfer if and only if each finitely generated semiregular ideal is Q_0 -invertible. Rings exist in which each finitely generated semiregular ideal is Q_0 -invertible but not all finitely generated semiregular ideals are regular [see, for example, [H, Example 17] or [Lu2, Example 14]]. Thus there are Gaussian polynomials which are regular but do not have invertible content in the traditional sense of invertibility with regard to the total quotient ring.

The first lemma is part of [LR, Proposition 1].

Lemma 1. *Let $f(x) \in R[x]$. If $f(x^n)$ is Gaussian for some positive integer n , then $f(x)$ is Gaussian.*

Proof. The proof follows immediately from noting that for each positive integer n , $c(f(x)g(x)) = c(f(x^n)g(x^n))$ and $c(f(x))c(g(x)) = c(f(x^n))c(g(x^n))$. \square

The proof of the next lemma is quite similar to that of the first. The proof hints at a useful technique.

Lemma 2. *Let $f(x) \in R[x]$. If $f(x)$ is Gaussian, then $f(-x)$ is Gaussian.*

Proof. Simply note that $f(x)g(x)$ and $f(-x)g(-x)$ have the same content. \square

The next lemma and its proof are embedded in the proof of [LR, Lemma 2]. Note that Lemma 2 of [LR] is stated in terms of quasilocal integral domains, while our Lemma 3 and Lemma 4 deal with arbitrary commutative rings. Lemma 3 follows from a rather clever combination of the previous two.

Lemma 3. *Let $f(x) = g(x^2) + xh(x^2)$ of degree n . If f is a Gaussian polynomial of R , then so are $f(x)f(-x) = g(x^2)^2 - x^2h(x^2)^2$ and $g(x)^2 - xh(x)^2$. Moreover, $c(g(x)^2 - xh(x)^2) = c(f(x))^2$ and $c(f(x))^2$ can be generated by $n + 1$ elements (or fewer).*

Proof. It is clear that a finite product of Gaussian polynomials is Gaussian—simply factor one out at a time, then put them back together. For example, if r and s are Gaussian, then $c(rst) = c(r)c(st) = c(r)c(s)c(t) = c(rs)c(t)$. Thus if f is Gaussian, so is $f(x)f(-x) = g(x^2)^2 - x^2h(x^2)^2$. Now apply Lemma 1 to see that $g(x)^2 - xh(x)^2$ is Gaussian. Since $c(r(x^n)) = c(r(x))$ for each n , we have $c(g(x)^2 - xh(x)^2) = c(g(x^2)^2 - x^2h(x^2)^2) = c(f(x)f(-x)) = c(f(x))c(f(-x)) = c(f)^2$. As R may contain nonzero nilpotents, the degree of $r(x)^2$ is less than or equal to the degree of $r(x^2)$ for each polynomial $r(x) \in R[x]$ (with “less than” a distinct possibility). It follows that $\deg(g(x)^2 - xh(x)^2) \leq \deg(g(x^2) + xh(x^2)) = n$. Hence $c(f)^2$ can be generated by $n + 1$ elements. \square

Lemma 4 follows from a recursive application of Lemma 3. As with Lemma 3, the statement and the essence of the proof are from the proof given for [LR, Lemma 2].

Lemma 4. *If $f \in R[x]$ is a Gaussian polynomial of degree n , then for each non-negative integer m , $c(f(x))^{2^m}$ can be generated by $n + 1$ elements.*

Proof. Lemma 3 takes care of the case $m = 1$. Adopting the notation of Lemma 3, write $f(x) = g(x^2) + xh(x^2)$ and let $f_1(x) = g(x)^2 - xh(x)^2 = g_1(x^2) + xh_1(x^2)$. Again, by Lemma 3, we have that f_1 is a Gaussian polynomial of degree less than or equal to n whose content equals $c(f)^2$. Recursively apply Lemma 3 to produce a sequence of Gaussian polynomials $f_2 = g_1(x)^2 - xh_1(x)^2 = g_2(x^2) + xh_2(x^2)$, $f_3 = g_2(x)^2 - xh_2(x)^2 = g_3(x^2) + xh_3(x^2)$, etc. For each m , we have that f_m is a Gaussian polynomial with $\deg(f_m) \leq n$ and $c(f_m) = c(f_{m-1})^2 = c(f_{m-2})^4 = \cdots = c(f)^{2^m}$. Thus $c(f)^{2^m}$ can be generated by $n + 1$ elements. \square

For completeness we include a proof of the following lemma.

Lemma 5. *Let f be a polynomial over a ring R . Then f is Gaussian if and only if the image of f in $R_M[x]$ is Gaussian for each maximal ideal M .*

Proof. Assume f is Gaussian and let M be a maximal ideal of R . Then each polynomial $h \in R_M[x]$ can be written in the form $\sum (g_i/r)x^i$ for some elements $g_0, g_1, \dots, g_m \in R$ and $r \in R \setminus M$. In R , we have $c(f)c(g) = c(fg)$ where $g = \sum g_i x^i$. Since r is a unit of R_M , $c(f)R_M c(h) = c(f)R_M c(h) = c(fg)R_M = c(fg)R_M = c(fh)$. Hence the image of f in $R_M[x]$ is a Gaussian polynomial.

Conversely, if the image of f is Gaussian in $R_M[x]$ for each maximal ideal M , then for each polynomial $g \in R[x]$, $c(fg)R_M = c(f)c(g)R_M$. Since each ideal is characterized by its localizations at maximal ideals, we must have $c(fg) = c(f)c(g)$. Hence f is a Gaussian polynomial of $R[x]$. \square

Now for the main theorem.

Theorem 6. *Let $f \in R[x]$ be such that $c(f)$ is semiregular. Then the following are equivalent.*

- (1) f is Gaussian.
- (2) $c(f)\text{Hom}_R(c(f), R) = R$.
- (3) $c(f)$ is Q_0 -invertible.
- (4) For each maximal ideal M , $c(f)R_M$ is an invertible ideal of R_M .
- (5) $c(f)R_M$ is principal for each maximal ideal M .

Proof. The proof that (3) implies (1) is essentially the same as the proof when $c(f)$ is invertible in the traditional sense. If $c(f)$ is Q_0 -invertible, then there is a (finitely generated) R -submodule J of $Q_0(R)$ such that $Jc(f) = R$. If $g \in R[x]$ is such that $c(f)^m c(fg) = c(f)^{m+1} c(g)$, then $c(fg) = J^m c(f)^m c(fg) = J^m c(f)^{m+1} c(g) = c(f)c(g)$.

The equivalence of (2)–(5) is from [A] and [Lu2, Lemma 6]. Thus it remains to show that (1) implies at least one of (2) through (5). We will show that (1) implies (4).

Assume f is a Gaussian polynomial of degree n . There is nothing to prove if $n = 0$. By Lemma 4, there is a nonnegative integer k such that $c(f)^{k+1}$ can be generated by $k+1$ elements (or fewer) – simply take $k+1 = 2^m$ where $2^m \geq n+1$. The same set of generators will work locally. Thus by Corollary 1 of [ES], for each maximal ideal M there is an integer q (perhaps dependent on M) and an element $r \in c(f)^q R_M$ such that $rc(f)^q R_M = c(f)^{2q} R_M$. Since $c(f)$ is semiregular, so is $c(f)R_M$ and each of its powers. Thus r must be a regular element of $c(f)^q R_M$. An additional application of the Eakin-Sathaye Corollary 1 yields that $c(f)^k R_M$ must be a stable ideal of R_M . Hence by Lemma 1.11 of [Li] (see also [ES, Lemma, page 447]), $c(f)^k R_M$ is a principal ideal of $(c(f)^k R_M : c(f)^k R_M)$. [Note: This is a bit of a roundabout argument as Eakin and Sathaye prove first that $c(f)^k R_M$ is a principal ideal of $(c(f)^k R_M : c(f)^k R_M)$ and then invoke Lipman's Lemma 1.11 [Li] to show that $c(f)^k R_M$ is stable.]

As $c(f)^k R_M$ is a regular ideal of R_M , $(c(f)^k R_M : c(f)^k R_M)$ is contained in $T(R_M)$ and each of its elements is integral over R_M . Thus $c(f)^k R_M$ generates a principal regular ideal of $(R_M)'$, the integral closure of R_M . It follows that $c(f)(R_M)'$ is an invertible ideal of $(R_M)'$. Write $f = f_0 + f_1 x + \cdots + f_n x^n$ and let h_0, h_1, \dots, h_n be elements of $(c(f)R_M)'^{-1}$ for which $\sum h_{n-i} f_i = 1$. Then for $h = \sum h_j x^j$, the product $fh = u$ is a polynomial in $(R_M)'$ with unit content (as an ideal of $(R_M)'$) since the coefficient on x^n is 1. By the proof of [GH, Theorem 3], there are polynomials $v \in R_M[x]$ and $w \in (R_M)'[x]$ such that $v = uw$, v has unit content in R_M and w has unit content in $(R_M)'$. It follows that $f(hw) = v$ (as polynomials over $T(R_M)$). Since f is a Gaussian polynomial of R , its image in $R_M[x]$ is a Gaussian polynomial of R_M . Thus $R_M = c(v) = c(f(hw)) = (c(f)R_M)c(hw)$, and therefore $c(f)R_M$ is an invertible ideal of R_M . \square

Corollary 7. *If each regular polynomial in $R[x]$ is Gaussian, then R is strongly Prüfer.*

Proof. Assume that each regular polynomial in $R[x]$ is Gaussian and let $I = (a_0, a_1, \dots, a_n)$ be a finitely generated semiregular ideal of R . Then the polynomial $f(x) = \sum a_i x^i$ is regular and obviously has content I . By Theorem 6, I is Q_0 -invertible (and locally principal). That R is strongly Prüfer now follows from [Lu2, Theorem 8] (or simply by the definition [AAM]). \square

If R is an arithmetical ring, then the ideals of R_M are linearly ordered for each maximal ideal M . This in turn implies that each finitely generated ideal of R_M is principal (in fact, both properties are equivalent to R being arithmetical). For such a ring R , every polynomial is Gaussian. By way of the following example, we show that the converse does not hold. [Note: The converse does hold for integral domains [G, Theorem 28.6] (and now also by [LR, Theorem 4]).] In some sense this example generalizes the statement following Theorem 1.1 in [AK] — if R is a quasilocal ring with a nonzero maximal ideal whose square is zero, then each polynomial over R is Gaussian.

Example 8. Let D be a Prüfer domain with quotient field K and let $R = D(+)(K \oplus K)$ be the ring formed by taking the idealization of $K \oplus K$ over D . Then each polynomial over R is Gaussian, but there exists a Gaussian polynomial whose content is not locally principal.

Proof. The ring R consists of all pairs (r, b) where $r \in R$ and $b = (b_1, b_2) \in K \oplus K$ with addition and multiplication defined as follows:

$$(r, b) + (s, c) = (r + s, b + c), \quad (r, b)(s, c) = (rs, rc + sb).$$

The polynomials over R have the form (f, g) where $f \in D[x]$ and $g \in (K \oplus K)[x]$. Clearly, no nonzero element of D is a zero divisor on $K \oplus K$. Thus the nilradical of R is also its set of zero divisors, namely $(0)(+)(K \oplus K)$. Also, since D is a Prüfer domain, R is a Prüfer ring and so each finitely generated regular ideal of R is invertible [Lu1, Proposition 3.1 and Corollary 3.2]. Hence if $h \in R[x]$ is a polynomial whose content is regular, then h is Gaussian. The only polynomials with content not regular are those whose coefficients are all nilpotent. Let g be such a polynomial and let h be an arbitrary polynomial of R . If $c(h)$ is regular, then $c(gh) = c(g)c(h)$ since h is Gaussian. On the other hand, if $c(h)$ is not regular, then all of the coefficients of h are nilpotent. In this case we have $c(gh) = c(g)c(h) = (0)$ since the square of the nilradical is zero. Thus each polynomial is Gaussian.

The maximal ideals of R are all of the form $MR = M(+)(K \oplus K)$ for some maximal ideal M of D . Localizing at such an ideal yields the ring $D_M(+)(K \oplus K)$. Let $g = (0, (1, 0)x + (0, (0, 1)))$. The content of g is $(0)(+)(D \oplus D)$, an ideal that requires at least two generators. Localizing $c(g)$ at the maximal ideal $M(+)(K \oplus K)$ yields the two-generator ideal $(0)(+)(D_M \oplus D_M)$ of R_{MR} . As with $c(g)$, $c(g)R_M$ requires at least two generators. Thus g is a Gaussian polynomial whose content is not locally principal. \square

The ring in the previous example is strongly Prüfer since it is a Prüfer ring (i.e., each finitely generated regular ideal is invertible) where each semiregular ideal is regular.

Corollary 9. *Let R be a Prüfer ring. If each zero divisor of R is nilpotent and the square of the nilradical is (0) , then R is strongly Prüfer and each polynomial in $R[x]$ is Gaussian.*

Proof. If each zero divisor of R is nilpotent, then each finitely generated ideal is either nilpotent or both regular and invertible. Thus R is strongly Prüfer. With the additional assumption that the square of the nilradical is zero, we may simply repeat the proof given for Example 8 to show that each polynomial is Gaussian. \square

Suppose n and m are (nonzero) nilpotent elements of R such that $n^2 = m^2 = 0$ but $nm \neq 0$. Then the polynomial $f(x) = nx + m$ is not Gaussian since $(nx + m)(nx - m) = n^2x^2 - m^2 = 0$ but $c(f(x))c(f(-x)) = (n, m)^2 = (nm)$, which is not the zero ideal. If $2nm$ is not zero, the polynomial $nx + m$ also illustrates the fact that a power of a polynomial may be a nonzero Gaussian polynomial even though the original polynomial is not. In particular, if K is a field not of characteristic 2, then each polynomial over $K[y, z]/(y, z)^2$ is Gaussian while the polynomial $yx + z$ is not Gaussian over $K[y, z]/(y^2, z^2)$ but $(yx + z)^2$ is.

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